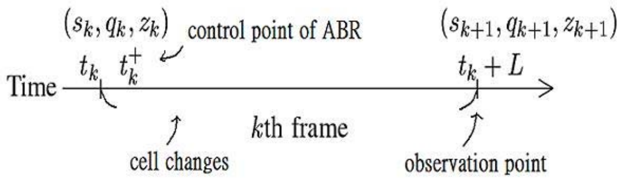
Chapter 4

Steady-State of ABR Cells at Buffer

We introduce a Markovian approach to construct a one-step transition probability and to calculate the stationary probability of the buffer size of ABR. Following [3], let $z_k = 1, 2, \ldots, M$ and $q_k = 0, 1, \ldots, C$ be random variables to denote the ABR sources status and the queue length of the buffer at the beginning time t_k of the kth frame, respectively. We define a 3-dimensional stochastic process $\{(s_k, q_k, z_k), k = 1, 2, \ldots\}$ which is embedded at the beginning of frame $\{t_k, k = 1, 2, \ldots\}$ where s_k is the arrival status of VBR. In this chapter, we construct the one step transition matrix and find the steady-state probability vector. From [3] it is assumed that changing the state occurs only at the beginning of the frame. Suppose that we observe this change of the state at departure. In addition, there would be a control point of z_k at instant time t_k^+, at which it adjusts a number of cells to enter into the buffer of ABR. If the buffer is full, those ABR sources will be blocked. In the transmission line, it is assumed the length of a frame is L. From time moving through L slots until $t_k + L$, we observe numbers of q_{k+1} in next state. During a frame, every slot may be occupied by VBR, ABR or no cell. The detailed description diagram is shown in Figure 4.1.

Figure 4.1: Change of the state occur only at the beginning of the frame



Given the buffer of ABR capacity C, it is clear that $0 \leq g_1 \leq g_2 \leq C$. According to the packet transition mechanism described in the previous section, the state transition is defined between t_k and $t_k + L$ at the kth frame. This is depicted at Figure 4.1. Assume denote by z_k the status indicator $z_k = 1, 2, \ldots, M$. Let B be a batch size of packets. Since the maximum number of the ABR buffer size is C and the frame length is L, if $C < L$, the maximum number of ABR packets that can be transmitted is C, implying there are at least $L - C$ slots left that can be used by VBR. Let

$$\beta = 1 - \sum_{s_{k+1}=0}^{1} \sum_{i=0}^{C} Pr\{s_{k+1}|s_k\} \binom{L}{i} (1 - v(s_k))^i v(s_k)^{L-i},$$

If $C \geq L$, β is zero.

Given s_k, we define $p_x(s_k)$ as follows

$$p_x(s_k) \quad = \quad Pr\{x \text{ ABR cells arrive at the frame given VBR sources is in } s_k \text{ level.}\}.$$

Notice the difference between $p_x(s_k)$ and $p(s_k)$ which was defined in Chapter 3. We give the following lemma,

Lemma 4.1 *Given the arrival status of VBR is s_k, the probability of x ABR transmitted to the frame is*

$$p_x(s_k) \quad = \quad \frac{1}{1-\beta} \binom{L}{x} (1 - v(s_k))^x v(s_k)^{L-x}, \quad 0 \leq x \leq C.$$

Remark 1 $\sum_{s_{k+1}=0}^{1} \sum_{x=0}^{C} Pr\{s_{k+1}|s_k\}p_x(s_k) = 1$ *given* s_k.

Since VBR cells have a higher priority, ABR cells can only be transmitted when VBR cells do not occupy the cells in the frame. For each VBR cell, $v(s_k)$ is the probability of VBR cell appearing at the slot. The transition probability is

$$Pr\{(s_{k+1}, q_{k+1}, z_{k+1})|(s_k, q_k, z_k)\} = Pr\{s_{k+1}|s_k\}p_x(s_k), 0 \leq x \leq C.$$

Depending on q_k and q_{k+1}, x is considered in the following

$$(i)\ 0 \leq q_k \leq g_1, \quad x = q_k + \alpha - q_{k+1} \text{ where } \alpha = \min\{(z_k + 1) \times B, C - q_k\}. \tag{4.1}$$

$$(ii)\ g_1 < q_k \leq g_2, \quad x = q_k + \alpha - q_{k+1} \text{ where } \alpha = \min\{(z_k + 0) \times B, C - q_k\} \tag{4.2}$$

$$(iii)\ g_2 < q_k \leq C, \quad x = q_k + \alpha - q_{k+1} \text{ where } \alpha = \min\{(z_k - 1) \times B, C - q_k\} \tag{4.3}$$

