Chapter 4

High Order Ordinary Differential Equation Equipped with A Kind of Three-Point Boundary Condtion

4.1 Introduction

In the last thirty years, a great deal of works has been done to study the positive solutions of two point boundary value problem for differential equations which are used to describe a number of physical, biological and chemical phenomena. For additional background and results, we refer the reader to the monograph by Agarwal and Wong [2, 3, 7, 8, 9], Agarwal, O'Regan and Wong [10] as well as the recent contributions by [12, 24, 29, 30, 50, 51, 59, 60, 52, 63].

Boundary value problems for higher order differential equations can arise, especially for fourth order equations. Recently, three point or multi-point boundary value problem of the differential equations were presented and studied by many authors, see [48, 49, 59, 60].

In this chapter, we attempt to establish some existence theorems of positive solutions for the following $n+2^{\text{th}}$ order nonlinear boundary value problem:

$$(HBVP)\begin{cases} (HE)\ [\phi_p(u^{(n)}(t))]'' = f(t, u(t), u^{(1)}(t), \cdots, u^{(n+1)}(t)) \\ (BC_5)\ \begin{cases} u^{(i)}(0) = 0,\ i = 0, 1, 2, \dots n-3, \\ u^{(n-2)}(0) = \xi u^{(n-2)}(1), \\ u^{(n-1)}(1) = \eta u^{(n-1)}(0), \\ u^{(n)}(0) = \mu u^{(n)}(\delta), \\ u^{(n)}(1) = \nu u^{(n)}(\delta), \end{cases} \end{cases}$$

where $f : (0,1) \times \mathbb{R}^{n+2} \to [0, +\infty)$ is a continuous function; $\mu, \nu \geq 0, \xi \neq 1, \eta \neq 1, 0 < \delta < 1, n \geq 2$ and $\phi_p(z) = |z|^{p-2}z$ for $p > 1$.

In 2006, Ma and Ge [52] has studied this topic for boundary value problem:

$$(HBVP_5^*)\begin{cases} (HE_5^*)\ [\phi_p(u''(t))]'' = a(t)f(u(t)) \\ (BC_5^*)\ \begin{cases} u(0) = \xi u(1), \\ u'(1) = \eta u'(0), \\ u''(0) = \mu u''(\delta), \\ u''(1) = \nu u''(\delta). \end{cases} \end{cases}$$

They applied a fixed-point theorem to establish the existence of at least three positive solutions of $(HBVP_5^*)$. Now, we consider the more general case (HBVP) and hope to obtain some extension of the excellent results of Ma and Ge [52].

4.2 Preliminaries

In order to abbreviate our discussion, we need the following observations and lemmas.

Throughout this chapter, we assume that

(C_5) q is a constant and satisfies $\frac{1}{p} + \frac{1}{q} = 1$;

and observe that

$(C_6)\quad (\phi_p)^{-1}(z) := \phi_q(z) = |z|^{q-2}z$

Definition 4.2.1 *Let X be a real Banach space and P be a cone of X. A map $\psi : P \to [0, +\infty)$ is called a nonnegative continuous concave functional map if ψ is nonnegative, continuous and satisfies for all $x, y \in P$ and $t \in [0, 1]$,*

$$\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y).$$

Definition 4.2.2 *Let X be a real Banach space and P be a cone of X. A map $\beta : P \to [0, +\infty)$ is called a nonnegative continuous convex functional map if β is nonnegative, continuous and satisfies for all $x, y \in P$ and $t \in [0, 1]$,*

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y).$$

Let γ, β and θ be nonnegative continuous convex functionals on P, and let α and ψ be nonnegative continuous concave functionals on P. For given nonnegative numbers h, a, b, d and c, we define the following sets:

$$\begin{aligned}
&P(\gamma, c) = \{x \in P \mid \gamma(x) < c\},\\
&P(\gamma, \alpha, a, c) = \{x \in P \mid a \leq \alpha(x), \gamma(x) \leq c\},\\
&Q(\gamma, \beta, d, c) = \{x \in P \mid \beta(x) \leq d, \gamma(x) \leq c\},\\
&P(\gamma, \theta, \alpha, a, b, c) = \{x \in P \mid a \leq \alpha(x), \theta(x) \leq b, \gamma(x) \leq c\},\\
&Q(\gamma, \beta, \psi, h, d, c) = \{x \in P \mid h \leq \psi(x), \beta(x) \leq d, \gamma(x) \leq c\}.
\end{aligned}$$

In order to obtain multiple positive solutions of $(HBVP)$, the following fixed-point theorem due to Avery which is a generalization of Leggett-Willliams fixed-point theorem will be fundamental.

