4 Meromorphic Functions Sharing Values with Their Derivatives

In this section, we study the problem on meromorphic functions sharing two values CM with their derivatives. We distinguish three theorems and give a corollary to show that $f \equiv f^{(k)}$ when f and $f^{(k)}$ share distinct finite values a and b CM.

Theorem 4.1 *[5] Let f be a non-constant meromorphic function, $b\ (\neq 0)$ be a finite value. If f and f' share the values 0 and b CM, then $f \equiv f'$.*

Proof. Suppose that $f \not\equiv f'$. Since f and f' share 0 CM, we know that 0 must be the Picard exceptional value of f and f'. For f and f' share ∞ IM. By Theorem 2.16 we have

$$T(r,f) = O(T(r,f')), \quad (r \notin E).$$

Using the second fundamental theorem for 0, b and ∞, we get

$$\begin{aligned}
T(r,f') &\le N\left(r,\frac{1}{f'}\right) + N\left(r,\frac{1}{f'-b}\right) + \overline{N}(r,f') + S(r,f')\\
&\le N\left(r,\frac{1}{\frac{f'}{f}-1}\right) + \overline{N}(r,f') + S(r,f')\\
&\le T\left(r,\frac{f'}{f}\right) + \overline{N}(r,f') + S(r,f')\\
&= m\left(r,\frac{f'}{f}\right) + N\left(r,\frac{f'}{f}\right) + \overline{N}(r,f') + S(r,f')\\
&= 2\overline{N}(r,f') + S(r,f')\\
&\le N(r,f') + S(r,f')\\
&\le T(r,f') + S(r,f').
\end{aligned}$$

Hence

$$2\overline{N}(r,f') = T(r,f') + S(r,f'), \tag{4.1}$$

and

$$2N\left(r,\frac{1}{f'-b}\right) = T(r,f') + S(r,f'). \tag{4.2}$$

Again using the second fundamental theorem for 0, b and ∞, we get

$$T(r,f') \leq N\left(r,\frac{1}{f'}\right) + N\left(r,\frac{1}{f'-b}\right) + \overline{N}(r,f') - N\left(r,\frac{1}{f''}\right) + S(r,f'),$$

with (4.1) and (4.2) we know

$$N\left(r,\frac{1}{f''}\right) = S(r,f').$$

By Lemma 2.17 with $\psi = (f+z)'$, we have

$$\begin{aligned}
\overline{N}_{1)}(r,f) &= \overline{N}_{1)}(r,f+z) \\
&\leq \overline{N}_{(2}(r,f+z) + \overline{N}\left(r,\frac{1}{f'}\right) + N_0\left(r,\frac{1}{f''}\right) + S(r,f+z) \\
&\leq \overline{N}_{(2}(r,f) + S(r,f'). \qquad (4.3)
\end{aligned}$$

In addition, from (4.1) and (4.3) we get

$$\overline{N}_{(2}(r,f) = S(r,f'), \qquad (4.4)$$

$$\overline{N}_{1)}(r,f) = S(r,f'). \qquad (4.5)$$

(4.4) and (4.5) lead to

$$\overline{N}(r,f') = \overline{N}(r,f) = S(r,f').$$

With (4.1) we obtain

$$T(r,f') = S(r,f'),$$

which is a contradiction, so $f \equiv f'$. ❑

Theorem 4.2 *[2] Let f be a non-constant meromorphic function, b $(\neq 0)$ be a finite value, k (≥ 2) be an integer. If f and $f^{(k)}$ share the values 0 and b CM, then $f \equiv f^{(k)}$.*

Proof. We assume that $b = 1$ without loss generality. If not, it follows by considering the function $\frac{f}{b}$. Suppose that $f \not\equiv f^{(k)}$. Under the hypothesis of Theorem 4.2, we know

$$\frac{f(f^{(k)}-1)}{(f-1)f^{(k)}} = e^{\alpha}, \qquad (4.6)$$

where α is an entire function. From (4.6) we have

$$T(r, e^{\alpha}) = O(T(r, f)), \quad (r \notin E),$$

and

$$\alpha' = \frac{(e^{\alpha})'}{e^{\alpha}} = \frac{-f'}{f(f-1)} + \frac{f^{(k+1)}}{f^{(k)}(f^{(k)}-1)}. \tag{4.7}$$