The transition probability matrix of (s_k, q_k, z_k) can be written as the $2M(C +$

$1) \times 2M(C+1)$ matrix, Γ,

$$\Gamma = \begin{bmatrix}
A_{1,0} & \cdots & A_{1,j} & \cdots & A_{2,j} & \cdots & A_{i,j} & \cdots & A_{e,2eB} & 0 & \cdots & 0 & 0 \\
A_{1,-1} & \cdots & A_{1,j-1} & \cdots & A_{2,j-1} & \cdots & A_{i,j-1} & \cdots & A_{e,j-1} & A_{e,2eB} & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots & \vdots \\
A_{1,-g_1} & \cdots & A_{1,-1} & \cdots & A_{1,j-1} & \cdots & A_{2,j-1} & \cdots & A_{i,j-1} & A_{i,j} & \cdots & 0 & 0 \\
F_{1,0} & \cdots & \cdots & \cdots & \cdots & \cdots & F_{i,j} & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
F_{1,d} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & F_{i,j} & \cdots & \cdots & 0 & 0 \\
G_{1,0} & \cdots & \cdots & \cdots & \cdots & \cdots & G_{i,j} & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
G_{1,l} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & G_{i,j} & \cdots & \cdots & \cdots & G_{e,C} \\
H_{c_1+1,0} & \cdots & \cdots & \cdots & \cdots & \cdots & H_{c_1+1,j} & \cdots & \cdots & \cdots & \cdots & \cdots & H_{c_1+1,C} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
H_{c_2-1,0} & \cdots & \cdots & \cdots & \cdots & \cdots & H_{c_2-1,j} & \cdots & \cdots & \cdots & \cdots & \cdots & H_{c_2-1,C} \\
J_0 & \cdots & \cdots & \cdots & \cdots & \cdots & J_j & \cdots & \cdots & \cdots & \cdots & \cdots & J_C \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
J_0 & \cdots & \cdots & \cdots & \cdots & \cdots & J_j & \cdots & \cdots & \cdots & \cdots & \cdots & J_C
\end{bmatrix},$$

where $e = (M-1), c_1 = C - eB, c_2 = C - B, a = g_1 - g_2, l = g_2 - C + eB + 1$.

Denote that

$$R_j = \begin{bmatrix} p_j(0)(1-p(0)) & p_j(0)p(0) \\ p_j(1)p(1) & p_j(1)(1-p(1)) \end{bmatrix};$$

for $j = -g_1, \ldots, 0, \ldots, MB$,

$$A_{1,j} = \begin{bmatrix}
\mathbf{0} & R_{2B-j} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & R_{3B-j} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & R_{(M-1)B-j} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & R_{MB-j} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & R_{MB-j}
\end{bmatrix};$$

for $i = 2, \ldots, M-2, j = (M+i-2)B+1, \ldots, (M+i-1)B$,

$$A_{i,j} = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & R_{iB-j} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & R_{(M-1)B-j} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & R_{MB-j} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & R_{MB-j} \end{bmatrix};$$

for $j = (2M-3)B+1, \ldots, 2(M-1)B$,

$$A_{e,j} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & R_{MB-j} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & R_{MB-j} \end{bmatrix};$$

for $j = g_1 - g_2, \ldots, 0, \ldots, (B + g_1 + 1)$,

$$F_{1,j} = \begin{bmatrix} R_{B+q_k-q_{k+1}} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & R_{2B+q_k-q_{k+1}} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & R_{(M-1)B+q_k-q_{k+1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & R_{MB+q_k-q_{k+1}} \end{bmatrix};$$

for $i = 2, \ldots, M-1, j = (M+i-2)B+1, \ldots, (M+i-1)B$,

$$F_{i,j} = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & R_{iB+q_k-q_{k+1}} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & R_{MB+q_k-q_{k+1}} \end{bmatrix};$$

for $j = g_2 - c_1 + 1, \ldots, 0, \ldots, MB + g_2 + 1$,

$$G_{1,j} = \begin{bmatrix} R_{B+q_k-q_{k+1}} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ R_{B+q_k-q_{k+1}} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & R_{2B+q_k-q_{k+1}} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & R_{(M-1)B+q_k-q_{k+1}} & \mathbf{0} \end{bmatrix};$$