Lemma 4.2.3 ([12], Theorem 2.4) *Let X be a real Banach space and P be a cone of X. Suppose γ, β and θ are three nonnegative continuous convex functionals* *on P and α, ψ are two nonnegative continuous concave functionals on P such that there are $c, L \in (0, \infty)$ satisfying*

$$\alpha(x) \leq \beta(x), \ ||x|| \leq L\gamma(x)$$

for $x \in \overline{P(\gamma, c)}$. Suppose further that

$$T : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$$

is completely continuous and there exist $h, d, a, b \geq 0$ with $0 < d < a$ such that each of the following is satisfied:

(i) $\{x \in P(\gamma, \theta, \alpha, a, b, c) \mid \alpha(x) > a\} \neq \emptyset$ *and* $\alpha(Tx) > a$ *for* $x \in P(\gamma, \theta, \alpha, a, b, c)$,

(ii) $\{x \in Q(\gamma, \beta, \psi, h, d, c) \mid \beta(x) < d\} \neq \emptyset$ *and* $\beta(Tx) < d$ *for* $x \in Q(\gamma, \beta, \psi, h, d, c)$,

(iii) $\alpha(Tx) > a$ *for* $x \in P(\gamma, \alpha, a, c)$ *with* $\theta(Tx) > b$,

(iv) $\beta(Tx) < d$ *for* $x \in Q(\gamma, \beta, d, c)$ *with* $\psi(Tx) < h$.

Then, T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such that $\beta(x_1) < d$, $a < \alpha(x_2)$, $d < \beta(x_3)$ with $\alpha(x_3) < a$.

Lemma 4.2.4 ([52]) *Suppose that H is continuous on $[0, 1]$, then the unique solution of boundary value problem*

$$\begin{cases} -y'' = H(t) \ in \ (0, 1) \\ y(0) = \phi_p(\mu) y(\delta), \ y(1) = \phi_p(\nu) y(\delta), \end{cases} \tag{4.2.1}$$

is

$$y(t) = \frac{1}{M} \int_0^1 g(t, s) H(s) ds,$$

where

$$M := 1 - \phi_p(\mu) - (\phi_p(\nu) - \phi_p(\mu))\delta \neq 0 \tag{4.2.2}$$

and

$$g(t, s) = \begin{cases} s(1-t) + \phi_p(\nu) s(t - \delta), \ 0 \leq s \leq t < \delta < 1 \ or \ 0 \leq s \leq \delta \leq t \leq 1, \\ t(1-s) + \phi_p(\nu) t(s - \delta) + \phi_p(\mu)(1-\delta)(s-t), \ 0 \leq t \leq s \leq \delta < 1, \\ s(1-t) + \phi_p(\nu)\delta(t-s) + \phi_p(\mu)(1-t)(\delta - s), \ 0 \leq \delta \leq s \leq t \leq 1, \\ (1-s)(t - \phi_p(\mu)t + \phi_p(\mu)\delta), \ 0 < \delta \leq t \leq s \leq 1 \ or \ 0 \leq t < \delta \leq s \leq 1. \end{cases}$$

Lemma 4.2.5 ([52]) *Suppose that H is continuous on $[0,1]$, then the unique solution of boundary value problem*

$$\begin{cases} -y'' = H(t) \ \textit{in}\ (0,1), \\ y(0) = \xi y(1),\ y'(1) = \eta y'(0), \end{cases} \tag{4.2.3}$$

is

$$y(t) = \frac{1}{M_1}\int_0^1 h(t,s)H(s)ds,$$

where

$$M_1 = (1-\xi)(1-\eta) \neq 0$$

and

$$h(t,s) = \begin{cases} s + \eta(t-s) + \xi\eta(1-t),\ 0 \le s \le t \le 1, \\ t + \xi(s-t) + \xi\eta(1-s),\ 0 \le t \le s \le 1. \end{cases}$$

Lemma 4.2.6 ([52]) *Suppose that $0 \le \xi,\ \eta < 1, 0 < t_1 < t_2 < 1$ and, $\delta \in (0,1)$. Then, for all $s \in [0,1]$,*

$$\frac{h(t_1,s)}{h(t_2,s)} \ge \frac{t_1}{t_2} \tag{4.2.4}$$

and

$$\frac{h(1,s)}{h(\delta,s)} \le \frac{1}{\delta} \tag{4.2.5}$$

hold.

Lemma 4.2.7 ([52]) *Suppose that $\xi,\ \eta > 1,\ 0 < t_1 < t_2 < 1$ and $\delta \in (0,1)$. Then, for all $s \in [0,1]$,*

$$\frac{h(t_2,s)}{h(t_1,s)} \ge \frac{1-t_2}{1-t_1} \tag{4.2.6}$$

and

$$\frac{h(0,s)}{h(\delta,s)} \le \frac{1}{1-\delta} \tag{4.2.7}$$

hold.

