Chapter 3

Second Order Functional Differential Equations with Boundary Condition of Sturm-Liouville's Type

3.1 Introduction

In this chapter, we deal with the existence of positive solutions to the functional differential equation

$$(FE)\ u''(t) + F(t, u_t) = 0,\ 0 < t < 1.$$

The solutions u must satisfy the initial function

$$u(s) = \phi(s),\ \ -r \le s \le 0,\ \text{for certain given } \phi,$$

and boundary condition of Sturm-Liouville's type

$$(BC_4) \begin{cases} u(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0, \end{cases}$$

where

$$\gamma, \delta \geq 0 \text{ and } \gamma + \delta > 0.$$

Our notations are stated as follows. We denote the set of all real numbers and the set of all nonnegative real numbers by $\mathbb{R}$ and $\mathbb{R}^+$ respectively. Now, fixed any $r \in \mathbb{R}^+$, let C_r denote the Banach space of all continuous functions $\psi : [-r, 0] \equiv J \to \mathbb{R}$ endowed with the suprenorm $||\psi||_J = \sup_{s\in J} |\psi(s)|$, and let

$$C_{r,0} = \{\psi \in C_r \mid \psi(0) = 0\}.$$

The notation u_t above denotes a function in C_r defined by

$$u_t(w) = u_t(w; \phi) := \begin{cases} u(t+w) & \text{if} \quad t + w \geq 0, \\ \phi(t+w) & \text{if} \quad t + w \leq 0, \end{cases}$$

here the given ϕ is an element of the space $C_{r,0}$.

From now on, we denote our problem mentioned in the following by $(FBVP)$. Moreover, given $w \in [-r, 0]$ fixed, by a solution of the $(FBVP)$ we mean a function $u \in C^2[0, 1]$ such that u satisfies the boundary condition (BC_4) and for a given ϕ the relation

$$u''(t) + F(t, u_t(w; \phi)) = 0$$

holds for all $t \in [0, 1]$.

There has recently been an increased interest in studying boundary value problems for functional differential equations, see, e.g. the books by Hale [34], Kolmanovskii and Myshkis [42] and Henderson [37]. Furthermore, as pointed out in [26], these problems have arisen from problems of physics and variational problems of control theory, as well as from much applied mathematics appeared early in the literature [31, 32]. We refer more detailed treatment to more interesting research [4, 5, 6, 13, 23, 35, 38, 39, 47, 54, 57, 61] and more references therein.

In next section, we state the key tool in establishing our main results, that is, the well-known Krasnoselkii fixed-point theorem [33, 43] and give a lemma that will be used in defining a positive operator in a cone. Then, in some function space, we construct a appropriate cone for applying the fixed-point theorem to our positive operator, this yields our existence results. Moreover, some remarks in section 3.3 will imply several corollaries of existence of multiple positive solutions, including the reduction to general ordinary differential equations with the boundary condition. Finally, one can see an example as applications.

3.2 Preliminaries and Existence Results

In order to abbreviate our discussion, throughout this chapter, we observe

(C_3) $k(t,s)$ is the Green's function of the differential equation

$$\begin{cases} u'' = 0, \\ (BC_4), \end{cases}$$

that is,

$$k(t,s) = \frac{1}{\gamma+\delta}\begin{cases} (\gamma+\delta-\gamma t)s & 0 \le s \le t \le 1, \\ (\gamma+\delta-\gamma s)t & 0 \le t \le s \le 1, \end{cases} \tag{3.2.1}$$

and suppose the following assumption hold:

(C_4) $F : [0,1] \times C_r \to \mathbb{R}^+$ is a continuous functional.

We now state the Krasnoselkii fixed-point theorem [33, 43] and a useful lemma which are required for the main result.