for $j = MB + g_2 + 1, \ldots, MB + g_2 + B$,

$$G_{2,j} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & R_{2B+q_k-q_{k+1}} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & R_{(M-1)B+q_k-q_{k+1}} & \mathbf{0} & \mathbf{0} \end{bmatrix};$$

for $i = 3, \ldots, M-1, j = (M+i-2)B + g_2 + 1, \ldots, (M+i-1)B + g_2$,

$$G_{i,j} = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cdots & R_{(i-1)B+q_k-q_{k+1}} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & R_{(M-1)B+q_k-q_{k+1}} & \mathbf{0} \end{bmatrix};$$

for $i = c_1 + 1, \ldots, c_2 - 1, j = 0, \ldots, C$,

$$H_{1,j} = \begin{bmatrix} R_{B+q_k-q_{k+1}} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ R_{B+q_k-q_{k+1}} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & R_{2B+q_k-q_{k+1}} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & R_{(M-1)B+q_k-q_{k+1}} & \mathbf{0} \end{bmatrix};$$

for $j = 0, \dots, C$,

$$J_j = \begin{bmatrix} R_{C-j} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ R_{C-j} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & R_{C-j} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & R_{C-j} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & R_{C-j} & \mathbf{0} \end{bmatrix}$$

in which $A_{ij}, F_{ij}, G_{ij}, H_{ij}$ and J_j are $2M \times 2M$ matrices.

In order to understand the properties of Γ matrix, we give several lemmas below,

Lemma 4.2 *The matrix Γ is a stochastic matrix, i.e.,*

$$0 \le \Gamma_{i,j} \le 1 \quad , \quad 1 \le i, j \le C,$$
$$\sum_{j=0}^{C} \Gamma_{i,j} = 1 \quad , \quad 1 \le i \le C.$$

where Γ_{ij} is the element at the ith row and the jth column of Γ.

Definition 1 *A markov chain is said to be irreducible if every state i can be reached from every other state j after a finite number of transitions – that is,*

$$\Gamma_{ij}^{(n)} > 0, \quad i \ne j, \quad \textit{for some nature number } n,$$

where $\Gamma_{ij}^{(n)}$ denotes the corresponding element of Γ after n-step transition.

Lemma 4.3 *Matrix Γ is irreducible.*

Proof:
It is easy to check from (4.1), (4.2) and (4.3) that $\Gamma_{ij}^{(n)} > 0$ for some $n > 0$ since $0 \le j = i + \alpha - x \le C$ where $j = q_{k+n}$ and $i = q_k$ for some $k > 0$. □

Definition 2 *A state is periodic with period ξ if a return is possible only in $\xi, 2\xi, 3\xi, \ldots$ steps. This means that $\Gamma_{jj}^{(n)} = 0$ whenever n is not divisible by ξ. In particular, a state is called aperiodic with period 1.*

Lemma 4.4 *If Γ is the transition matrix for an irreducible, aperiodic Markov chain, then there exists a unique invariant probability vector $\vec{\boldsymbol{\pi}}$ satisfying*

$$\vec{\boldsymbol{\pi}}\Gamma = \vec{\boldsymbol{\pi}}.$$

If $\vec{\boldsymbol{\kappa}}$ is any initial probability vector,

$$\lim_{n\to\infty} \vec{\boldsymbol{\kappa}}\Gamma^{(n)} = \vec{\boldsymbol{\pi}}.$$

Lemma 4.5 *In Γ, when $q_k \leq g_1$ and $(z_k+1)C < (q_{k+1} - q_k)$, there is no transition. So is the case $g_1 < q_k \leq C$ and $(z_k+1)B < (q_{k+1} - q_k)$.*

