2 Some Definitions and Theorems

In this section, we review some basic definitions, the first and the second fundamental theorems, and some well-known results in the theory of value distribution whose proofs can be found in [4, 6, 11].

Definition 2.1 *For $x \geq 0$, define*

$$\log^+ x = \max(\log x, 0) = \begin{cases} \log x, & \text{if} \quad x \geq 1 \\ 0, & \text{if} \quad 0 \leq x < 1, \end{cases}$$

which is called the positive logarithmic function.

Definition 2.2 *Let $f(z)$ be a non-constant meromorphic function in the disc $|z| \leq R$ $(0 < R < \infty)$. For $0 < r < R$, define*

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

which is the average of the positive logarithm of $|f(z)|$ on the circle $|z| = r$.

Definition 2.3 *Let $f(z)$ be a non-constant meromorphic function in the disc $|z| \leq R$ $(0 < R < \infty)$. For $0 < r < R$, define*

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

where $n(t, f)$ denotes the number of poles of $f(z)$ in the disc $|z| \leq t$, multiple poles are counted according to their multiplicities. $n(0, f)$ denotes the multiplicity of poles of $f(z)$ at the origin. $N(r, f)$ is called the counting function of poles of $f(z)$.

Definition 2.4 *Let $f(z)$ be a non-constant meromorphic function in the disc $|z| \leq R$ $(0 < R < \infty)$. For $0 < r < R$, define*

$$T(r, f) = m(r, f) + N(r, f),$$

which is called the characteristic function of $f(z)$.

Definition 2.5 *Let $f(z)$ be a non-constant meromorphic function in the disc $|z| \leq R$ $(0 < R < \infty)$. For $0 < r < R$, define*

$$\overline{N}(r, f) = \int_0^r \frac{\overline{n}(t, f) - \overline{n}(0, f)}{t} dt + \overline{n}(0, f) \log r,$$

where $\overline{n}(t, f)$ denotes the number of distinct poles of $f(z)$ in the disc $|z| \leq t$,and any of them be counted only once. $\overline{N}(r, f)$ is called the reduced counting function of poles of $f(z)$.

Definition 2.6 *Let $f(z)$ be a non-constant meromorphic function in the complex plane. The order λ of $f(z)$ is defined by*

$$\lambda(f) \equiv \lambda = \varlimsup_{r \to \infty} \frac{\log^+ T(r, f)}{\log r}.$$

Definition 2.7 *Let $f(z)$ be a non-constant meromorphic function in the complex plane and a be any complex number. The deficiency of a with respect to $f(z)$ is defined by*

$$\delta(a, f) = 1 - \varlimsup_{r \to \infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r, f)}.$$

If $\delta(a, f) > 0$, then the complex number a is called a deficient value of $f(z)$. The deficient value is also called exceptional value in the sense of Nevanlinna.

Theorem 2.8 (The first fundamental theorem) *Suppose that $f(z)$ is meromorphic in $|z| < R$ $(\leq \infty)$ and a is any complex number. Then for $0 < r < R$ we have*

$$T\left(r, \frac{1}{f - a}\right) = T(r, f) + \log |c_\lambda| + \varepsilon(a, r), \tag{2.1}$$

where c_λ is the first non-zero cofficient of the Laurent expansion of $\frac{1}{f-a}$ at the origin, and

$$|\varepsilon(a, r)| \leq \log^+ |a| + \log 2.$$

We can simply write formula (2.1) as

$$T\left(r, \frac{1}{f - a}\right) = T(r, f) + O(1),$$

which means that, for any complex number a, the difference of $T\left(r, \frac{1}{f-a}\right)$ and $T(r,f)$ is a bounded quantity.

Theorem 2.9 (The second fundamental theorem) *Suppose that $f(z)$ is a non-constant meromorphic function in $|z| < R$ and a_j $(j = 1, 2, ..., q)$ are q (≥ 2) distinct finite complex numbers. Then for $0 < r < R$, we have*

$$m(r,f) + \sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) \leq 2T(r,f) - N_1(r) + S(r,f), \tag{2.2}$$

where

$$N_1(r) = 2N(r,f) - N(r,f') + N\left(r, \frac{1}{f'}\right),$$

and

$$S(r,f) = m\left(r, \frac{f'}{f}\right) + m\left(r, \sum_{j=1}^{q} \frac{f'}{f - a_j}\right) + O(1).$$

If the order of $f(z)$is finite, then

$$S(r,f) = O(\log r) = o(T(r,f)), \quad (r \to \infty).$$

If the order of $f(z)$is infinite, then

$$S(r,f) = O(\log(rT(r,f))) = o(T(r,f)), \quad (r \to \infty, r \notin E),$$

where $E \subseteq (0, \infty)$ is a set of finite linear measure.
When $q \geq 3$, (2.2) can be restated as

$$(q-2)T(r,f) < \sum_{j=1}^{q} N\left(r, \frac{1}{f - a_j}\right) - N_1(r) + S(r,f)$$

or

$$(q-2)T(r,f) < \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f - a_j}\right) + S(r,f).$$

Theorem 2.10 *Let $f_j(z)$ $(j = 1, 2, 3)$ be meromorphic functions and $f_1(z)$ be non-constant. If*

$$\sum_{j=1}^{3} f_j(z) \equiv 1,$$

and
$$\sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) + 2\sum_{j=1}^{3} \overline{N}(r, f_j) < (\lambda + o(1))T(r) \quad (r \in I),$$
where $\lambda < 1$, $T(r) = max_{1 \le j \le 3}\{T(r, f_j)\}$, *and* $I \subseteq (0, \infty)$ *is a set of infinite linear measure, then* $f_2(z) \equiv 1$ *or* $f_3(z) \equiv 1$.