Theorem 3.2.1 ([33, 43]) *Let E be a Banach space, and let $K \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $||Au|| \le ||u||,\ u \in K \cap \partial\Omega_1$ *and* $||Au|| \ge ||u||,\ u \in K \cap \partial\Omega_2$; *or*

(ii) $||Au|| \ge ||u||,\ u \in K \cap \partial\Omega_1$ *and* $||Au|| \le ||u||,\ u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Lemma 3.2.2 *Suppose that $k(t,s)$ is defined as in* (3.2.1). *Then, for any p_1, p_2 with $0 \le p_1 < p_2 \le 1$, we have the following results:*

$$\begin{cases} \frac{k(t,s)}{k(s,s)} \le 1, & \text{for } t \in [0,1] \text{ and } s \in [0,1], \\ \frac{k(t,s)}{k(s,s)} \ge \min\{\frac{(1-p_2)\gamma+\delta}{\gamma+\delta}, p_1\}, & \text{for } t \in [p_1,p_2] \text{ and } s \in [0,1]. \end{cases}$$

Proof. From (3.2.1), we have

$$\frac{k(t,s)}{k(s,s)} = \begin{cases} \frac{\gamma+\delta-\gamma t}{\gamma+\delta-\gamma s}, & 0 \le s \le t \le 1, \\ \frac{t}{s}, & 0 \le t \le s \le 1. \end{cases}$$

Hence, we obtain the desired results:

$$\frac{k(t,s)}{k(s,s)} \le 1 \text{ for } t \in [0,1],\ s \in [0,1],$$

and

$$\frac{k(t,s)}{k(s,s)} \ge \begin{cases} \frac{(1-p_2)\gamma+\delta}{\gamma+\delta}, & 0 \le s \le t \le p_2, \\ p_1, & p_1 \le t \le s \le 1. \end{cases}$$

❑

From Lemma 3.2.1, we define a number

$$M = M(p_1,p_2) := \min\{\frac{(1-p_2)\gamma+\delta}{\gamma+\delta}, p_1\}$$

and next, state and prove our main results.

Theorem 3.2.3 (Existence result for $-1 < w \le 0$) *Suppose the following hypotheses hold:*

(H_1) *there exists a positive constant λ such that, for any $t \in [0,1]$ and any $\psi \in C_r$ with $||\psi||_J \le \lambda$,*

$$F(t,\psi) \le \lambda \left(\int_0^1 k(s,s)ds \right)^{-1},$$

and

(H_2) *there exist p_1, p_2 with $0 \le -w \le p_1 < p_2 \le 1$ and a positive constant $\eta \ne \lambda$ such that, for any $t \in [p_1,p_2]$ and any $\psi \in C_r$ with $M\eta \le ||\psi||_J \le \eta$,*

$$F(t,\psi) \ge \eta \left(\int_{p_1}^{p_2} k(\frac{p_1+p_2}{2}, s)ds \right)^{-1}.$$

Then for any given $\phi \in C_{r,0}$ with $||\phi||_J \leq \lambda$, $(FBVP)$ has at least one positive solution u such that $||u||$ between λ and η.

Proof. Without loss of generality, we assume $\lambda < \eta$. It is clear that $(FBVP)$ has a solution $u = u(t)$ if and only if u is the solution of the operator equation

$$u(t) = \int_0^1 k(t,s)F(s, u_s(w;\phi))ds := A_\phi u(t), \ \ u \in C[0,1].$$

Let K be a cone in $C_0[0,1] := \{u \in C[0,1] \mid u(0) = 0\}$ defined by

$$K = \{u \in C_0[0,1] \mid u(t) \geq 0, \ \min_{t\in[p_1,p_2]} u(t) \geq M||u||\}.$$

Following from the definition of K and Lemma 3.2.2 we have

$$\begin{aligned}\min_{t\in[p_1,p_2]}(A_\phi u)(t) = &\min_{t\in[p_1,p_2]} \int_0^1 k(t,s)F(s,u_s(w;\phi))ds \\ \geq &M \int_0^1 k(s,s)F(s,u_s(w;\phi))ds \\ \geq &M \int_0^1 k(t,s)F(s,u_s(w;\phi))ds.\end{aligned}$$