Proof:
Because $q_{k+1} = q_k + \alpha - x$ from (4.1), (4.2) and (4.3), one may check if $q_k \leq g_1$, then

$$q_{k+1} - q_k = \alpha - x \leq (z_k + 1)B - x \leq (z_k + 1)C.$$

In case of $g_1 < q_k \leq C$, we have

$$q_{k+1} - q_k = \alpha - x \leq (z_k + 1)B. \qquad \square$$

Lemma 4.6 *z_k is determined by $z_k \geq \frac{q_{k+1} - q_k}{B} - 1$.*

When g_1 and g_2 are given, we can construct the transition matrix Γ, and find a steady-state probability vector. Let $\vec{\mathbf{e}} = (1, 1, \ldots, 1)$ be $2M$-dimensional column vector and $\vec{\boldsymbol{\pi}} = (\vec{\pi_0}, \vec{\pi_1}, \ldots, \vec{\pi_C})$, where $\vec{\pi_q} = (\pi_{(0,q,1)}, \pi_{(1,q,1)}, \ldots, \pi_{(0,q,M)}, \pi_{(1,q,M)})$ is a row vector with $2M$ elements. $\vec{\boldsymbol{\pi}}$ be a $2M(C+1)$-dimensional row vector of the steady-state probability of Γ. Those probabilities, as they exist, are computed from the equations

$$\boldsymbol{\vec{\pi}}\Gamma = \boldsymbol{\vec{\pi}},$$

$$\sum_{i=0}^{C} \vec{\pi}_i \vec{\mathbf{e}} = 1.$$

From the above equations, we can rewrite them as follows.

$$\begin{aligned}
\sum_{j=0}^{g_1} A_{1,j}\vec{\pi_j} + \sum_{j=g_1+1}^{d} F_{1,j}\vec{\pi_j} + \sum_{j=d+1}^{l} G_{1,j}\vec{\pi_j} + \sum_{i=c_1+1}^{c_2-1} H_{i,0}\vec{\pi_i} + J_0 \sum_{j=c_2}^{C} \vec{\pi_j} &= \vec{\pi_0} \\
\vdots &= \vdots \\
G_{e,C}\vec{\pi_{c_1}} + \sum_{i=c_1+1}^{c_2-1} H_{i,C}\vec{\pi_i} + J_C \sum_{j=c_2}^{C} \vec{\pi_j} &= \vec{\pi_C} \\
\sum_{i=0}^{C} \vec{\pi_i}\vec{\mathbf{e}} &= 1
\end{aligned}$$

In order to illustrate the structure of the Γ matrix, we give a simple numerical example to describe detailed equations where $C = 10, B = 2$ and $M = 3$. Its state diagram may refer to Appendix A.

$$\Gamma = \left[\begin{array}{ccccccccccccccccccccccccccccccccc}
\mathbf{0} & R_4^+ & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & R_6^+ & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & 0 & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & R_6^+ & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & 0 & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & R_5^+ & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & R_7^+ & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & R_7^+ & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & R_6^+ & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & R_8^+ & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & R_8^+ & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & R_7^+ & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & R_9^+ & \mathbf{0} & \mathbf{0} & R_8 & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & R_9^+ & \mathbf{0} & \mathbf{0} & R_8 & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
R_6^+ & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & R_8^+ & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & R_{10} & \mathbf{0} & \mathbf{0} & R_9 & \mathbf{0} & \mathbf{0} & R_8 & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 \\
R_7^+ & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & R_9^+ & \mathbf{0} & \mathbf{0} & R_8 & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & R_{10} & \mathbf{0} & \mathbf{0} & R_9 & \mathbf{0} & \mathbf{0} & R_8 & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 \\
R_8^+ & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & R_{10} & \mathbf{0} & \mathbf{0} & R_9 & \mathbf{0} & \mathbf{0} & R_8 & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & R_{10} & \mathbf{0} & \mathbf{0} & R_9 & \mathbf{0} & \mathbf{0} & R_8 & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 \\
R_9^+ & \mathbf{0} & \mathbf{0} & R_8 & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
R_9^+ & \mathbf{0} & \mathbf{0} & R_8 & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & R_{10} & \mathbf{0} & \mathbf{0} & R_9 & \mathbf{0} & \mathbf{0} & R_8 & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} \\
R_{10} & \mathbf{0} & \mathbf{0} & R_9 & 0 & \mathbf{0} & R_8 & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} \\
R_{10} & \mathbf{0} & \mathbf{0} & R_9 & 0 & \mathbf{0} & R_8 & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & R_{10} & \mathbf{0} & \mathbf{0} & R_9 & 0 & \mathbf{0} & R_8 & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} \\
R_{10} & \mathbf{0} & \mathbf{0} & R_9 & 0 & \mathbf{0} & R_8 & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} \\
R_{10} & \mathbf{0} & \mathbf{0} & R_9 & 0 & \mathbf{0} & R_8 & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & R_{10} & \mathbf{0} & \mathbf{0} & R_9 & \mathbf{0} & \mathbf{0} & R_8 & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & 0 & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} \\
R_{10} & \mathbf{0} & \mathbf{0} & R_9 & 0 & \mathbf{0} & R_8 & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} \\
R_{10} & \mathbf{0} & \mathbf{0} & R_9 & 0 & \mathbf{0} & R_8 & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & R_{10} & \mathbf{0} & \mathbf{0} & R_9 & \mathbf{0} & \mathbf{0} & R_8 & \mathbf{0} & \mathbf{0} & R_7 & \mathbf{0} & \mathbf{0} & R_6 & \mathbf{0} & \mathbf{0} & R_5 & \mathbf{0} & \mathbf{0} & R_4 & \mathbf{0} & \mathbf{0} & R_3 & \mathbf{0} & \mathbf{0} & R_2 & \mathbf{0} & \mathbf{0} & R_1 & \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0}
\end{array}\right].$$

