3 Entire Functions Sharing Values with Their Derivatives

In this section, we study the problem on entire functions sharing two values (CM and IM) with their first derivatives.

Theorem 3.1 *[10] Let f be a non-constant entire function. If f and f' share values a and b CM, then $f \equiv f'$.*

Proof. When only one of a and b is zero, we suppose that $a = 0$ and $b \neq 0$ without loss of generality, then 0 must be the Picard exceptional value of f and f'. Set

$$f(z) = e^{\alpha(z)}, \quad f'(z) = e^{\beta(z)}, \tag{3.1}$$

where $\alpha(z)$ and $\beta(z)$ are non-constant entire functions. Then we have

$$e^{\beta(z)} = \alpha'(z)e^{\alpha(z)}. \tag{3.2}$$

Notice that b is a CM value shared by f and f', we have

$$\frac{f-b}{f'-b} = e^{\gamma}, \tag{3.3}$$

where $\gamma(z)$ is an entire function. Combining (3.1) with (3.3) we can get

$$\frac{1}{b}e^{\alpha} + e^{\gamma} - \frac{1}{b}e^{\beta+\gamma} = 1. \tag{3.4}$$

Using Theorem 2.10 to this, we know $e^{\gamma} \equiv 1$ or $\frac{-1}{b}e^{\beta+\gamma} \equiv 1$. If $e^{\gamma} \equiv 1$, (3.3) implies $f \equiv f'$. If $\frac{-1}{b}e^{\beta+\gamma} \equiv 1$, then $e^{\beta} = -be^{-\gamma}$ and (3.4) leads to $e^{\beta} = b^2e^{-\alpha}$. So from (3.2), we can get $e^{2\alpha} = \frac{b^2}{\alpha'}$, then $ff' = b^2$, which is impossible.

Now we assume that $a \neq 0$ and $b \neq 0$, then we have

$$\frac{f'-a}{f-a} = e^{\alpha(z)}, \quad \frac{f'-b}{f-b} = e^{\beta(z)}, \tag{3.5}$$

where $\alpha(z)$ and $\beta(z)$ are entire functions. Suppose that $f \neq f'$. We can solve from (3.5)

$$f = \frac{be^{\beta} - ae^{\alpha} + a - b}{e^{\beta} - e^{\alpha}}, \qquad f' = \frac{be^{\alpha} - ae^{\beta} + (a-b)e^{\beta+\alpha}}{e^{\alpha} - e^{\beta}}.$$

Hence,

$$ae^{2\beta} + be^{2\alpha} + (a-b)e^{2\alpha+\beta} - (a-b)e^{\alpha+2\beta} + \{(b-a)(\beta' - \alpha') - (a+b)\}e^{\alpha+\beta}$$
$$+(a-b)\beta' e^{\beta} - (a-b)\alpha' e^{\alpha} = 0. \tag{3.6}$$

Clearly,

$$T(r,f) < 2T(r,e^{\alpha}) + 2T(r,e^{\beta}) + O(1). \tag{3.7}$$

Assume that $e^{\alpha} \equiv c$, where c $(\neq 0,1)$ is a constant. Then from (3.7) we know that e^{β} is not a constant, and (3.6) implies

$$Ae^{2\beta} + Be^{\beta} + bc^2 = 0, \tag{3.8}$$

where

$$A = a - (a-b)c, \qquad B = (a-b)c^2 + \{(b-a)\beta' - (a+b)\}c + (a-b)\beta'.$$

By Theorem 2.11, we have

$$T(r,B) = S(r,e^{\beta})$$

and by Theorem 2.12, (3.8) can not hold, so e^{α} is not a constant.
Assume that $e^{\beta} \equiv c$, where c $(\neq 0,1)$ is a constant. Then (3.6) implies

$$Ae^{2\alpha} + Be^{\alpha} + ac^2 = 0, \tag{3.9}$$

where

$$A = b + (a-b)c, \qquad B = (b-a)c^2 + \{(a-b)\alpha' - (a+b)\}c - (a-b)\alpha'.$$

By Theorem 2.11, we have

$$T(r,B) = S(r,e^{\alpha})$$

and by Theorem 2.12, (3.9) can not hold, so e^{β} is not a constant.