Thus, $\min_{t\in[p_1,p_2]}(A_\phi u)(t) \geq M||Au||$, which implies $A_\phi K \subset K$. Furthermore, it is easy to check $A_\phi : K \to K$ is completely continuous. To complete the proof, we separate the rest of our proof into the following two steps:

Step 1. Let $\Omega_1 = \{u \in K \mid ||u|| < \lambda\}$. It follows from (H_1) and Lemma 3.2.2 that for $u \in \partial\Omega_1$,

$$\begin{aligned}(A_\phi u)(t) = &\int_0^1 k(t,s)F(s,u_s(w;\phi))ds \\ \leq &\int_0^1 k(s,s)F(s,u_s(w;\phi))ds \\ \leq &\lambda \left(\int_0^1 k(s,s)ds\right)^{-1} \left(\int_0^1 k(s,s)ds\right) \frac{||u||}{\lambda} \\ = &||u||.\end{aligned}$$

Hence,

$$||A_\phi u|| \leq ||u|| \text{ for } u \in \partial\Omega_1 \cap K.$$

Step 2. Let $\Omega_2 = \{u \in K \mid ||u|| < \eta\}$. It follows from the definitions of $||u||$ and K that

$$\begin{cases} u(t) \le ||u|| = \eta \text{ for } t \in [0,1], \\ u(t) \ge \min_{t\in[p_1,p_2]} u(t) \ge M||u|| = M\eta \text{ for } t \in [p_1, p_2], \end{cases}$$

for $u \in \partial\Omega_2$, which implies

$$M\eta \le u(t) \le \eta \text{ for } t \in [p_1, p_2].$$

Moreover, it follows from $0 \le -w \le p_1 < p_2 \le 1$ that $s + w \ge 0$ for $s \in [p_1, p_2]$. This implies $u_s(w; \phi) = u(s + w)$ for $s \in [p_1, p_2]$. Hence,

$$\begin{aligned} (A_\phi u)(\frac{p_1+p_2}{2}) &= \int_0^1 k(\frac{p_1+p_2}{2}, s)F(s; u_s(w,\phi))ds \\ &\ge \int_{p_1}^{p_2} k(\frac{p_1+p_2}{2}, s)F(s; u_s(w,\phi))ds \\ &\ge \eta \left(\int_{p_1}^{p_2} k(\frac{p_1+p_2}{2}, s)ds\right)^{-1} \left(\int_{p_1}^{p_2} k(\frac{p_1+p_2}{2}, s)ds\right) \frac{||u||}{\eta} \\ &= ||u||, \end{aligned}$$

which implies

$$||A_\phi u|| \ge ||u|| \text{ for } u \in \partial\Omega_2.$$

Therefore, by Theorem 3.2.1, we complete this proof. ❑

Note this given w may not belong to $(-1, 0]$, hence, we conclude the following result.

Theorem 3.2.4 (Existence result for $-r \le w \le 0$**)** *Suppose the following hypotheses hold:*

(H_1) *there exists a positive constant* λ *such that, for any* $t \in [0,1]$ *and any* $\psi \in C_r$ *with* $||\psi||_J \le \lambda$,

$$F(t, \psi) \le \lambda \left(\int_0^1 k(s,s)ds\right)^{-1},$$

and

(H_3) *there exist* p_1, p_2 *with* $0 \le p_1 < p_2 \le 1$ *and a positive constant* $\eta \ne \lambda$ *such* *that, for any $t \in [p_1, p_2]$ and $\psi \in C_r$ with $||\psi||_J \le \eta$,*

$$F(t,\psi) \ge \eta \left(\int_{p_1}^{p_2} k(\frac{p_1+p_2}{2}, s) ds \right)^{-1}.$$

Then for any given $\phi \in C_{r,0}$ with $||\phi||_J \le \min\{\lambda, \eta\}$, $(FBVP)$ has at least one positive solution u such that $||u||$ between λ and η.