4.3 Three Positive solutions

Here, we consider the classical Banach space $X = C^n([0,1])$ which is equipped with the usual norm. The cones P_1 and $P_2 \subset X$ are defined as follows:

$$P_1 = \{u \in X \mid u^{(n-2)}(t) \text{ is nonnegative concave and nondecreasing on } [0,1]\}$$

and

$$P_2 = \{u \in X \mid u^{(n-2)}(t) \text{ is nonnegative concave and nonincreasing on } [0,1]\}.$$

Next, let $t_1, t_2, t_3 \in (0,1)$ with $t_1 < t_2$ be fixed. Moreover, we shall define the nonnegative continuous concave functionals α, ψ and nonnegative convex functionals β, θ, γ on P_1 by

$$\gamma(x) = \max_{t\in[0,t_3]} x^{(n-2)}(t) = x^{(n-2)}(t_3), x \in P_1,$$

$$\psi(x) = \min_{t\in[\delta,1]} x^{(n-2)}(t) = x^{(n-2)}(\delta), x \in P_1,$$

$$\beta(x) = \max_{t\in[\delta,1]} x^{(n-2)}(t) = x^{(n-2)}(1), x \in P_1,$$

$$\alpha(x) = \min_{t\in[t_1,t_2]} x^{(n-2)}(t) = x^{(n-2)}(t_1), x \in P_1,$$

$$\theta(x) = \max_{t\in[t_1,t_2]} x^{(n-2)}(t) = x^{(n-2)}(t_2), x \in P_1.$$

Theorem 4.3.1 *Suppose that $M > 0$ and the following assumptions hold:*

(A_4) $f : (0,1) \times \mathbb{R}^{n+2} \to [0,\infty)$ *is continuous,*

(A_5) $\xi, \eta \in [0,1)$ *and* $a, b, c \in (0,\infty)$ *satisfying* $0 < a < b < \frac{t_2}{t_1} b \le c$

and

(A_6) *there are three positive constants k_1, k_2, k_3 satisfy the following conditions:*

(1°) $f(t, y_1, \cdots, y_{n+2}) > k_1 \phi_p(\frac{b}{B})$,

for $(t, y_1, \cdots, y_{n+2}) \in [t_1, t_2] \times \prod_{k=0}^{n-2} [b t_1^{n-2-k}, \frac{t_2^{n-1-k}}{t_1} b] \times \mathbb{R}^3$,

(2°) $f(t, y_1, \cdots, y_{n+2}) < k_2 \phi_p(\frac{a}{C})$,

for $(t, y_1, \cdots, y_{n+2}) \in [0,1] \times [0,a]^{n-1} \times \mathbb{R}^3$,

(3°) $f(t, y_1, \cdots, y_{n+2}) \le k_3 \phi_p(\frac{c}{A})$,

for $(t, y_1, \cdots, y_{n+2}) \in [0,1] \times [0, \frac{1}{t_3} c]^{n-1} \times \mathbb{R}^3$, *where A, B and C are defined as follows:*

$$A = \tfrac{1}{M_1\phi_q(M)} \int_0^1 h(t_3, s)\phi_q[\int_0^1 k_3 \cdot g(s,r)dr]ds,$$
$$B = \tfrac{1}{M_1\phi_q(M)} \int_0^1 h(t_1, s)\phi_q[\int_{t_1}^{t_2} k_1 \cdot g(s,r)dr]ds,$$
$$C = \tfrac{1}{M_1\phi_q(M)} \int_0^1 h(1, s)\phi_q[\int_0^1 k_2 \cdot g(s,r)dr]ds.$$

Then, the boundary value problem $(HBVP)$ has at least three positive solutions.

Proof. By Lemma 4.2.4, we obtain

$$u^{(n)}(t) = -\frac{1}{\phi_q(M)}\phi_q\left(\int_0^1 g(t,s)f(s,u(s),\cdots,u^{(n+1)}(s))ds\right).$$

Moreover, it follows from Lemma 4.2.5, we see that

$$u^{(n-2)}(t) = \frac{1}{M_1\phi_q(M)}\int_0^1 h(t,s)\phi_q\left(\int_0^1 g(s,\tau)f(\tau,u(\tau),\cdots,u^{(n+1)}(\tau))d\tau\right)ds.$$

Inspired by the above-mentioned result, we can define a completely continuous operator $T : P_1 \to X$ via

$$(Tu)^{(n-2)}(t) = \frac{1}{M_1\phi_q(M)}\int_0^1 h(t,s)\phi_q\left(\int_0^1 g(s,r)f(r,u(r),\cdots,u^{(n+1)}(r))dr\right)ds.$$