Assume that $e^{\beta-\alpha} \equiv c$, where c ($\neq 0, 1$) is a constant. Then (3.6) implies

$$Ae^{3\alpha} + Be^{2\alpha} + Ce^{\alpha} = 0, \tag{3.10}$$

where

$$A = (a-b)(1-c)c, \quad B = ac^2 + b - (a+b)c, \quad C = (a-b)(c-1)\alpha'.$$

By Theorem 2.11, we have

$$T(r, C) = S(r, e^{\alpha})$$

and by Theorem 2.12, (3.10) can not hold, so $e^{\beta-\alpha}$ is not a constant.

Assume that $e^{\beta-2\alpha} \equiv c$, where c ($\neq 0$) is a constant. Then (3.6) implies

$$Ae^{5\alpha} + Be^{4\alpha} + Ce^{3\alpha} + De^{2\alpha} + Ee^{\alpha} = 0, \tag{3.11}$$

where

$$A = -(a-b)c^2, \quad B = ac^2 + (a-b)c, \quad C = \{(b-a)\alpha' - (a+b)\}c,$$

$$D = b + 2(a-b)c\alpha', \quad E = -(a-b)\alpha'.$$

By Theorem 2.11, we have

$$T(r, C) = S(r, e^{\alpha}), \quad T(r, D) = S(r, e^{\alpha}), \quad T(r, E) = S(r, e^{\alpha})$$

and by Theorem 2.12, (3.11) can not hold, so $e^{\beta-2\alpha}$ is not a constant.

Assume that $e^{2\beta-\alpha} \equiv c$, where c ($\neq 0$) is a constant. Then (3.6) implies

$$Ae^{5\beta} + Be^{4\beta} + Ce^{3\beta} + De^{2\beta} + Ee^{\beta} = 0, \tag{3.12}$$

where

$$A = \frac{a-b}{c^2}, \quad B = \frac{b-(a-b)c}{c^2}, \quad C = \frac{(a-b)\beta' - (a+b)}{c},$$

$$D = \frac{ac - 2(a-b)\beta'}{c}, \quad E = (a-b)\beta'.$$

By Theorem 2.11, we have

$$T(r, C) = S(r, e^{\beta}), \quad T(r, D) = S(r, e^{\beta}), \quad T(r, E) = S(r, e^{\beta})$$

and by Theorem 2.12, (3.12) can not hold, so $e^{2\beta-\alpha}$ is not a constant.

Again from (3.6) we have

$$ae^{\beta}+be^{2\alpha-\beta}+(a-b)e^{2\alpha}-(a-b)e^{\alpha+\beta}+\{(b-a)(\beta'-\alpha')-(a+b)\}e^{\alpha}$$
$$-(a-b)\alpha'e^{\alpha-\beta}=-(a-b)\beta'. \qquad (3.13)$$

Applying Theorem 2.13 to (3.13), there exist not all zero constants c_1,c_2,...,c_6 such that

$$c_1e^{\beta}+c_2e^{2\alpha-\beta}+c_3e^{2\alpha}+c_4e^{\alpha+\beta}+c_5\{(b-a)(\beta'-\alpha')-(a+b)\}e^{\alpha}+c_6\alpha'e^{\alpha-\beta}=0.$$

It leads to

$$c_1e^{\beta-\alpha}+c_2e^{\alpha-\beta}+c_3e^{\alpha}+c_4e^{\beta}+c_6\alpha'e^{-\beta}=-c_5\{(b-a)(\beta'-\alpha')-(a+b)\}.$$

Using Theorem 2.13 again to this, we have not all zero constants d_1,d_2,...,d_5 such that

$$d_1e^{\beta-\alpha}+d_2e^{\alpha-\beta}+d_3e^{\alpha}+d_4e^{\beta}+d_5\alpha'e^{-\beta}=0.$$