Proof. This proof follows in similar fashion to that of Theorem 3.2.3. One just need to modify Step 2 in the process of the demonstration of Theorem 3.2.3 as the following:

Step 2. Let $\Omega_2 := \{u \in K \mid ||u|| < \eta\}$. It follows from the definitions of $||u||$ and K that

$$\begin{cases} u(t) \le ||u|| = \eta \text{ for } t \in [0,1], \\ u(t) \ge \min\limits_{t \in [p_1,p_2]} u(t) \ge M||u|| = M\eta \text{ for } t \in [p_1,p_2], \end{cases}$$

for $u \in \partial\Omega_2$, which implies

$$M\eta \le u(t) \le \eta \text{ for } t \in [p_1, p_2].$$

Moreover, for $s \in [p_1, p_2]$,

$$u_s(w;\phi) := \begin{cases} u(s+w) & \text{if} \quad s+w \ge 0, \\ \phi(s+w) & \text{if} \quad s+w \le 0, \end{cases}$$

This implies, for $s \in [p_1, p_2]$,

$$||u_s(w;\phi)|| \le \eta.$$

Hence,

$$\begin{aligned} (A_\phi u)(\frac{p_1+p_2}{2}) &= \int_0^1 k(\frac{p_1+p_2}{2}, s) F(s; u_s(w,\phi)) ds \\ &\ge \int_{p_1}^{p_2} k(\frac{p_1+p_2}{2}, s) F(s; u_s(w,\phi)) ds \\ &\ge \eta \left(\int_{p_1}^{p_2} k(\frac{p_1+p_2}{2}, s) ds \right)^{-1} \left(\int_{p_1}^{p_2} k(\frac{p_1+p_2}{2}, s) ds \right) \frac{||u||}{\eta} \\ &= ||u||, \end{aligned}$$

which implies

$$||A_\phi u|| \ge ||u|| \text{ for } u \in \partial\Omega_2.$$

❑

3.3 Applications

Remark. Assume that $F(t,\psi)$ satisfies the following property $\mathbb{P}$:

If $\max\limits_{t\in[0,1]} F(t,\psi)$ *is unbounded, then there exists a* ϕ *with* $||\phi||_J$ *large enough such that for any* $\psi \in C_r$ *with* $||\psi||_J \le ||\phi||_J$*, we have* $\max\limits_{t\in[0,1]} F(t,\psi) \le \max\limits_{t\in[0,1]} F(t,\phi)$.

Given p_1, p_2 with $0 \le p_1 < p_2 \le 1$ and let

$$\max F_0 := \lim_{||\psi||_J\to 0} \max_{t\in[0,1]} \frac{F(t,\psi)}{||\psi||_J},$$

$$\min F_0 := \lim_{||\psi||_J\to 0} \min_{t\in[p_1,p_2]} \frac{F(t,\psi)}{||\psi||_J},$$

$$\max F_\infty := \lim_{||\psi||_J\to \infty} \max_{t\in[0,1]} \frac{F(t,\psi)}{||\psi||_J},$$

and

$$\min F_\infty := \lim_{||\psi||_J\to \infty} \min_{t\in[p_1,p_2]} \frac{F(t,\psi)}{||\psi||_J}.$$

Since

$$\left(\int_0^1 k(s,s)ds\right)^{-1} := A = \frac{6(\gamma+\delta)}{\gamma+3\delta}, \tag{3.3.1}$$

and

$$\left(\int_{p_1}^{p_2} k(\frac{p_1+p_2}{2},s)ds\right)^{-1} := B = \frac{16(\gamma+\delta)}{(p_2-p_1)(L_1L_2+L_3L_4)}, \tag{3.3.2}$$

where

$$L_1 = p_2+3p_1,\ L_2 = 2\gamma - p_1\gamma - p_2\gamma + 2\delta,$$

$$L_3 = 4\gamma+4\delta-3\gamma p_2 - \gamma p_1,\ L_4 = p_1+p_2,$$

we have the following results:

