Chapter 5

Nonlinear Second Order Elliptic Equations in An Exterior Domain

5.1 Introduction

We consider the nonlinear second order elliptic equation

$$(PE)\ \Delta u + f(x, u, \nabla u) = 0.\ x \in G_A,$$

in an exterior domain $G_A = \{x \in \mathbb{R}^n \mid |x| > A\}$, where $n \geq 3$ and $A > 0$. We try to prove that under quite general assumptions on function f, the equation (PE) has a positive solution in $G_B = \{x \in \mathbb{R}^n \mid |x| > B\}$ for some $B \geq A$, that is, there exists a function $u \in C^2(G_B)$ such that u satisfies (PE) at every point $x \in G_B$. A lower solution of (PE), is a function $u \in C^2(G_B)$ satisfies $\Delta u + f(x, u, \nabla u) \geq 0$, and an upper solution of (PE) is a function $u \in C^2(G_B)$ satisfies $\Delta u + f(x, u, \nabla u) \leq 0$.

In 1997, A. Constantin [18, 19] proved the existence of the equation

$$(PE^*)\ \Delta u + p(x, u) + q(|x|)x \cdot \nabla u = 0.$$

in the exterior domain G_A as follows:

Theorem 5.1.1 ([18, 19]) *Assume that p is locally Hölder continuous in $G_A \times \mathbb{R}$ ([28]) and q is of $C^1(\mathbb{R}^+)$. If*

$$0 \le p(x,t) \le a(|x|)w(t),\ t \in \mathbb{R}^+,\ x \in \mathbb{R}^n,$$

where $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $w \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ with $w(0) = 0$, then there is a positive solution $u(x)$ to (PE^*) on G_B for some $B \ge A$ with $\lim_{|x|\to\infty} u(x) = 0$ provided q is bounded and

$$\int_0^\infty s[a(s) + |q(s)|]ds < \infty.$$

We shall extend Theorem 5.1.1 to a more general result in the next section.

5.2 Construction of Upper and Lower Solutions

Denote $S_B = \{x \in \mathbb{R}^n \mid |x| = B\}$ for $B \ge A$. In order to prove our main result, we need the following excellent lemma, see Noussair and Swanson [55].

Lemma 5.2.1 *Assume that f is locally Hölder continuous in $G_A \times \mathbb{R} \times \mathbb{R}^n$. If there are a positive lower solution w and a positive upper solution v to (PE) in G_B such that $w(x) \le v(x)$ for all $x \in G_B \cup S_B$, then (PE) has a solution u in G_B satisfying $w(x) \le u(x) \le v(x)$ in $G_B \cup S_B$ and $u(x) = v(x)$ on S_B.*

We are now in a position to state and prove our main result.

Theorem 5.2.2 *Suppose that f is locally Hölder continuous in $G_A \times \mathbb{R} \times \mathbb{R}^n$ and satisfies*

$$0 \le f(x,t,z) \le k(|x|,t) + g(|x|, x \cdot z),\ t \in \mathbb{R}^+, x \in \mathbb{R}^n, z \in \mathbb{R}^n,$$

where k and g satisfy the following conditions:
(A_{10}) *$k \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ with $k(\cdot, 0) = 0$ satisfies a Lipschitz condition with respect to the second variable, that is, there exists a bounded function $M_1 \in L^1(\mathbb{R}^+; (0,\infty))$*

such that $|k(a,b)| \le M_1(a)|b|$ *on* $\mathbb{R}^+ \times \mathbb{R}^+$,

(A_{11}) $g \in C(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$ *with* $g(\cdot, 0) = 0$ *satisfies the following condition: there exists a bounded function* $M_2 \in L^1(\mathbb{R}^+; (0,\infty))$ *such that* $g(a,b) \le M_2(a)b$ *on* $\mathbb{R}^+ \times \mathbb{R}$. *Then there is a positive solution* $u(x)$ *to* (PE) *on* G_B *for some* $B \ge A$ *with* $\lim_{|x|\to\infty} u(x) = 0$ *if* $\int_0^\infty s[M_1(s) + M_2(s)]ds < \infty$.