Therefore

$$d_1e^{2\beta-\alpha}+d_2e^{\alpha}+d_3e^{\alpha+\beta}+d_4e^{2\beta}=-d_5\alpha'.$$

Using Theorem 2.13 again to this, we have not all zero constants t_1,t_2,...,t_4 such that

$$t_1e^{2\beta-\alpha}+t_2e^{\alpha}+t_3e^{\alpha+\beta}+t_4e^{2\beta}=0.$$

Suppose that $t_4\neq 0$ without loss of generality, we derive from above equality

$$-\frac{t_1}{t_4}e^{-\alpha}-\frac{t_2}{t_4}e^{\alpha-2\beta}-\frac{t_3}{t_4}e^{\alpha-\beta}=1.$$

Note that $e^{-\alpha}$, $e^{\alpha-2\beta}$, $e^{\alpha-\beta}$ are all not constants, and $\delta(0,e^{-\alpha})=1$, $\delta(0,e^{\alpha-2\beta})=1$, $\delta(0,e^{\alpha-\beta})=1$. This contradicts Theorem 2.14, so $f(z)\equiv f'(z)$. ❑

Theorem 3.2 *[8] Let $f(z)$ be a non-constant entire function, a and b be distinct finite values. If f and f' share the value a and b IM, then $f\equiv f'$.*

Proof. We distinguish the following two cases.

Case 1. Assume that $ab \neq 0$.

Since a and b are shared by f and f' IM, the zeros of $f-a$ and $f-b$ must be simple zeros. Suppose that $f(z) \not\equiv f'$. Then

$$\begin{aligned} N\left(r, \frac{1}{f-f'}\right) &\leq T(r, f-f') + O(1) \\ &= m\left(r, f\left(1-\frac{f'}{f}\right)\right) + O(1) \\ &\leq m(r,f) + S(r,f) \\ &= T(r,f) + S(r,f). \end{aligned}$$

Therefore

$$N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-b}\right) \leq N\left(r, \frac{1}{f-f'}\right) \leq T(r,f) + S(r,f). \tag{3.14}$$

Note

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) + m\left(r, \frac{1}{f-b}\right) &= m\left(r, \frac{1}{f-a} + \frac{1}{f-b}\right) + O(1) \\ &\leq m\left(r, \frac{f'}{f-a} + \frac{f'}{f-b}\right) + m\left(r, \frac{1}{f'}\right) + O(1) \\ &= m\left(r, \frac{f'}{f-a}\right) + m\left(r, \frac{f'}{f-b}\right) + m\left(r, \frac{1}{f'}\right) + O(1) \\ &= m\left(r, \frac{1}{f'}\right) + S(r,f). \end{aligned}$$

This and (3.14) we have

$$T(r,f) \leq m\left(r, \frac{1}{f'}\right) + S(r,f). \tag{3.15}$$

Combining

$$\overline{N}\left(r, \frac{1}{f'-a}\right) + \overline{N}\left(r, \frac{1}{f'-b}\right) \leq N\left(r, \frac{1}{f-f'}\right) \leq T(r,f) + S(r,f)$$

with

$$N\left(r, \frac{1}{f'-a}\right) - \overline{N}\left(r, \frac{1}{f'-a}\right) + N\left(r, \frac{1}{f'-b}\right) - \overline{N}\left(r, \frac{1}{f'-b}\right) \leq N\left(r, \frac{1}{f''}\right),$$

we get

$$N\left(r,\frac{1}{f'-a}\right)+N\left(r,\frac{1}{f'-b}\right)\le T(r,f)+N\left(r,\frac{1}{f''}\right)+S(r,f). \tag{3.16}$$