Theorem 2.11 *Let* $h(z)$ *be a non-constant entire function and* $f(z) = e^{h(z)}$. *Then*

(i) $T(r, h) = o(T(r, f)) \quad (r \to \infty)$,

(ii) $T(r, h') = S(r, f)$.

Theorem 2.12 *Suppose that* $f_1(z), f_2(z), ..., f_n(z)$ $(n \ge 2)$ *are meromorphic functions and* $g_1(z), g_2(z), ..., g_n(z)$ *are entire functions satisfying the following conditions.*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$.

(ii) $g_j(z) - g_k(z)$ *are not constants for* $1 \le j < k \le n$.

(iii) *For* $1 \le j \le n$, $1 \le h < k \le n$, $T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \quad (r \to \infty, r \notin E)$.

Then $f_j(z) \equiv 0$ $(j = 1, 2, ..., n)$.

Theorem 2.13 *Let* $g_j(z)$ $(j = 1, 2, ..., n)$ *be entire functions and* $a_j(z)$ $(j = 0, 1, ..., n)$ *be meromorphic functions satisfying*
$$T(r, a_j) = o\left(\sum_{k=1}^{n} T(r, e^{g_k})\right) \quad (r \to \infty, r \notin E) \quad (j = 0, 1, ..., n).$$
If
$$\sum_{j=1}^{n} a_j(z)e^{g_j(z)} \equiv a_0(z),$$
then there exist constants c_j $(j = 1, 2, ..., n)$, *at least one of them is not zero, such that*
$$\sum_{j=1}^{n} c_j a_j(z)e^{g_j(z)} \equiv 0.$$

Theorem 2.14 *[9] Let $g_j(z)$ $(j = 1, 2, ..., p)$ be transcendental entire functions and a_j $(j = 0, 1, ..., p)$ be non-zero constants. If $\sum_{j=1}^{p} a_j g_j(z) = 1$, then $\sum_{j=1}^{p} \delta(0, g_j) \leq p - 1$.*

Theorem 2.15 *[7] Suppose that $f(z)$ is a non-constant meromorphic function satisfying the differential equation*

$$(f')^n = a_0(z) + a_1(z)f + ...a_{2n}(z)f^{2n} = P(z, f),$$

where n is a positive integer and $a_0, a_1, ..., a_{2n}$ $(\neq 0)$ are meromorphic functions satifying

$$\sum_{j=0}^{2n} T(r, a_j) = S(r, f).$$

Then $m(r, f) = S(r, f)$ and

$$m\left(r, \frac{1}{f - \alpha}\right) = S(r, f),$$

where α is a finite value, and $P(z, \alpha) \neq 0$.

Theorem 2.16 *Let f and g be non-constant meromorphic functions. If f and g share distinct values a_1, a_2 and a_3 IM, then*

$$T(r, f) < 3T(r, g) + S(r, f).$$

Lemma 2.17 *Let f be a non-constant meromorphic function, $\psi(z) = f^{(k)}(z)$, where k is a positive integer. Then*

$$k\overline{N}_{1)}(r, f) \leq \overline{N}_{(2}(r, f) + \overline{N}\left(r, \frac{1}{\psi - 1}\right) + N_0\left(r, \frac{1}{\psi'}\right) + S(r, f),$$

where $N_0\left(r, \frac{1}{\psi'}\right)$ denotes the counting function of zeros of ψ' which are not the multiple zeros of $\psi - 1$. $\overline{N}_{1)}(r, f)$ denotes the counting function of single poles of $f(z)$. $\overline{N}_{(2}(r, f)$ denotes the counting function of multiple poles of $f(z)$, and any of them be counted only once.

Theorem 2.18 *[3] Suppose $f(z)$ is a transcendental meromorphic function in the complex plane and k is a positive integer. Then*

$$(k-1)\overline{N}(r,f) \leq (1+\varepsilon)N\left(r, \frac{1}{f^{(k)}}\right) + (1+\varepsilon)[N(r,f) - \overline{N}(r,f)] + S(r,f),$$

where ε is any fixed positive number.

Theorem 2.19 *Suppose $f(z)$ is a non-constant meromorphic function in the complex plane and k is a positive integer. Then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k)}) - T(r,f) + N\left(r, \frac{1}{f}\right) + S(r,f),$$

and

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f).$$