Suppose that $\max F_0 = C_1 \in [0,A)$. Taking $\epsilon = A - C_1$, there exists a $\lambda_1 > 0$ (λ_1 can be chosen small arbitrary) such that for any $\psi \in C_r$ with $||\psi||_J \le \lambda_1$, we have

$$\max_{t\in[0,1]} \frac{F(t,\psi)}{||\psi||_J} \le \epsilon + C_1 = A.$$

Hence, for $t \in [0,1]$ and $\psi \in C_r$ with $||\psi||_J \in [0,\lambda_1]$,

$$F(t,\psi) \le A||\psi||_J \le A\lambda_1,$$

which satisfies the hypothesis (H_1) of Theorem 3.2.3.

Suppose that $\min F_\infty = C_2 \in (\frac{B}{M}, \infty]$. Taking $\epsilon = C_2 - \frac{B}{M} > 0$, there exists an $\eta_1 > 0$ (η_1 can be chosen large arbitrary) such that for any $\psi \in C_r$ with $||\psi||_J \geq M\eta_1$, we have

$$\min_{t\in[p_1,p_2]} \frac{F(t,\psi)}{||\psi||_J} \geq -\epsilon + C_2 = \frac{B}{M}.$$

Hence, for $t \in [p_1, p_2]$ and $\psi \in C_r$ with $||\psi||_J \in [M\eta_1, \eta_1]$,

$$F(t,\psi) \geq \frac{M}{B}||\psi||_J \geq \frac{B}{M}M\eta_1 = B\eta_1,$$

which satisfies the hypothesis (H_2) of Theorem 3.2.3.

Suppose that $\min F_0 = C_3 \in (\frac{B}{M}, \infty]$. Taking $\epsilon = C_3 - \frac{B}{M} > 0$, there exists an $\eta_2 > 0$ (η_2 can be chosen small enough) such that for any $\psi \in C_r$ with $||\psi||_J \leq \eta_2$, we have

$$\min_{t\in[p_1,p_2]} \frac{F(t,\psi)}{||\psi||_J} \geq -\epsilon + C_3 = \frac{B}{M}.$$

Hence, for any $t \in [p_1, p_2]$ and $\psi \in C_r$ with $||\psi||_J \in [M\eta_2, \eta_2]$,

$$F(t,\psi) \geq \frac{B}{M}||\psi||_J \geq \frac{B}{M}M\eta_2 = B\eta_2,$$

which satisfies the hypothesis (H_2) of Theorem 3.2.3.

Suppose that $\max F_\infty = C_4 \in [0, A)$. Taking $\epsilon = A - C_4 > 0$, there exists a $\theta > 0$ (θ can be chosen large arbitrary) such that for any $\psi \in C_r$ with $||\psi||_J \geq \theta$, we have

$$\max_{t\in[0,1]} \frac{F(t,\psi)}{||\psi||_J} \leq \epsilon + C_4 = A. \tag{3.3.3}$$

Now we have the following two cases:

Case 1. Assume that $\max_{t\in[0,1]} F(t,\psi)$ is bounded, that is, there exists a constant $L > 0$ such that

$$F(t,\psi) \leq L, \quad \text{for} \quad t \in [0,1] \quad \text{and} \quad \psi \in C_r.$$

Taking $\lambda_2 = \frac{L}{A}$, hence, for $t \in [0,1]$ and $\psi \in C_r$ with $||\psi||_J \in [0, \lambda_2]$,

$$F(t,\psi) \leq L = A\lambda_2.$$

Case 2. Assume that $\max_{t\in[0,1]} F(t,\psi) := G_t(\psi)$ is unbounded. Then, by $\mathbb{P}$, there exists a ϕ with $||\phi||_J := \lambda_2 \geq \theta$ such that for any $\psi \in C_r$ with $||\psi||_J \leq \lambda_2$, we have

$$\max_{t\in[0,1]} F(t,\psi) = G_t(\psi) \leq G_t(\phi) = \max_{t\in[0,1]} F(t,\phi).$$