It is clear that u is a positive solution of $(HBVP)$ if and only if u is a fixed point of T on cone P_1. Since M and $M_1 > 0$,

$$(Tu)^{(n-2)} \geq 0 \text{ for } u \in P_1.$$

Furthermore,

$$\begin{aligned}((Tu)^{(n-2)}(t))' =& \frac{1-\xi}{M_1\phi_q(M)}\{\eta\int_0^t \phi_q\left(\int_0^1 g(s,r)f(r,u(r),u^{(1)}(r),\cdots,u^{(n+1)}(r))dr\right)ds \\ &+ \int_t^1 \phi_q\left(\int_0^1 g(s,r)f(r,u(r),u^{(1)}(r),\cdots,u^{(n+1)}(r))dr\right)ds\} \\ \geq& 0\end{aligned}$$

and

$$((Tu)^{(n-2)}(t))'' = -\frac{1}{\phi_q(M)}\phi_q\left(\int_0^1 g(t,r)f(r,u(r),u^{(1)}(r),\cdots,u^{(n+1)}(r))dr\right) \leq 0,$$

which imply $TP_1 \subset P_1$. Follows from $u \in P_1$, we see that

$$\gamma(u) = u^{(n-2)}(t_3) = u^{(n-2)}(t_3 \times 1 + (1-t_3) \times 0)$$
$$\geq t_3 u^{(n-2)}(1) + (1-t_3)u^{(n-2)}(0) \geq t_3 \times \max_{t\in[0,1]} |u^{(n-2)}(t)|.$$

Therefore, for $u \in \overline{P_1(\gamma, c)}$, we have

$$0 \leq u^{(n-2)}(t) \leq \max_{t\in[0,1]} |u^{(n-2)}(t)| \leq \frac{1}{t_3}\gamma(u) \leq \frac{1}{t_3}c$$

on $[0,1]$ which implies

$$0 \leq u^{(k)}(t) \leq \frac{1}{t_3}c \text{ on } [0,1],\ k = 0, \cdots, n-2.$$

This and $(A_6) - (3^{\circ})$ imply

$$\begin{aligned}
\gamma((Tu)) &= \max_{t\in[0,t_3]} (Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(t_3) \\
&= \frac{1}{M_1\phi_q(M)} \int_0^1 h(t_3, s)\phi_q\left(\int_0^1 g(s,r)f(r, u(r), u^{(1)}(r), \cdots, u^{(n+1)}(r))dr\right) ds \\
&\leq \frac{1}{M_1\phi_q(M)} \int_0^1 h(t_3, s)\phi_q\left(\int_0^1 k_3 \cdot g(s,r) \cdot \phi_p(\frac{c}{A})dr\right) ds \\
&= \frac{c}{A}\left\{\frac{1}{M_1\phi_q(M)} \int_0^1 h(t_3, s)\phi_q\left(\int_0^1 k_3 \cdot g(s,r)dr\right) ds\right\} = c.
\end{aligned}$$

Therefore, we obtain that $T : \overline{P_1(\gamma, c)} \to \overline{P_1(\gamma, c)}$. Next, we separate the rest proof into the following steps:

Step 1 For some $\varepsilon_1 \in (0, \frac{t_2}{t_1}b - b)$, let

$$u_1^{(n-2)}(t) := b + \varepsilon_1,$$

then,

$$u_1 \in P_1(\gamma, \theta, \alpha, b, \frac{t_2}{t_1}b, c) = \{x \in P_1 \mid b \leq \alpha(x),\ \theta(x) \leq \frac{t_2}{t_1}b,\ \gamma(x) \leq c\}.$$

This means that $u_1 \in \{u \in P_1(\gamma, \theta, \alpha, b, \frac{t_2}{t_1}b, c) \mid \alpha(u) > b\} \neq \emptyset$ is well-defined. Moreover, we have

$$u^{(n-2)}(t) \geq u^{(n-2)}(t_1) = \alpha(u) \geq b$$

and

$$u^{(n-2)}(t) \leq u^{(n-2)}(t_2) = \theta(u) \leq \frac{t_2}{t_1}b$$

on $[t_1, t_2]$, which implies, for $u \in P_1(\gamma, \theta, \alpha, b, \frac{t_2}{t_1}b, c)$,

$$bt_1^{n-2-k} \leq u^{(k)}(t) \leq \frac{t_2^{n-1-k}}{t_1}b \text{ on } [t_1, t_2],\ k = 0, \ldots n-2.$$

This and $(A_6) - (1^\circ)$ imply

$$\begin{aligned}
\alpha((Tu)) &= \min_{t\in[t_1,t_2]} (Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(t_1) \\
&= \frac{1}{M_1\phi_q(M)} \int_0^1 h(t_1, s)\phi_q \left(\int_0^1 g(s, r)f(r, u(r), u^{(1)}(r), \cdots, u^{(n+1)}(r))dr \right) ds \\
&> \frac{1}{M_1\phi_q(M)} \int_0^1 h(t_1, s)\phi_q \left(\int_{t_1}^{t_2} k_1 \cdot g(s, r) \cdot \phi_p(\frac{b}{B})dr \right) ds \\
&= \frac{b}{B} \left\{ \frac{1}{M_1\phi_q(M)} \int_0^1 h(t_1, s)\phi_q \left(\int_{t_1}^{t_2} k_1 \cdot g(s, r)dr \right) ds \right\} = b.
\end{aligned}$$