Suppose $p(0) = p(1) = r(0) = r(1) = 0.5, L = 20, C = 10$. By (3.1) we have $y(0) = y(1) = 0.5$ which implies that $v(s_k) = 0.25$ and $\beta = 0.986$. By Lemma 4.1, we can compute $p_x(s_k), x = 0, \ldots, 10$. From the above equations, the solution yields $\pi_0 = 9.998 \times 10^{-1}, \pi_1 = 2.000 \times 10^{-4}, \pi_3 = 0, \pi_4 = 0, \pi_5 = 0, \pi_6 = 0, \pi_7 = 0, \pi_8 = 0, \pi_9 = 0, \pi_{10} = 0$. It gives probabilities $p_x(s_k), 0 \leq x \leq C$ since $\beta = 0.986$. It implies the most of time the buffer of ABR is close to be empty, that is $\pi_0 \cong 0.99$.

Let $E[X]$ represent the expected numbers of ABR cells arriving in the channel, which is given by

$$E[X] = (\sum_{z=0}^{M} Bz \sum_{q=0}^{C} \sum_{s=0}^{1} \pi_{(s,q,z)})(1 - \sum_{z=0}^{M} \sum_{s=0}^{1} \pi_{(s,C,z)}). \tag{4.4}$$

Suppose $E[Q]$ represents the expected average numbers of queue length of ABR cells at buffer, which is given by

$$E[Q] = \sum_{s=0}^{1} \sum_{z=0}^{M} \sum_{q=0}^{C} q \cdot \pi_{(s,q,z)}. \tag{4.5}$$

From (4.5), $E[Q]$ is clearly a function of g_1 and g_2. Hence, (4.5) can be written as

$$E[Q] = E_Q(g_1, g_2).$$

Meanwhile, let $V[Q]$ denote the variance of queue length of ABR cells at buffer. Hence, it may be written as follows

$$V[Q] = E[Q^2] - E^2[Q].$$

We can easily verify that $V[Q]$ is also a function of g_1 and g_2, denoted as $V_Q(g_1, g_2)$.

Now, we assume $p(0)$ and $p(1)$ are fixed and define the probability of occurrence of VBR and ABR at channel, respectively, by

$$\begin{aligned} u_V &= Pr\{\text{A VBR cell arrives at a slot in the channel}\} \\ &= v(0) + v(1) \\ &= r(0)y(0) + r(1)y(1) \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} u_A &= Pr\{\text{An ABR cell arrives at a slot in the channel}\} \\ &= Pr\{\text{An ABR cell arrives at a slot in the channel} \mid \text{no VBR arrival at a slot}\} \\ &\quad \times Pr\{\text{no VBR arrival at a slot}\} \\ &= \frac{E[X]}{L}(1 - u_V). \end{aligned} \tag{4.7}$$

Lemma 4.7 *The utilization of ABR in the channel is* $\frac{E[X]}{L}(1 - u_V)$ *which is also the probability that the VBR buffer is not empty. Therefore the probability of no VBR cell at VBR buffer is* $1 - \frac{E[X]}{L}(1 - u_V)$

Proof:

