Chapter 2

Second Order Ordinary Differential Equations with Various Boundary Conditions

2.1 Introduction

In this chapter, we shall attempt to construct an excellent existence criterion for the boundary value problem (BVP_j)

$$(E)\ u'' + f(t, u, u') = 0,\ 0 < t < 1,$$

equipped with suitable boundary conditions (BC_j), $j = 1, 2, 3$, as follows:

$$(BC_1)\ \begin{cases} \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0, \end{cases}$$

with $\alpha, \beta, \gamma, \delta \geq 0$, $\gamma\beta + \alpha\gamma + \alpha\delta > 0$,

$$(BC_2)\ u(0) = 0, \quad u(1) - \sum_{i=1}^{m-2} k_i u(\xi_i) = 0,$$

with $k_i > 0$ $(i = 1, 2, \ldots, m-2)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, \sum_{i=1}^{m-2} k_i \xi_i < 1$, and

$$(BC_3)\ u(0) = cu(\xi),\ \ u(1) = bu(\sigma),$$

with $0 < \xi < \sigma < 1$, $0 \leq c \leq \frac{1}{1-\xi}$, $c\xi(1-b) + (1-c)(1-b\sigma) > 0$ and $0 \leq b \leq \frac{1}{\sigma}$.

The motivation for the present work stems from many the investigations in [1, 2, 3, 25, 56]. In fact, particular cases of the boundary value problems (BVP_j) occur in various physical phenomena [11, 15, 16, 17, 20, 21, 25], specially such as gas diffusion through porous media, thermal self ignition of a chemically active mixture of gases in a vessel [17], catalysis theory [20], chemically reacting systems, as well as adiabatic tubular reactor processes.

For the other related works, we refer to recent contributions of Agarwal and Wong [1, 2, 3], Anuradaha, Hai and Shivaji [11], Bailey, Shampine and Waltman [15], Erbe and Wang [25], Lee and O'Regan [44], Henderson [36], Kelevedjiev [40, 41] and Wong, Lian, Lin and Yu [58] and the references therein.

2.2 Existence of Positive Solution for Three Kinds of BVP

First, we note that $-u'' = 0$ with boundary condition (BC_j) has Green functions $G_j(t, s)$ on $[0, 1] \times [0, 1]$, respectively. For $j = 1$, one can see that

$$G_1(t,s) = \frac{1}{\rho} \begin{cases} (\gamma + \delta - \gamma t)(\beta + \alpha s) & 0 \leq s \leq t \leq 1, \\ (\beta + \alpha t)(\gamma + \delta - \gamma s) & 0 \leq t \leq s \leq 1, \end{cases} \tag{2.2.1}$$

where $\rho := \gamma\beta + \alpha\gamma + \alpha\delta$, and $G_1(t,s) \geq 0$ on $[0,1]\times[0,1]$. For $j=2$, it follows from [46] that

$$G_2(t,s) = \begin{cases} \frac{s(1-t)-\sum_{j=1}^{m-2} k_j(\xi_j - t)s + \sum_{j=1}^{i-1} k_j \xi_j (t-s)}{1-\sum_{i=1}^{m-2} k_i \xi_i}, \text{ if} \\ 0 \leq t \leq 1, \xi_{i-1} \leq s \leq \min\{\xi_i, t\},\ i = 1,2,\ldots,m-1; \\ \frac{t[(1-s)-\sum_{j=i}^{m-2} k_j(\xi_j - s)]}{1-\sum_{i=1}^{m-2} k_i\xi_i}, \text{ if} \\ 0 \leq t \leq 1, \max\{\xi_{i-1}, t\} \leq s \leq \xi_i,\ i = 1,2,\ldots,m-1, \end{cases} \tag{2.2.2}$$

and $G_2(t,s) \geq 0$ on $[0,1]\times[0,1]$. For $j=3$, it follows from [14] that

$$G_3(t,s) = \begin{cases} s\in[0,\xi]: & \begin{cases} \frac{s}{\zeta}[(1-b\sigma)+(b-1)t], & s \leq t, \\ \frac{t}{\zeta}[(1-b\sigma)+(b-1)s] + \frac{c(1-\xi+b\xi-b\sigma)}{\zeta}, & t \leq s, \end{cases} \\ s\in[\xi,\sigma]: & \begin{cases} \frac{1}{\zeta}[(1-b\sigma)+(b-1)t](c\xi - cs + s), & s \leq t, \\ \frac{1}{\zeta}[(1-b\sigma)+(b-1)s](c\xi - ct + t), & t \leq s, \end{cases} \\ s\in[\sigma,1]: & \begin{cases} \frac{1-s}{\zeta}(t-ct+c\xi)+(s-t), & s \leq t, \\ \frac{1-s}{\zeta}(c\xi - ct + t), & t \leq s, \end{cases} \end{cases} \tag{2.2.3}$$

where $\zeta := c\xi(1-b)+(1-c)(1-b\sigma)$, and $G_3(t,s)$ is also nonnegative on $[0,1]\times[0,1]$.