Step 2 For some $\varepsilon_2 \in (0, a - \delta a)$, let

$$u_2^{(n-2)}(t) := a - \varepsilon_2,$$

then,

$$u_2 \in Q_1(\gamma, \beta, \psi, \delta a, a, c) = \{x \in P_1 \mid \delta a \leq \psi(x), \beta(x) \leq a, \gamma(x) \leq c\}.$$

This means that $u_2 \in \{u \in Q_1(\gamma, \beta, \psi, \delta a, a, c) \mid \beta(u) < a\} \neq \emptyset$ is well-defined. Moreover, we have $0 \leq u^{(n-2)}(t) \leq u^{(n-2)}(1) = \beta(u) \leq a$ on $[0, 1]$, which implies, for $u \in Q_1(\gamma, \beta, \psi, \delta a, a, c)$,

$$0 \leq u^{(k)}(t) \leq a \text{ on } [0, 1],\ k = 0, \cdots, n-2.$$

This and $(A_6) - (2^\circ)$ imply

$$\begin{aligned}
\beta((Tu)) &= \max_{t\in[\delta,1]} (Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(1) \\
&= \frac{1}{M_1\phi_q(M)} \int_0^1 h(1, s)\phi_q \left(\int_0^1 g(s, r)f(r, u(r), u^{(1)}(r), \cdots, u^{(n+1)}(r))dr \right) ds \\
&< \frac{1}{M_1\phi_q(M)} \int_0^1 h(1, s)\phi_q \left(\int_0^1 k_2 \cdot g(s, r) \cdot \phi_p(\frac{a}{C})dr \right) ds \\
&= \frac{a}{C} \left\{ \frac{1}{M_1\phi_q(M)} \int_0^1 h(1, s)\phi_q \left(\int_0^1 k_2 \cdot g(s, r)dr \right) ds \right\} = a.
\end{aligned}$$

Step 3 For $u \in P_1(\gamma, \alpha, b, c) = \{x \in P_1 \mid b \leq \alpha(x), \gamma(x) \leq c\}$ with $\theta(Tu) > \frac{t_2}{t_1} b$, it follows from Lemma 4.2.6 that

$$\begin{aligned}
\alpha((Tu)) &= \min_{t\in[t_1,t_2]} (Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(t_1) \\
&= \frac{1}{M_1\phi_q(M)} \int_0^1 h(t_1, s)\phi_q\left(\int_0^1 g(s,r)f(r,u(r),\cdots,u^{(n+1)}(r))dr\right) ds \\
&= \frac{1}{M_1\phi_q(M)} \int_0^1 \frac{h(t_1,s)}{h(t_2,s)} h(t_2, s)\phi_q\left(\int_0^1 g(s,r)f(r,u(r),\cdots,u^{(n+1)}(r))dr\right) ds \\
&\geq \frac{t_1}{t_2}\left\{\frac{1}{M_1\phi_q(M)} \int_0^1 h(t_2, s)\phi_q\left(\int_0^1 g(s,r)f(r,u(r),\cdots,u^{(n+1)}(r))dr\right) ds\right\} \\
&= \frac{t_1}{t_2}\{(Tu)^{(n-2)}(t_2)\} = \frac{t_1}{t_2} \max_{t\in[t_1,t_2]} (Tu)^{(n-2)}(t) \\
&= \frac{t_1}{t_2}\theta(Tu) > \frac{t_1}{t_2}\frac{t_2}{t_1} b = b.
\end{aligned}$$

Step 4 For $u \in Q_1(\gamma, \beta, a, c) = \{x \in P_1 \mid \beta(x) \leq a, \gamma(x) \leq c\}$ with $\psi(Tu) < \delta a$, it follows from Lemma 4.2.6 that

$$\begin{aligned}
\beta((Tu)) &= \max_{t\in[\delta,1]} (Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(1) \\
&= \frac{1}{M_1\phi_q(M)} \int_0^1 h(1, s)\phi_q\left(\int_0^1 g(s,r)f(r,u(r),\cdots,u^{(n+1)}(r))dr\right) ds \\
&= \frac{1}{M_1\phi_q(M)} \int_0^1 \frac{h(1,s)}{h(\delta,s)} h(\delta, s)\phi_q\left(\int_0^1 g(s,r)f(r,u(r),\cdots,u^{(n+1)}(r))dr\right) ds \\
&\leq \frac{1}{\delta}\left\{\frac{1}{M_1\phi_q(M)} \int_0^1 h(\delta, s)\phi_q\left(\int_0^1 g(s,r)f(r,u(r),\cdots,u^{(n+1)}(r))dr\right) ds\right\} \\
&= \frac{1}{\delta}\{(Tu)^{(n-2)}(\delta)\} = \frac{1}{\delta} \min_{t\in[\delta,1]} (Tu)^{(n-2)}(t) \\
&= \frac{1}{\delta}\psi(Tu) < \frac{1}{\delta}\delta a = a.
\end{aligned}$$