Note

$$\begin{aligned}
&m\left(r,\frac{1}{f'}\right)+m\left(r,\frac{1}{f'-a}\right)+m\left(r,\frac{1}{f'-b}\right)\\
&=m\left(r,\frac{1}{f'}+\frac{1}{f'-a}+\frac{1}{f'-b}\right)+O(1)\\
&\le m\left(r,\frac{f''}{f'}+\frac{f''}{f'-a}+\frac{f''}{f'-b}\right)+m\left(r,\frac{1}{f''}\right)+O(1)\\
&=m\left(r,\frac{f''}{f'}\right)+m\left(r,\frac{f''}{f'-a}\right)+m\left(r,\frac{f''}{f'-b}\right)+m\left(r,\frac{1}{f''}\right)+O(1)\\
&=m\left(r,\frac{1}{f''}\right)+S(r,f).
\end{aligned}$$

This and (3.16) we get

$$\begin{aligned}
m\left(r,\frac{1}{f'}\right)+2T(r,f')&\le T(r,f)+T(r,f'')+S(r,f)\\
&\le T(r,f)+T(r,f')+S(r,f).
\end{aligned}$$

This and (3.15) we obtain

$$T(r,f')=S(r,f).$$

In addition, (3.15) implies

$$T(r,f)\le T(r,f')+S(r,f)=S(r,f).$$

It is a contradiction, thus $f\equiv f'$

Case 2. Assume that $ab=0$.

We suppose that $a=0$ and $b=1$ without loss of generality. Since 0 and 1 are shared values of f and f', we know that the multiplicity of zeros of f is great than one and the zeros of $f-1$ are the simple zeros. Suppose that $f\not\equiv f'$. Set

$$g=\frac{f'(f'-f)}{f(f-1)}. \tag{3.17}$$

Then g is an entire function and

$$T(r,g) = m\left(r, \frac{f'}{f-1}\left(\frac{f'}{f}-1\right)\right) \leq m\left(r, \frac{f'}{f-1}\right) + m\left(r, \frac{f'}{f}\right) + O(1) = S(r,f). \tag{3.18}$$

Moreover, (3.17) leads to

$$(f')^2 - ff' = g(f^2 - f).$$

Taking derivative twice, we derive

$$2f'f'' - (f')^2 - ff'' = g'(f^2 - f) + g(2ff' - f') \tag{3.19}$$

and

$$\begin{aligned} &2(f'')^2 + 2f'f''' - 3f'f'' - ff''' \\ = &\; g''(f^2 - f) + 2g'(2ff' - f') + g[2(f')^2 + 2ff'' - f'']. \end{aligned} \tag{3.20}$$

Let z_1 be a zero of $f-1$. Then $f(z_1) = f'(z_1) = 1$. So from (3.19) and (3.20) we have

$$f''(z_1) = 1 + g(z_1),$$

$$f'''(z_1) = 2g'(z_1) - g^2(z_1) + 2g(z_1) + 1.$$

Set

$$\phi = \frac{f'' - (g+1)f'}{f-1}, \tag{3.21}$$

$$\psi = \frac{f''' - (2g' - g^2 + 2g + 1)f'}{f-1}. \tag{3.22}$$

Since $f-1$ has only simple zeros, we know that ϕ and ψ are entire functions. Hence,

$$\begin{aligned} T(r,\phi) &= m(r,\phi) \\ &\leq m\left(r, \frac{f''}{f-1}\right) + m\left(r, \frac{f'}{f-1}\right) + m(r,g) + O(1) \\ &= S(r,f) \end{aligned}$$

and

$$\begin{aligned} T(r,\psi) &= m(r,\psi) \\ &\leq m\left(r, \frac{f'''}{f-1}\right) + m\left(r, \frac{f'}{f-1}\right) + m(r,g') + m(r,g^2+2g) + O(1) \\ &= S(r,f). \end{aligned}$$

Combining (3.21) with (3.22), we get

$$f'(2g^2 - g' + \phi) = (f-1)(\psi - \phi' - (1+g)\phi). \tag{3.23}$$

If $2g^2 - g' + \phi \neq 0$, (3.23) implies

$$N\left(r, \frac{1}{f-1}\right) \leq N\left(r, \frac{1}{2g^2 - g' + \phi}\right) = S(r,f)$$

and

$$\overline{N}\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{\psi - \phi' - (1+g)\phi}\right) = S(r,f).$$