Moreover, for $j = 1,2,3$, we see that

(C_1) $\frac{\partial}{\partial t}G_j(t,s)$ exists almost everywhere on $[0,1]\times[0,1]$

and

(C_2) $G_j(t,\cdot)$, $\frac{\partial}{\partial t}G_j(t,\cdot) \in L^1([0,1])$ for each fixed $t \in [0,1]$

hold, hence, define two numbers as follows:

$$A_j^{-1} := \max_{0\leq t\leq 1} \int_0^1 G_j(t,s)ds, \tag{2.2.4}$$

and

$$k_j := A_j \max_{0\leq t\leq 1} \int_0^1 \left| \frac{\partial}{\partial t} G_j(t,s) \right| ds. \tag{2.2.5}$$

Theorem 2.2.1 *Suppose*

(A_1) *$f(t,0,0)$ is not identical to zero on $[0,1]$*

and there exists a constant $a > 0$ satisfying

(A_2) $\quad f : [0,1] \times [0,a] \times [-k_j a, k_j a] \to [0,\infty)$ *is continuous,*

(A_3) $\quad \sup \{ f(t,x,y) \mid (t,x,y) \in [0,1] \times [0,a] \times [-k_j a, k_j a] \} \leq aA_j$.

Then (BVP_j) *has at least one positive solution, for* $j = 1,2,3$, *respectively.*

Proof. Given $j \in \{1,2,3\}$, and denote $G := G_j$, $A := A_j$, $k := k_j$ and $(BVP) := (BVP_j)$. Without loss of generality, we assume that

$(A_2)'$ $\quad f : [0,1] \times [0,\infty) \times (-\infty,\infty) \to [0,\infty)$ is continuous.

Next, we consider the Banach space $C^1[0,1]$ with norm

$$|||u||| = \max\{||u||,\ k^{-1}||u'||\},\ \text{where } ||w|| := \max_{t\in[0,1]} |w(t)|.$$

Define

$$C^1_+[0,1] = \{u \in C^1[0,1] \mid u(t) \geq 0,\ 0 \leq t \leq 1\}.$$

and

$$T(u(t)) = \int_0^1 G(t,s) f(s,u(s),u'(s))ds,\ u \in C^1_+[0,1].$$

By assumption $(A_2)'$, we know that $T : C^1_+[0,1] \to C^1_+[0,1]$ and satisfies

$$(T(u(t)))' = \int_0^1 \frac{\partial}{\partial t} G(t,s) f(s,u(s),u'(s))ds.$$

Standard applications of Arzelà-Ascoli theorem, we can prove that $T : C^1_+[0,1] \to C^1_+[0,1]$ is completely continuous and let

$$V_a = \{u \in C^1_+[0,1] \mid |||u||| \leq a\}.$$

It is clear that if $u \in V_a$, then

$$||u|| \leq a \text{ and } ||u'|| \leq ka.$$

Thus, we have

$$0 \leq u(t) \leq a \text{ and } -ka \leq u'(t) \leq ka,\ 0 \leq t \leq 1.$$

By our assumption (A_3), we obtain

$$||Tu|| = \max_{0\leq t\leq 1} \int_0^1 G(t,s) f(s,u(s),u'(s))ds \leq aA \max_{0\leq t\leq 1} \int_0^1 G(t,s)ds = aAA^{-1} = a$$

and

$$||(Tu)'|| \leq \max_{0\leq t\leq 1}\int_0^1 \left|\frac{\partial}{\partial t}G(t,s)\right| f(s,u(s),u'(s))ds \leq aA\max_{0\leq t\leq 1}\int_0^1 \left|\frac{\partial}{\partial t}G(t,s)\right| ds = ka.$$

Therefore, we have

$$|||Tu||| \leq a \text{ and } T: V_a \to V_a.$$

Since V_a is closed, bounded and convex, then we can apply Schauder's fixed-point theorem [22] to obtain a $u^*(t)$ such that

$$u^*(t) = (Tu^*)(t) = \int_0^1 G(t,s)f(s,u^*(s),(u^*)'(s))ds,\ 0\leq t\leq 1$$

and u^* satisfies the boundary condition (BC). This implies that u^* is a solution of the (BVP). Furthermore, by assumption (A_1), we see that the zero function is not a solution of (BVP) which shows that

$$||u^*|| \equiv u^*(c) > 0 \text{ for some } c \in [0,1].$$

Since

$$(u^*(t))'' = -f(t,u^*(t),(u^*(t))') \leq 0 \text{ on } [0,1],$$

we see that u^* is a nonnegative concave function on $[0,1]$. Therefore, we can separate the rest proof into the following:

Case1. If $t\in[0,c]$, then

$$\begin{aligned} u^*(t) &= u^*(\frac{c-t}{c}\times 0 + \frac{t}{c}\times c) \geq \frac{c-t}{c}u^*(0) + \frac{t}{c}u^*(c) \\ &\geq \frac{t}{c}u^*(c) = \frac{t}{c}||u^*|| \geq t||u^*||. \end{aligned}$$

Case2. If $t\in[c,1]$, then

$$\begin{aligned} u^*(t) &= u^*(\frac{t-c}{1-c}\times 1 + \frac{1-t}{1-c}\times c) \geq \frac{t-c}{1-c}u^*(1) + \frac{1-t}{1-c}u^*(c) \\ &\geq \frac{1-t}{1-c}u^*(c) = \frac{1-t}{1-c}||u^*|| \geq (1-t)||u^*||. \end{aligned}$$

Thus, we have

$$u^*(t) \geq ||u^*||\min\{\, t, 1-t\,\} > 0,\ 0 < t < 1,$$

which implies u^* is a positive solution and the proof is completed. ❑

2.3 Remarks and An Example

We first give some remarks as follows. Note that Q. Yao's result [62] is generalized in the following.