Therefore, the hypotheses of Lemma 4.2.3 are fulfilled. Thus, there exist three positive solutions u_1, u_2, u_3 for $(HBVP)$. ❑

Let $t_1, t_2, t_3 \in (0, 1)$ with $t_1 < t_2$ fixed. Moreover, we shall define the nonnegative continuous concave functionals α, ψ and nonnegative convex functionals β, θ, γ

on P_2 by

$$\gamma(x) = \max_{t\in[t_3,1]} x^{(n-2)}(t) = x^{(n-2)}(t_3),\ x \in P_2,$$

$$\psi(x) = \min_{t\in[0,\delta]} x^{(n-2)}(t) = x^{(n-2)}(\delta),\ x \in P_2,$$

$$\beta(x) = \max_{t\in[0,\delta]} x^{(n-2)}(t) = x^{(n-2)}(0),\ x \in P_2,$$

$$\alpha(x) = \min_{t\in[t_1,t_2]} x^{(n-2)}(t) = x^{(n-2)}(t_2),\ x \in P_2,$$

$$\theta(x) = \max_{t\in[t_1,t_2]} x^{(n-2)}(t) = x^{(n-2)}(t_1),\ x \in P_2.$$

Theorem 4.3.2 *Suppose that $M > 0$ and the following assumptions hold:*

(A_7) $\quad f : (0,1)\times\mathbb{R}^{n+2} \to [0,\infty)$ *is continuous,*

(A_8) $\quad \xi,\eta \in (1,\infty)$ *and* $a,b,c \in (0,\infty)$ *satisfying* $0 < a < b < \frac{1-t_1}{1-t_2}b \le c,$

and

(A_9) $\quad$ *there are three positive constants* k_1,k_2,k_3 *satisfy* $(2^\circ),(3^\circ)$*, and*

$$(4^\circ)\quad f(t,y_1,\cdots,y_{n+2}) > k_1\phi_p(\tfrac{b}{B}),$$
$$(t,y_1,\cdots,y_{n+2}) \in [t_1,t_2]\times\prod_{k=0}^{n-2}[bt_1^{n-2-k},\tfrac{1-t_1}{1-t_2}t_2^{n-2-k}b]\times\mathbb{R}^3.$$

Then, the boundary value problem $(HBVP)$ has at least three positive solutions.

Proof. By Lemma 4.2.4, we obtain

$$u^{(n)}(t) = -\frac{1}{\phi_q(M)}\phi_q\left(\int_0^1 g(t,s)f(s,u(s),\cdots,u^{(n+1)}(s))ds\right).$$

Moreover, it follows from Lemma 4.2.5, we see that

$$u^{(n-2)}(t) = \frac{1}{M_1\phi_q(M)}\int_0^1 h(t,s)\phi_q\left(\int_0^1 g(s,\tau)f(\tau,u(\tau),\cdots,u^{(n+1)}(\tau))d\tau\right)ds.$$

Inspired by the above-mentioned result, we can define a completely continuous operator $T : P_2 \to X$ via

$$(Tu)^{(n-2)}(t) = \frac{1}{M_1\phi_q(M)}\int_0^1 h(t,s)\phi_q\left(\int_0^1 g(s,r)f(r,u(r),\cdots,u^{(n+1)}(r))dr\right)ds.$$

It is clear that u is a positive solution of $(HBVP)$ if and only if u is a fixed point of T on cone P_2. Since M and $M_1 > 0$,

$$(Tu)^{(n-2)} \ge 0 \text{ for } u \in P_2.$$

Furthermore,

$$
\begin{aligned}
((Tu)^{(n-2)}(t))' =& \frac{1-\xi}{M_1\phi_q(M)}\{\eta\int_0^t \phi_q\left(\int_0^1 g(s,r)f(r,u(r),u^{(1)}(r),\cdots,u^{(n+1)}(r))dr\right)ds \\
&+\int_t^1 \phi_q\left(\int_0^1 g(s,r)f(r,u(r),u^{(1)}(r),\cdots,u^{(n+1)}(r))dr\right)ds\} \\
\leq& 0,
\end{aligned}
$$

and

$$
((Tu)^{(n-2)}(t))'' = -\frac{1}{\phi_q(M)}\phi_q\left(\int_0^1 g(t,r)f(r,u(r),u^{(1)}(r),\cdots,u^{(n+1)}(r))dr\right) \leq 0,
$$

which imply $TP_2 \subset P_2$. Follows from $u \in P_2$, we see that

$$
\begin{aligned}
\gamma(u) = u^{(n-2)}(t_3) &= u^{(n-2)}(t_3 \times 1 + (1-t_3)\times 0) \\
&\geq t_3 u^{(n-2)}(1) + (1-t_3)u^{(n-2)}(0) \geq t_3 \times \max_{t\in[0,1]}|u^{(n-2)}(t)|.
\end{aligned}
$$