Proof. Let us consider the differential equation

$$(r^{n-1}y')' + r^{n-1}k_0(r,y) + r^{n-1}g(r, ry') = 0, \ r > 1, \tag{5.2.1}$$

where

$$k_0(a,b) := \begin{cases} k(a,b) & \text{if} \quad b > 0, \\ -k(a,|b|) & \text{if} \quad b \le 0, \end{cases}$$

Clearly, k_0 still satisfies (A_{10}). The change of variables

$$r = \beta(s) := (\frac{1}{n-2}s)^{\frac{1}{n-2}}, \ h(s) := sy(\beta(s))$$

transforms (5.2.1) into

$$h''(s) + \frac{1}{n-2}\beta'(s)\beta(s)k_0\left(\beta(s), \frac{h(s)}{s}\right) + \frac{\beta(s)^2}{(n-2)^2 s}g\left(\beta(s), (n-2)h'(s) - \frac{h(s)}{\beta(s)^{n-2}}\right) = 0.$$

It follows from (A_{10}) and (A_{11}) that, for each $s \in \mathbb{R}^+$, we have

$$\left|k_0\left(\beta(s), \frac{h(s)}{s}\right)\right| \le M_1(\beta(s))\left|\frac{h(s)}{s}\right| \ \text{for} \ \left|\frac{h(s)}{s}\right| \le 2, \tag{5.2.2}$$

and

$$g\left(\beta(s), (n-2)h'(s) - \frac{h(s)}{\beta(s)^{n-2}}\right) \le M_2(\beta(s))\left\{(n-2)h'(s) - \frac{h(s)}{\beta(s)^{n-2}}\right\}. \tag{5.2.3}$$

From (5.2.1), (5.2.2) and (5.2.3), it is natural to consider

$$h''(s) + \frac{1}{n-2}\beta'(s)\beta(s)M_1(\beta(s))\left|\frac{h(s)}{s}\right| + M_2(\beta(s))\left\{h'(s) - \frac{h(s)}{s}\right\}\beta'(s)\beta(s) = 0.$$

Let

$$b(s) = \frac{1}{n-2}\frac{\beta'(s)\beta(s)M_1(\beta(s))}{s} + \frac{\beta'(s)\beta(s)M_2(\beta(s))}{s}, \ s \ge 1,$$

$$c(s) = \beta'(s)\beta(s)M_2(\beta(s)), \ s \ge 1.$$

It follows from $\int_0^\infty s[M_1(s)+M_2(s)]ds < \infty$ that

$$\int_1^\infty c(s)ds < \infty \text{ and } \int_1^\infty sb(s) < \infty,$$

which yields

$$\int_1^\infty \int_t^\infty b(s)dsdt < \infty.$$

Let $T_0 \geq \max\{1, (n-2)A^{n-2}\}$ satisfy

$$2e^{2\int_{T_0}^\infty c(\xi)d\xi} \int_{T_0}^\infty \int_t^\infty b(s)dsdt \leq 1.$$

Now, we will show that

$$h''(s) + c(s)h'(s) + b(s)h(s) = 0, \ s \geq T_0 \tag{5.2.4}$$

has a solution $h(s)$ such that $|h(s)-1| \leq 1$ for all $s \geq T_0$ and $\lim_{s\to\infty} h(s) = 1$. Consider the Banach space $X = \{x \in C([T_0,\infty),\mathbb{R}) \mid x(t) \text{ is bounded}\}$ with supremum norm. Let $K = \{x \in X \mid |x(t)-1| \leq 1, \ t \geq T_0\}$ and define the operator $F : K \to X$ by

$$(Fx)(t) = 1 - \int_t^\infty e^{\int_s^\infty c(\xi)d\xi} \int_s^\infty e^{-\int_r^\infty c(\xi)d\xi} b(r)x(r)drds, \ t \geq T_0.$$

Since $0 \leq x(t) \leq 2$ for $x \in K$ and $t \geq T_0$,

$$\begin{aligned} 0 &\leq \int_t^\infty e^{\int_s^\infty c(\xi)d\xi} \int_s^\infty e^{-\int_r^\infty c(\xi)d\xi} b(r)x(r)drds \\ &\leq 2e^{2\int_{T_0}^\infty c(\xi)d\xi} \int_t^\infty \int_s^\infty b(r)drds \\ &\leq 1, \ t \geq T_0. \end{aligned}$$

Thus $F(K) \subseteq K$. Next, we prove that F is compact. Let $\{x_n\}_{n=1}^\infty$ be a sequence in K. Denote

$$f_n(s) = e^{\int_s^\infty c(\xi)d\xi} \int_s^\infty e^{-\int_r^\infty c(\xi)d\xi} b(r)x_n(r)dr, \text{ for } s \geq T_0.$$