Consider that ABR cells arrive in the system when no VBR cells occur. These ABR cells would be served with probability $\frac{u_A}{1-u_V}$. Therefore, from (4.4) it could be written as

$$E[X] = \sum_{i=0}^{L} i \binom{L}{i} \left(\frac{u_A}{1-u_V}\right)^i \left(1-\frac{u_A}{1-u_V}\right)^{L-i}.$$

By the mean of Binomial distribution, we get

$$\begin{aligned} E[X] = & \; L\left(\frac{u_A}{1-u_V}\right) \\ \Rightarrow \quad u_A = & \; \frac{E[X]}{L}(1-u_V). \end{aligned} \qquad \square$$

From (4.7), we understand that u_A is a function of g_1 and g_2, because u_A can be determined by parameters g_1, g_2, $r(\cdot)$ and $y(\cdot)$. In other words, the matrix is constructed and $\vec{\pi}$ is calculated while parameters g_1, g_2, $r(\cdot)$ and $y(\cdot)$ are given. Here, we may consider that the domain of g_1 and g_2 is compact that is relaxed on lattice $\{0, 1, 2, \ldots, C\}$. Assume that $E_Q(g_1, g_2)$ is continuous on this relaxed domain. So we regard all functions with respect to g_1 and g_2 as continuous on this relaxed domain. Then, the u_A is obtained immediately. Consider it is written as

$$u_A = u_A(g_1, g_2). \tag{4.8}$$

Provided g_2 fixed, the partial derivative of (4.8) is given with respect to g_1,

$$\begin{aligned} \frac{\partial u_A(g_1,g_2)}{\partial g_1} = & \; \frac{(1-u_V)}{L}[\left(\sum_{z=0}^{M} Bz \sum_{q=0}^{C}\sum_{s=0}^{1} \frac{\partial \pi_{(s,q,z)}(g_1,g_2)}{\partial g_1}\right)\left(1-\sum_{z=0}^{M}\sum_{s=0}^{1}\pi_{(s,C,z)}(g_1,g_2)\right) \\ & - \left(\sum_{z=0}^{M} Bz \sum_{q=0}^{C}\sum_{s=0}^{1}\pi_{(s,q,z)}(g_1,g_2)\right)\left(\sum_{z=0}^{M}\sum_{s=0}^{1}\frac{\partial \pi_{(s,C,z)}(g_1,g_2)}{\partial g_1}\right)]. \end{aligned} \tag{4.9}$$

Denote that

$$\theta(g_1,g_2) = \sum_{z=0}^{M} Bz \sum_{q=0}^{C}\sum_{s=0}^{1}\pi_{(s,q,z)}(g_1,g_2); \qquad \Delta\theta_1(g_1,g_2) = \sum_{z=0}^{M} Bz \sum_{q=0}^{C}\sum_{s=0}^{1}\frac{\partial \pi_{(s,q,z)}(g_1,g_2)}{\partial g_1},$$

$$\gamma(g_1,g_2) = \sum_{z=0}^{M}\sum_{s=0}^{1}\pi_{(s,C,z)}(g_1,g_2); \qquad \Delta\gamma_1(g_1,g_2) = \sum_{z=0}^{M}\sum_{s=0}^{1}\frac{\partial \pi_{(s,C,z)}(g_1,g_2)}{\partial g_1}.$$

Similarly, provided g_1 fixed, the partial derivative of (4.8) is written with respect to g_2,

$$\begin{aligned}\frac{\partial u_A(g_1,g_2)}{\partial g_2} &= \frac{(1-u_V)}{L}[\left(\sum_{z=0}^{M} Bz\sum_{q=0}^{C}\sum_{s=0}^{1}\frac{\partial\pi_{(s,q,z)}(g_1,g_2)}{\partial g_2}\right)\left(1-\sum_{z=0}^{M}\sum_{s=0}^{1}\pi_{(s,C,z)}(g_1,g_2)\right)\\ &\quad -\left(\sum_{z=0}^{M} Bz\sum_{q=0}^{C}\sum_{s=0}^{1}\pi_{(s,q,z)}(g_1,g_2)\right)\left(\sum_{z=0}^{M}\sum_{s=0}^{1}\frac{\partial\pi_{(s,C,z)}(g_1,g_2)}{\partial g_2}\right)].\end{aligned}$$