Remark. In our proof of Theorem 2.2.1, we defined operator T on $C_+^1[0,1]$ with respect to (BC_j), $j=1,2,3$, respectively. Note that the complete continuity of T on $C_+^1[0,1]$ depends on the form of the Green's functions $G_j(t,s)$. We remark that for other boundary conditions, if the corresponding Green functions are nonnegative, satisfy (C_1), (C_2), moreover, and make the operator T be completely continuous on $C_+^1[0,1]$, Theorem 2.2.1 is still valid.

Remark. One can compute A_j and k_j, for $j=1,2,3$, respectively. For example, for (BVP_1), from (2.2.1), (2.2.4) and (2.2.5), we have

$$A_1^{-1} = \frac{1}{\rho}\left[\beta(\frac{\gamma}{2}+\delta) + \frac{\alpha^2(\frac{\gamma}{2}+\delta)^2}{2\rho}\right] \tag{2.3.1}$$

and

$$k_1 = \left[\beta(\frac{\gamma}{2}+\delta) + \frac{\alpha^2(\frac{\gamma}{2}+\delta)^2}{2\rho}\right]^{-1} \times \max\{\frac{\alpha\gamma}{2}+\alpha\delta, \frac{\alpha\gamma}{2}+\gamma\beta\}, \tag{2.3.2}$$

where $\rho := \gamma\beta + \alpha\gamma + \alpha\delta$.

Remark. For (BVP_2), in particular, it is well-known three-point boundary condition,

$$(BC_2^*)\ u(0)=0, \quad u(1)=ku(\theta),$$

where k and θ are both constants satisfying $0<\theta<1$ and $0<k<\frac{1}{\theta}$.
Note that from (2.2.2),

$$G_2^*(t,s) = \frac{1}{1-k\theta}t(1-s) - U(t,s) - \frac{k}{1-k\theta}V(t,s), \quad 0\le t,s\le 1, \tag{2.3.3}$$

where

$$U(t,s) = \begin{cases} t-s, & s\le t, \\ 0, & t\le s \end{cases}$$

and

$$V(t,s) = \begin{cases} t(\theta - s), & s \leq \theta, \\ 0, & \theta \leq s. \end{cases}$$

So, by (2.3.3), (2.2.4) and (2.2.5), we have

$$A_2^* = \begin{cases} 8(\frac{1-k\theta}{1-k\theta^2})^2, & \text{if } k\theta(2-\theta) \leq 1, \\ \frac{2(1-k\theta)}{k\theta(1-\theta)}, & \text{if } k\theta(2-\theta) \geq 1 \end{cases} \tag{2.3.4}$$

and

$$k_2^* = \frac{3 - 2k\theta + k\theta^2}{2(1-k\theta)} A_2^*. \tag{2.3.5}$$

and conclude that $(E) - (BC_2^*)$ has at least one positive solution if (A_1), (A_2) and (A_3) hold. This also generalizes the result of Q. Yao [62] since we do not need the "increasing property" of the source term f.

Last but not least, we afford an example to observe that giving the same one differential equation equipped with different boundary conditions may effect the existence of positive solutions.

Example 2.3.1 Consider

$$(e)\ u'' + (\pi^2 - 7)u + \sin u + \frac{1}{2} = 0,\ 0 < t < 1,$$

equipped with different boundary conditions. Set

$$f(t,u,u') = (\pi^2 - 7)u + \sin u + \frac{1}{2}.$$

If (e) is equipped with

$$(BC_2^*)\ u(0) = 0,\ u(\frac{1}{3}) = u(1),$$

from (2.3.4) and (2.3.5), we have

$$A_2^* = \frac{9}{2} \text{ and } k_2^* = \frac{5}{3}.$$

Set $a = 1$, it follows from Theorem 2.2.1 that $(e) - (BC_2^*)$ has at least one position. However, if we assign

$$(BC_1^*)\ u'(0) = 0,\ u(1) = 0$$

to (e), we can obtain, from (2.3.1) and (2.3.2),

$$A_1^* = 2 \text{ and } k_1^* = \frac{1}{2}.$$

In this problem, by contradiction, it is clear that there is no $a > 0$ such that f satisfying (A_3). Moreover, one can compute that $\frac{\pi^2}{4}$ is the first eigenvalue of the linear eigenvalue problem

$$u''(t) + \lambda u(t) = 0, \;\; 0 < t < 1,$$

with (BC_1^*). It follows from Theorem 2 [45], that $(e) - (BC_1^*)$ has no positive solution.