Then $f_n \in L^1([T_0,\infty),\mathbb{R})$ satisfies $\lim_{p\to\infty} \int_p^\infty |f_n(s)|ds = 0$ and

$$\int_{T_0}^\infty |f_n(s)|ds \leq 2e^{2\int_{T_0}^\infty c(\xi)d\xi} \int_{T_0}^\infty \int_t^\infty b(s)dsdr \leq 1, \ n \geq 1.$$

By the Lebesgue dominated convergence theorem,

$$\lim_{\delta\to 0}\int_{T_0}^{\infty}\int_{s}^{s+\delta} e^{-\int_r^{\infty} c(\xi)d\xi} b(r)drds = 0$$

and

$$\lim_{\delta\to 0}\int_{T_0}^{\infty} |e^{\int_{s+\delta}^{\infty} c(\xi)d\xi} - e^{\int_s^{\infty} c(\xi)d\xi}| \int_{s}^{\infty} b(r)drds = 0.$$

Therefore, for any given $\epsilon > 0$, there is a $\gamma > 0$ such that

$$2e^{\int_{T_0}^{\infty} c(\xi)d\xi}\int_{T_0}^{\infty}\int_{s}^{s+\delta} e^{-\int_r^{\infty} c(\xi)d\xi} b(r)drds < \frac{\epsilon}{2},\ |\delta| \le \gamma,$$

and

$$2e^{\int_{T_0}^{\infty} c(\xi)d\xi}\int_{T_0}^{\infty} |e^{\int_{s+\delta}^{\infty} c(\xi)d\xi} - e^{\int_s^{\infty} c(\xi)d\xi}| \int_{s}^{\infty} b(r)drds < \frac{\epsilon}{2},\ |\delta| \le \gamma.$$

Since $0 \le x_n(t) \le 2$ for all $t \ge T_0$ and $n > 1$, the previous choice of γ enables us to deduce that

$$\begin{aligned}\int_{T_0}^{\infty} |f_n(s+\delta) - f_n(s)| \le & 2e^{\int_{T_0}^{\infty} c(\xi)d\xi}\int_{T_0}^{\infty} |e^{\int_{s+\delta}^{\infty} c(\xi)d\xi} - e^{\int_s^{\infty} c(\xi)d\xi}| \int_{s}^{\infty} b(r)drds \\ & + 2e^{\int_{T_0}^{\infty} c(\xi)d\xi}\int_{T_0}^{\infty} |e^{\int_{s+\delta}^{\infty} c(\xi)d\xi} - e^{\int_s^{\infty} c(\xi)d\xi}| \int_{s}^{\infty} b(r)drds \\ < & \epsilon,\ n \ge 1,\ |\delta| < \gamma.\end{aligned}$$

By Riesz's theorem (see [53]), the sequence $\{f_n\}_{n=1}^{\infty}$ is compact in $L^1([T_0,\infty),\mathbb{R})$. It follows from

$$Fx_n(t) = 1 - \int_t^{\infty} f_n(s)ds,\ t \ge T_0,\ n \ge 1$$

that $\{Fx_n\}_{n=1}^{\infty}$ is compact in K. This implies that F is a compact mapping. By the Schauder fixed-point theorem, the mapping F has a fixed point $h \in K$. It is easy to verify that h is a nonnegative solution of (5.2.4) in $[T_0,\infty)$ and satisfies $\lim_{s\to\infty} h(s) = 1$. Take $T_1 > T_0$ so that $h(s) > 0$ for $s \ge T_1$ and let $B = (\frac{1}{n-2}T_1)^{\frac{1}{n-2}} \ge A$. Define $v(x) = y(r) = \frac{h(s)}{s}$ for $r = |x| \ge B$, where $r = \beta(s)$.