Therefore, we have

$$
0 \leq u^{(n-2)}(t) \leq \max_{t\in[0,1]}|u^{(n-2)}(t)| \leq \frac{1}{t_3}\gamma(u) \leq \frac{1}{t_3}c
$$

on $[0,1]$, which implies, for $u \in \overline{P_2(\gamma,c)}$,

$$
0 \leq u^{(k)}(t) \leq \frac{1}{t_3}c \text{ on } [0,1],\ k = 0,\cdots,n-2.
$$

This and $(A_9) - (3^\circ)$ imply

$$
\begin{aligned}
\gamma((Tu)) &= \max_{t\in[t_3,1]}(Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(t_3) \\
&= \frac{1}{M_1\phi_q(M)}\int_0^1 h(t_3,s)\phi_q\left(\int_0^1 g(s,r)f(r,u(r),u^{(1)}(r),\cdots,u^{(n+1)}(r))dr\right)ds \\
&\leq \frac{1}{M_1\phi_q(M)}\int_0^1 h(t_3,s)\phi_q\left(\int_0^1 k_3 g(s,r)\phi_p(\frac{c}{A})dr\right)ds \\
&= \frac{c}{A}\left\{\frac{1}{M_1\phi_q(M)}\int_0^1 h(t_3,s)\phi_q\left(\int_0^1 k_3 g(s,r)dr\right)ds\right\} = c.
\end{aligned}
$$

Therefore, we obtain that $T: \overline{P_2(\gamma,c)} \to \overline{P_2(\gamma,c)}$. Next, we separate the rest proof into the following steps:

Step 1 For some $\varepsilon_1 \in (0, \frac{1-t_1}{1-t_2}b - b)$, let

$$
u_1^{(n-2)}(t) := b + \varepsilon_1,
$$

then,

$$u_1 \in P_2(\gamma,\theta,\alpha,b,\frac{1-t_1}{1-t_2}b,c) = \{x \in P_2 \mid b \le \alpha(x), \theta(x) \le \frac{1-t_1}{1-t_2}b, \gamma(x) \le c\}.$$

This means that $u_1 \in \{u \in P_2(\gamma,\theta,\alpha,b,\frac{1-t_1}{1-t_2}b,c) \mid \alpha(u) > b\} \neq \emptyset$ is well-defined. Moreover, we have

$$u^{(n-2)}(t) \ge u^{(n-2)}(t_2) = \alpha(u) \ge b$$

and

$$u^{(n-2)}(t) \le u^{(n-2)}(t_1) = \theta(u) \le \frac{1-t_1}{1-t_2}b$$

on $[t_1,t_2]$, which implies, for $u \in P_2(\gamma,\theta,\alpha,b,\frac{1-t_1}{1-t_2}b,c)$,

$$b_1^{n-2-k} \le u^{(k)}(t) \le \frac{1-t_1}{1-t_2}t_2^{n-2-k}b \text{ on } [t_1,t_2],\ k=0,\cdots,n-2.$$

This and $(A_9)-(4^\circ)$ imply

$$\begin{aligned}
\alpha((Tu)) &= \min_{t\in[t_1,t_2]}(Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(t_2)\\
&= \frac{1}{M_1\phi_q(M)}\int_0^1 h(t_2,s)\phi_q\left(\int_0^1 g(s,r)f(r,u(r),u^{(1)}(r),\cdots,u^{(n+1)}(r))dr\right)ds\\
&> \frac{1}{M_1\phi_q(M)}\int_0^1 h(t_2,s)\phi_q\left(\int_{t_1}^{t_2} k_1\cdot g(s,r)\cdot\phi_p(\frac{b}{B})dr\right)ds\\
&= \frac{b}{B}\left\{\frac{1}{M_1\phi_q(M)}\int_0^1 h(t_2,s)\phi_q\left(\int_{t_1}^{t_2} k_1\cdot g(s,r)dr\right)ds\right\} = b.
\end{aligned}$$

Step 2 For some $\varepsilon_2 \in (0, a-(1-\delta)a)$, let

$$u_2^{(n-2)}(t) := a-\varepsilon_2,$$

then

$$u_2 \in P_2(\gamma,\beta,\psi,(1-\delta)a,a,c) = \{x \in P_2 \mid (1-\delta)a \le \psi(x), \beta(x) \le a, \gamma(x) \le c\}.$$

