Chapter 1

Introduction

The study of differential equations have aroused great interest in such areas of pure and applied mathematics, as Riemnnian geometry, fluid mechanics, population genetics, chemistry, ecology, astronomy, and meteorology. Many problems can be described as second order differential equations with suitable boundary conditions, and solutions to such problems transformed by a natural phenomena usually represent the magnitude of the phenomena. Hence, the most essential objectives of such study is first to make sure the existence of solutions. After we solve the major problem, we will consider the uniqueness and further analyze the behavior characteristics of those positive solutions. The topic of this dissertation is to systematically cope with the existence of positive solutions of some boundary value problems.

In the mathematical literature, plenty of fixed-point theorems, such as Schauder's fixed-point theorem [22], Krasnoselkii's fixed-point theorem [33, 43], and so on, are introduced to solve the existence of solutions for boundary value problems. For this approach, we usually write boundary value problems as integral forms and fixed points of integral operators usually are solutions of original problems. In this thesis, they also play an important role. Another powerful tool recently receiving considerable attention is upper and lower solution theory. Note that in this dissertation we also develop a result by applying this theory studied by Noussair and Swanson [55] in a so-called exterior domain $G_A = \{x \in \mathbb{R}^n \mid |x| > A\}$, where $n \geq 3$ and $A > 0$.

This thesis is organized as follows. In chapter 1, we give an introduction. In chapter 2, under some assumptions on f, we consider the following second order equation

$$(E)\ u'' + f(t, u, u') = 0$$

on $[0, 1]$, equipped with three kinds of boundary conditions listed as follows:

$$(BC_1) \begin{cases} \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0, \end{cases}$$

with $\alpha, \beta, \gamma, \delta \geq 0$, $\gamma\beta + \alpha\gamma + \alpha\delta > 0$,

$$(BC_2)\ u(0) = 0, \quad u(1) - \sum_{i=1}^{m-2} k_i u(\xi_i) = 0,$$

with $k_i > 0$ $(i = 1, 2, \ldots, m-2)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, \sum_{i=1}^{m-2} k_i \xi_i < 1$, and

$$(BC_3)\ u(0) = cu(\xi),\ u(1) = bu(\sigma),$$

with $0 < \xi < \sigma < 1$, $0 \leq c \leq \frac{1}{1-\xi}$, $c\xi(1-b) + (1-c)(1-b\sigma) > 0$ and $0 \leq b \leq \frac{1}{\sigma}$. We show that they all have at least one positive solution, respectively. Moreover, there still are several interesting remarks and an example in the last section of this chapter.

In chapter 3, we consider the following second order functional differential equation,

$$(FE)\ u''(t) + F(t, u_t) = 0$$

on $[0, 1]$, another form of differential equation arisen from problems of physics and variational problems of control theory. Under general assumptions on the source term and with boundary condition of Sturm-Liouville's type:

$$(BC_4) \begin{cases} u(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0, \end{cases}$$

where

$$\gamma, \delta \geq 0 \text{ and } \gamma + \delta > 0.$$

we also prove that there exist positive solutions for the problem. Furthermore, we will note that when reducing functional differential equation to differential equation, more applications can work.

In chapter 4, we focus on a high order $p-$Laplician equation with another kind of three-point boundary condition:

$$(BC_5) \begin{cases} u^{(i)}(0) = 0, \ i = 0, 1, 2, \ldots n-3, \\ u^{(n-2)}(0) = \xi u^{(n-2)}(1), \\ u^{(n-1)}(1) = \eta u^{(n-1)}(0), \\ u^{(n)}(0) = \mu u^{(n)}(\delta), \\ u^{(n)}(1) = \nu u^{(n)}(\delta), \end{cases}$$

with $\mu, \nu \geq 0, \xi \neq 1, \eta \neq 1, 0 < \delta < 1, n \geq 2$, which is not of multi-point type mentioned as (BC_2). It is worthy to point out that a well-known generalization of Leggett-Williams' fixed-point theorem, that is, Lemma 4.2.3, will be applied to get at least three positive solutions for this problem.

In chapter 5, we study the second order nonlinear elliptic equation

$$(PE) \ \Delta u + f(x, u, \nabla u) = 0. \ x \in G_A,$$

where $G_A = \{x \in \mathbb{R}^n | \ |x| > A\}$, $n \geq 3$ and $A > 0$ is an "exterior domain." By constructing respective upper and lower solutions, we apply Noussair and Swanson's research [55] on the upper and lower solution theory in G_A to obtain solutions between our upper and lower solutions.