Since $\lim_{s\to\infty} h(s) = 1$, $\lim_{|x|\to\infty} v(x) = 0$. Hence, $v(x) > 0$ on $S_B \cup G_B$ and

$$\begin{aligned}
\Delta v + f(x, v(x), \nabla v(x)) \leq & (r^{n-1}y')' + r^{n-1}k(r, y) + r^{n-1}g(r, ry') \\
= & h''(s) + \frac{1}{n-2}\beta'(s)\beta(s)k_0(\beta(s), \frac{h(s)}{s}) \\
& + \frac{\beta(s)^2}{(n-2)^2 s} g(\beta(s), (n-2)h'(s) - \frac{h(s)}{\beta(s)^{n-2}}) \\
\leq & h''(s) + c(s)h'(s) + b(s)h(s) \\
= & 0, \ r \geq B,
\end{aligned}$$

which implies v is an upper solution of (PE) on G_B. Clearly, $w(x) \equiv 0$ satisfies

$$\Delta w(x) + f(x, w(x), \nabla w(x)) \geq 0, \ x \in G_B.$$

By the Lemma 5.2.1 we see that (PE) has a solution $u(x)$ in G_B with $w(x) \leq u(x) \leq v(x)$ for $|x| > B$ and $u(x) = v(x)$ for $|x| = B$.

Finally, we will show that u is positive. We choose a positive number $k > \frac{n}{2B^2}$. For any given $\epsilon > 0$, we define

$$u_\epsilon = \inf_{x \in S_B} \{u(x)\} + \epsilon e^{-k|x|^2}, \ x \in S_B \cup G_B,$$

where $u(x)$ is a solution of (E) in G_B. If $x \in G_B$, then it follows from

$$\begin{aligned}
(\Delta u_\epsilon)(x) &= \epsilon(4k^2|x|^2 - 2kn)e^{-k|x|^2} \\
&> 0 \geq -f(x, u, \nabla u) \\
&= (\Delta(u + \epsilon e^{-kB^2}))(x),
\end{aligned}$$

that $(\Delta(u + \epsilon e^{-kB^2} - u_\epsilon))(x) < 0$. On the other hand, by using the fact $|x| \geq B$, we get

$$u(x) + \epsilon e^{-kB^2} - u_\epsilon(x) \geq 0, \ x \in G_B.$$

Since $u(x) \geq 0$ on G_B and $u_\epsilon(x)$ is bounded on G_B, the function

$$z_\epsilon(x) = u(x) + \epsilon e^{-kB^2} - u_\epsilon(x), \ x \in G_B \cup S_B$$

has a finite infimum in $G_B \cup S_B$. For any $C > B$,

$$\inf z_\epsilon(x) = \min z_\epsilon(x) \text{ on } G_{BC} = \{x \in \mathbb{R}^n \mid B \leq |x| \leq C\}.$$

If there exists a $x_0 \in \{x \in \mathbb{R}^n \mid B < |x| \leq C\}$ with $z_\epsilon(x_0) = \min_{x \in G_{BC}}\{z_\epsilon(x)\}$, then $(\Delta z_\epsilon)(x_0) \geq 0$, which is a contradiction. Thus, $\min_{x \in G_{BC}} z_\epsilon(x)$ lies on $\{x \in \mathbb{R}^n \mid |x| = B\}$ for all $C > B$. It follows from

$$\inf_{x \in G_B \cup S_B} z_\epsilon(x) = \min_{x \in S_B} z_\epsilon(x) \geq 0$$

that

$$u_\epsilon(x) \leq u(x) + \epsilon e^{-kB^2}, \ x \in G_B \cup S_B.$$

Letting $\epsilon \to 0$ in the previous relation, we get

$$u(x) \geq \inf_{x \in S_B} u(x) = \inf_{x \in S_B} v(x) = y(B) = \frac{h((n-2)B^{n-2})}{(n-2)B^{n-2}} = \frac{h(T_1)}{T_1} > 0, \ x \in G_B,$$

and this shows that $u(x)$ is positive in G_B. It follows from $u(x) \leq v(x)$ for $|x| \geq B$ and $\lim_{|x| \to \infty} v(x) = 0$ that $\lim_{|x| \to \infty} u(x) = 0$. This completes the proof. ❑

Remark. Let $f(x, u, \nabla u) = p(x, u) + q(|x|)x \cdot \nabla u$, where p is locally Hölder continuous in $G_A \times \mathbb{R}$ satisfying

$$0 \leq p(x, t) \leq a(|x|)w(t), \ t \in \mathbb{R}^+, \ x \in \mathbb{R}^n.$$

Here $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $w \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ with $w(0) = 0$, q is a bounded C^1 function and $\int_0^\infty s[a(s) + |q(s)|]ds < \infty$. Moreover, we let $k(|x|, t) = a(|x|)w(t)$ and $g(|x|, x \cdot z) = q(|x|)x \cdot z$, then our Theorem 5.2.2 is reduced to Theorem 5.1.1.