By the second fundamental theorem for 1, 0, ∞, we obtain

$$T(r,f) < \overline{N}\left(r, \frac{1}{f-1}\right) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r,f) + S(r,f) = S(r,f).$$

This is a contradiction. Therefore

$$2g^2 - g' + \phi = 0. \tag{3.24}$$

Let z_0 be a zero of $f(z)$, then from (3.20) and (3.21) we know

$$2f''(z_0) = -g(z_0), \quad f''(z_0) = -\phi(z_0),$$

hence $\phi(z_0) = \frac{1}{2}g(z_0)$. In addition, (3.24) leads to

$$2g^2(z_0) + \frac{1}{2}g(z_0) - g'(z_0) = 0. \tag{3.25}$$

If $2g^2 + \frac{1}{2}g - g' \neq 0$, from (3.18) and (3.25) we have

$$\overline{N}\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{2g^2 + \frac{1}{2}g - g'}\right) = S(r,f).$$

Note that

$$\begin{aligned}
N\left(r, \frac{1}{f-1}\right) &\leq N\left(r, \frac{f}{f'-f}\right) \\
&= N\left(r, \frac{1}{\frac{f'}{f}-1}\right) \\
&\leq T\left(r, \frac{f'}{f}\right) + O(1) \\
&= N\left(r, \frac{f'}{f}\right) + m\left(r, \frac{f'}{f}\right) + O(1) \\
&= \overline{N}\left(r, \frac{1}{f}\right) + S(r,f) \\
&= S(r,f).
\end{aligned}$$

By the second fundamental theorem for 1, 0, ∞, we obtain

$$T(r,f) < \overline{N}\left(r, \frac{1}{f-1}\right) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r,f) + S(r,f) = S(r,f).$$

This is a contradiction. Thus

$$g' = \frac{1}{2}g + 2g^2. \tag{3.26}$$

Suppose that g is a non-constant entire function, then by Theorem 2.15 we know

$$T(r,g) = S(r,g),$$

which is a contradiction. So g must be a constant. From (3.26), we know that $g \equiv 0$ or $g \equiv -\frac{1}{4}$. Since $f \neq f'$, hence $g \equiv -\frac{1}{4}$, and from (3.17) we obtain

$$(2f' - f)^2 = f.$$

Set $h = 2f' - f$. Then $f = h^2$ and $f' = 2hh'$. Hence,

$$h' = \frac{1}{4}(1 + h). \tag{3.27}$$

Solving (3.27), we get

$$h(z) = Ae^{\frac{1}{4}z} - 1,$$

where A ($\neq 0$) is a constant. Let $z^* = 4\pi i - 4\log A$. Then $h(z^*) = -2$, and from (3.27) we get $h'(z^*) = -\frac{1}{4}$. Hence

$$f(z^*) = h^2(z^*) = 4$$

and

$$f'(z^*) = 2h(z^*)h'(z^*) = 1.$$

This contradicts the fact that 1 is a shared value of f and f' IM.

Thus $f \equiv f'$. ❑

For the shared value problem of an entire function f with its higher-order derivatives, the following results are well-known.

Theorem 3.3 *[12] Let $f(z)$ be a non-constant entire function, k (≥ 2) be an integer and a $(\neq 0)$ be a finite value. Suppose that 0 is the Picard exceptional value of f and $f^{(k)}$, and that a is a IM share value of f and $f^{(k)}$. Then $f(z) = e^{Az+B}$, where A and B are constants satisfying $A^k = 1$, that is $f \equiv f^{(k)}$.*

Theorem 3.4 *[12] Let f be a non-constant entire function, k be a positive integer, a and b be two distinct finite values. If f and $f^{(k)}$ share values a and b CM, then $f \equiv f^{(k)}$.*