This implies that there exists some $t_0 \in [0,1]$ such that,

$$F(t,\psi) \leq F(t_0,\phi), \text{ for } t \in [0,1] \text{ and } \psi \in C_r \text{ with } ||\psi||_J \leq \lambda_2.$$

It follows from $\lambda_2 \geq \theta$ and (3.3.3) that, for $t \in [0,1]$ and $\psi \in C_r$ with $||\psi||_J \in [0,\lambda_2]$,

$$F(t,\psi) \leq F(t_0,\phi) \leq A||\phi||_J = A\lambda_2.$$

By Case 1 and 2, the hypothesis (H_1) of Theorem 3.2.3 is satisfied.

It follows from the above Remark that the following corollaries hold.

Corollary 3.3.1 *Assume that F satisfies $\mathbb{P}$ and suppose there exist p_1 and p_2 with $0 \leq -w \leq p_1 < p_2 \leq 1$, A and B are defined as* (3.3.1) *and* (3.3.2) *respectively. Then in the case*

(H_4) $\max F_0 = C_1 \in [0,A)$ *and* $\min F_\infty = C_2 \in (\frac{B}{M},\infty]$, *or*

(H_5) $\min F_0 = C_3 \in (\frac{B}{M},\infty]$ *and* $\max F_\infty = C_4 \in [0,A)$,

we have following corresponding results (i) *and* (ii) *respectively.*

(i) *For any given $\phi \in C_{r,0}$ with $||\phi||_J$ small enough, $(FBVP)$ has at least one positive solution.*

(ii) *For any given $\phi \in C_{r,0}$, $(FBVP)$ has at least one positive solution.*

Proof. It follows from our Remark and Theorem 3.2.3 that the desired result holds, immediately. ❑

Corollary 3.3.2 *Assume that F satisfies $\mathbb{P}$ and suppose there exist p_1 and p_2 with $0 \leq -w \leq p_1 < p_2 \leq 1$, A and B are defined as* (3.3.1) *and* (3.3.2) *respectively. If the following hypotheses hold:*

(H_6) $\min F_\infty = C_2,\ \min F_0 = C_3 \in (\frac{B}{M}, \infty]$,

(H_7) *there exists* $\lambda^* > 0$ *such that*

$$F(t,\psi) \le A\lambda^*, \text{ for } t \in [0,1] \text{ and } \psi \in C_r \text{ with } ||\psi||_J \in [0, \lambda^*],$$

then, for any given $\phi \in C_{r,0}$ *with* $||\phi||_J \le \lambda^*$, $(FBVP)$ *has at least two positive solutions* u_1 *and* u_2 *such that* $0 < ||u_1|| < \lambda^* < ||u_2||$.

Proof. It follows from our Remark that there exist two real numbers η_1 and η_2 satisfying

$$0 < \eta_2 < \lambda^* < \eta_1,$$

$$F(t,\psi) \ge B\eta_1, \text{ for } t \in [p_1, p_2] \text{ and } \psi \in C_r \text{ with } ||\psi||_J \in [M\eta_1, \eta_1],$$

and

$$F(t,\psi) \ge B\eta_2, \text{ for } t \in [p_1, p_2] \text{ and } \psi \in C_r \text{ with } ||\psi||_J \in [M\eta_2, \eta_2].$$

Thus, by Theorem 3.2.3, we see for any given $\phi \in C_{r,0}$ with $||\phi||_J \in [0, \lambda^*]$, $(FBVP)$ has two positive solutions u_1 and u_2 such that $\eta_2 < ||u_1|| < \lambda^* < ||u_2|| < \eta_1$. Hence, we complete this proof. ❑