Denote that

$$\Delta\theta_2(g_1,g_2)=\sum_{z=0}^{M} Bz\sum_{q=0}^{C}\sum_{s=0}^{1}\frac{\partial\pi_{(s,q,z)}(g_1,g_2)}{\partial g_2};\quad \Delta\gamma_2(g_1,g_2)=\sum_{z=0}^{M}\sum_{s=0}^{1}\frac{\partial\pi_{(s,C,z)}(g_1,g_2)}{\partial g_2}.$$

Since the system is stable, every steady-state probability is positive. In particularly, $\pi_{(s,C,z)}(g_1,g_2)$ increases as g_i increases, $i=1,2$ under certain conditions. Therefore, we have a lemma

Lemma 4.8 *If* $\frac{\Delta\theta_i(g_1,g_2)}{\theta(g_1,g_2)} > \frac{\Delta\gamma_i(g_1,g_2)}{1-\gamma(g_1,g_2)}$, *then* $\frac{\partial u_A(g_1,g_2)}{\partial g_i} > 0, i=1,2.$

Under this condition, the utilization of ABR cells is nondecreasing when g_1 increases. Similarly, the utilization of ABR cells is nondecreasing when g_2 increases.

Clearly, the utilization of the channel of this system is defined as $\rho = u_V + u_A$, then ρ is also a function of g_1 and g_2 and is denoted by $\rho(g_1,g_2)$. The proportion of idleness of the transmission link of at this system is

$$\begin{aligned} u_{idle} &= Pr\{\text{A slot contains no ABR and VBR in the channel}\}\\ &= 1-u_V-u_A. \end{aligned}\tag{4.10}$$

We continue to discuss the case of ρ. Provided g_2 fixed, the partial derivative of $\rho(g_1,g_2)$ with respect to g_1,

$$\begin{aligned}\frac{\partial \rho(g_1,g_2)}{\partial g_1} &= \frac{(1-u_V)}{L}[\left(\sum_{z=0}^{M} Bz\sum_{q=0}^{C}\sum_{s=0}^{1}\frac{\partial\pi_{(s,q,z)}(g_1,g_2)}{\partial g_1}\right)\left(1-\sum_{z=0}^{M}\sum_{s=0}^{1}\pi_{(s,C,z)}(g_1,g_2)\right)\\ &\quad -\left(\sum_{z=0}^{M} Bz\sum_{q=0}^{C}\sum_{s=0}^{1}\pi_{(s,q,z)}(g_1,g_2)\right)\left(\sum_{z=0}^{M}\sum_{s=0}^{1}\frac{\partial\pi_{(s,C,z)}(g_1,g_2)}{\partial g_1}\right)].\end{aligned}\tag{4.11}$$

Next, provided g_2 fixed, the partial derivative of $\rho(g_1, g_2)$ with respect to g_1,

$$\begin{aligned}\frac{\partial\rho(g_1,g_2)}{\partial g_2} &= \frac{(1-u_V)}{L}[\left(\sum_{z=0}^{M} Bz \sum_{q=0}^{C}\sum_{s=0}^{1} \frac{\partial\pi_{(s,q,z)}(g_1,g_2)}{\partial g_2}\right)\left(1-\sum_{z=0}^{M}\sum_{s=0}^{1}\pi_{(s,C,z)}(g_1,g_2)\right) \\ &\quad -\left(\sum_{z=0}^{M} Bz \sum_{q=0}^{C}\sum_{s=0}^{1}\pi_{(s,q,z)}(g_1,g_2)\right)\left(\sum_{z=0}^{M}\sum_{s=0}^{1}\frac{\partial\pi_{(s,C,z)}(g_1,g_2)}{\partial g_2}\right)]. \end{aligned} \tag{4.12}$$

Lemma 4.9 *If* $\dfrac{\Delta\theta_i(g_1,g_2)}{\theta(g_1,g_2)} > \dfrac{\Delta\gamma_i(g_1,g_2)}{1-\gamma(g_1,g_2)}$, *then* $\frac{\partial\rho(g_1,g_2)}{\partial g_i} > 0, i = 1, 2.$

(4.11) and (4.12) have similar results with (4.8) and (4.9). We have that $\rho(g_1, g_2)$ increases as g_1 increases. This is because the increase of the low critical number g_1 causes an increase in the number of the low congestion status of buffer. From (4.12), we obtain that $\rho(g_1, g_2)$ increases as g_2 increases. This is because the increase of the upper critical number g_2 causes an decrease in the number of the high congestion status of buffer.

The above expression equations verify the numerical results in the experiments of [3].