This means that $u_2 \in \{u \in Q_2(\gamma,\beta,\psi,(1-\delta)a,a,c) \mid \beta(u) < a\} \neq \emptyset$ is well-defined. Moreover, we have

$$0 \le u^{(n-2)}(t) \le u^{(n-2)}(0) = \beta(u) \le a$$

on $[0,1]$, which implies, for $u \in Q_2(\gamma, \beta, \psi, (1-\delta)a, a, c)$,

$$0 \leq u^{(k)}(t) \leq a \text{ on } [0,1],\ k = 0, \cdots, n-2.$$

This and $(A_9) - (2^\circ)$ imply

$$\begin{aligned}
\beta((Tu)) &= \max_{t\in[0,\delta]} (Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(0) \\
&= \frac{1}{M_1\phi_q(M)} \int_0^1 h(0,s)\phi_q\left(\int_0^1 g(s,r)f(r,u(r),\cdots,u^{(n+1)}(r))dr\right) ds \\
&< \frac{1}{M_1\phi_q(M)} \int_0^1 h(0,s)\phi_q\left(\int_0^1 k_2\cdot g(s,r)\cdot\phi_p(\frac{a}{C})dr\right) ds \\
&= \frac{a}{C}\left\{\frac{1}{M_1\phi_q(M)} \int_0^1 h(0,s)\phi_q\left(\int_0^1 k_2\cdot g(s,r)dr\right) ds\right\} = a.
\end{aligned}$$

Step 3 For $u \in P_2(\gamma, \alpha, b, c) = \{x \in P_2 \mid b \leq \alpha(x), \gamma(x) \leq c\}$ with $\theta(Tu) > \frac{1-t_1}{1-t_2}b$, it follows from Lemma 4.2.7 that

$$\begin{aligned}
\alpha(Tu) &= \min_{t\in[t_1,t_2]} (Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(t_2) \\
&= \frac{1}{M_1\phi_q(M)} \int_0^1 h(t_2,s)\phi_q\left(\int_0^1 g(s,r)f(r,u(r),\cdots,u^{(n+1)}(r))dr\right) ds \\
&= \frac{1}{M_1\phi_q(M)} \int_0^1 \frac{h(t_2,s)}{h(t_1,s)}h(t_1,s)\phi_q\left(\int_0^1 g(s,r)f(r,u(r),\cdots,u^{(n+1)}(r))dr\right) ds \\
&\geq \frac{1-t_2}{1-t_1}\left\{\frac{1}{M_1\phi_q(M)} \int_0^1 h(t_2,s)\phi_q\left(\int_0^1 g(s,r)f(r,u(r),\cdots,u^{(n+1)}(r))dr\right) ds\right\} \\
&= \frac{1-t_2}{1-t_1}\{(Tu)^{(n-2)}(t_1)\} = \frac{1-t_2}{1-t_1}\max_{t\in[t_1,t_2]}(Tu)^{(n-2)}(t) \\
&= \frac{1-t_2}{1-t_1}\theta(Tu) > \frac{1-t_2}{1-t_1}\times\frac{1-t_1}{1-t_2}b = b.
\end{aligned}$$

Step 4 For $Q_2(\gamma, \beta, a, c) = \{x \in P_2 \mid \beta(x) \leq a, \gamma(x) \leq c\}$ with $\psi(Tu) < (1-\delta)a$, it follows from Lemma 4.2.7 that

$$\begin{aligned}
\beta(Tu) &= \max_{t\in[0,\delta]}(Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(0)\\
&= \frac{1}{M_1\phi_q(M)}\int_0^1 h(0,s)\phi_q\left(\int_0^1 g(s,r)f(r,u(r),\cdots,u^{(n+1)}(r))dr\right)ds\\
&= \frac{1}{M_1\phi_q(M)}\int_0^1 \frac{h(0,s)}{h(\delta,s)}h(\delta,s)\phi_q\left(\int_0^1 g(s,r)f(r,u(r),\cdots,u^{(n+1)}(r))dr\right)ds\\
&\le \frac{1}{(1-\delta)}\left\{\frac{1}{M_1\phi_q(M)}\int_0^1 h(\delta,s)\phi_q\left(\int_0^1 g(s,r)f(r,u(r),\cdots,u^{(n+1)}(r))dr\right)ds\right\}\\
&= \frac{1}{(1-\delta)}\{(Tu)^{(n-2)}(\delta)\} = \frac{1}{(1-\delta)}\min_{t\in[0,\delta]}(Tu)^{(n-2)}(t)\\
&= \frac{1}{(1-\delta)}\psi(Tu) < \frac{1}{(1-\delta)}(1-\delta)a = a.
\end{aligned}$$

Therefore, the hypotheses of Lemma 4.2.3 are fulfilled and there exist three positive solutions u_1, u_2, u_3 for $(HBVP)$. ❑