Corollary 3.3.3 *Assume that* F *satisfies* $\mathbb{P}$ *and suppose there exist* p_1 *and* p_2 *with* $0 \le -w \le p_1 < p_2 \le 1$, A *and* B *are defined as* (3.3.1) *and* (3.3.2) *respectively. If the following hypotheses hold:*

(H_8) $\max F_0 = C_1,\ \max F_\infty = C_4 \in [0, A)$,

(H_9) *there exists* $\eta^* > 0$ *such that*

$$F(t,\psi) \ge B\eta^*, \text{ for } t \in [p_1, p_2] \text{ and } \psi \in C_r \text{ with } ||\psi||_J \in [M\eta^*, \eta^*],$$

then, for any given $\phi \in C_{r,0}$ *with* $||\phi||_J$ *small enough,* $(FBVP)$ *has at least two positive solutions* u_1 *and* u_2 *such that* $0 < ||u_1|| < \eta^* < ||u_2||$.

Proof. It follows from our Remark that there exist two real numbers λ_1 and λ_2 satisfying

$$0 < \lambda_1 < \eta^* < \lambda_2,$$

$$F(t,\psi) \le A\lambda_1, \text{ for } t \in [0,1] \text{ and } \psi \in C_r \text{ with } ||\psi||_J \in [0,\lambda_1],$$

$$F(t,\psi) \le A\lambda_2, \text{ for } t \in [0,1] \text{ and } \psi \in C_r \text{ with } ||\psi||_J \in [0,\lambda_2].$$

Thus, by Theorem 3.2.3, we see for any given $\phi \in C_{r,0}$ with $||\phi||_J \in [0,\lambda_1]$, $(FBVP)$ has two positive solutions u_1 and u_2 such that $\lambda_1 < ||u_1|| < \eta^* < ||u_2|| < \lambda_2$. Hence, we complete this proof. ❑

Remark. We note that in the limiting case $r = 0$, C_r is reduced to $\mathbb{R}$. Then $(FBVP)$ can be reduced to a general boundary value problem as follows:

$$(BVP_4) \begin{cases} u''(t) + f(t,u(t)) = 0, \ t \in (0,1), \\ (BC_4) \end{cases}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ is continuous. It is easy to check that our Theorems can appropriately apply on (BVP_4). Furthermore, in this case, $\mathbb{P}$ automatically holds for this function $f(t,u)$ on $[0,1] \times [0,\infty)$. Hence, all corollaries are applicable to (BVP_4). Note that for many source terms, we can easily compute corresponding "$\max f_0, \min f_0, \max f_\infty, \min f_\infty$" in appropriate ranges, for example, $f(t,u) := \frac{e^u-1}{1+t^2}(\max f_0 = 1 \text{ and } \min f_0 = \frac{1}{2})$, $f(t,u) := u + t^2 e^{-u}(\max f_0 = \infty, \ \min f_0 = \max f_\infty = \min f_\infty = 1)$.

To illustrate the usage of our results, we present the following example.

Example 3.3.4 Consider the boundary value problem

$$u''(t) + p(t)\sqrt{u(t-\frac{1}{3})} + C = 0, \ t \in [0,1],$$

and

$$\begin{cases} u(t) = \phi(t), \ t \in [-\frac{1}{3}, 0] \\ (BC_4), \end{cases}$$

where $p(t)$ is a positive continuous function on $[0,1]$, $C > 0$, $\phi \in C([-\frac{1}{3},0],\mathbb{R}^+)$ with $\phi(0) = 0$ is arbitrarily given. Then, we have

$$F(t,\psi) := p(t)\sqrt{\psi(-\frac{1}{3})} + C,$$

which implies F satisfies $\mathbb{P}$. One can compute

$$\max F_\infty = 0$$

and for any p_1 and p_2 with $0 \leq -\frac{1}{3} \leq p_1 < p_2 \leq 1$,

$$\min F_0 = \infty.$$

Applying Corollary 3.3.1 to this example, we can conclude that there is at least one positive solution to this boundary value problem.