By Theorem 2.11, we know

$$T(r, \alpha') = S(r, e^{\alpha}) = S(r, f).$$

If f is an entire function, then from Theorem 3.1 and Theorem 3.4 , $f \equiv f^{(k)}$, which is a contradiction. So f is not an entire function. Let z_0 be a pole of order p of f. If $e^{\alpha} \equiv c \ (\neq 0)$ is a constant. Taking $z = z_0$ in (4.6), we have $c = 1$, so from (4.6), $f \equiv f^{(k)}$. This is a contradiction. Hence e^{α} is not a constant and $\alpha' \neq 0$. From (4.7) we know that z_0 is a zero of order at least $p-1$ of α', thus

$$N(r, f) - \overline{N}(r, f) \leq N\left(r, \frac{1}{\alpha'}\right) \leq T(r, \alpha') + O(1) = S(r, f).$$

That is

$$N(r, f) = \overline{N}(r, f) + S(r, f). \tag{4.8}$$

Set

$$g = \frac{f}{f^{(k)}}, \tag{4.9}$$

then g is an entire function. From (4.6) and (4.9), we have

$$f^{(k)} = \frac{1}{g} - \frac{1}{e^{\alpha}-1}\left(1 - \frac{1}{g}\right). \tag{4.10}$$

By second fundamental theorem for 0, 1 and ∞, we get

$$\begin{aligned} T(r, e^{\alpha}) &\leq N\left(r, \frac{1}{e^{\alpha}}\right) + N\left(r, \frac{1}{e^{\alpha}-1}\right) + N(r, e^{\alpha}) + S(r, e^{\alpha}) \\ &\leq N\left(r, \frac{1}{e^{\alpha}-1}\right) + S(r, f) \\ &\leq T\left(r, \frac{1}{e^{\alpha}-1}\right) + S(r, f) \\ &\leq T(r, e^{\alpha}) + S(r, f). \end{aligned}$$

So

$$N\left(r,\frac{1}{e^{\alpha}-1}\right)=T(r,e^{\alpha})+S(r,f)=T\left(r,\frac{1}{e^{\alpha}-1}\right)+S(r,f),$$

then

$$m\left(r,\frac{1}{e^{\alpha}-1}\right)=S(r,f).$$

(4.9) implies

$$m\left(r,\frac{1}{g}\right)=S(r,f),$$

so from (4.10) we obtain

$$m(r,f^{(k)})\leq 2m\left(r,\frac{1}{g}\right)+m\left(r,\frac{1}{e^{\alpha}-1}\right)+O(1)=S(r,f).$$

Hence

$$(k+1)\overline{N}(r,f)\leq T(r,f^{(k)})=m(r,f^{(k)})+N(r,f^{(k)})\leq (k+1)\overline{N}(r,f)+S(r,f). \tag{4.11}$$

Set

$$\begin{aligned}N^{*}(r)&=N\left(r,\frac{1}{f^{(k)}-f}\right)-N\left(r,\frac{1}{f}\right)-N\left(r,\frac{1}{f-1}\right)\\&=N\left(r,\frac{1}{f^{(k)}-f}\right)-N\left(r,\frac{1}{f^{(k)}}\right)-N\left(r,\frac{1}{f^{(k)}-1}\right),\end{aligned} \tag{4.12}$$

and

$$F=\frac{[f(f-1)]^{k+1}[f^{(k)}(f^{(k)}-1)]^{k-1}}{(f^{(k)}-f)^{2k}},$$

$$G=\frac{F'}{F}+(k+1)\frac{\alpha''}{\alpha'}.$$

If z^{*} is a pole of order p of f, when $p=1$, z^{*} is not pole or zero of F, and when $p>1$, z^{*} is a pole of F. So F has no zeros and

$$\begin{aligned}\overline{N}(r,F)&\leq N\left(r,\frac{1}{f^{(k)}-f}\right)-N\left(r,\frac{1}{f}\right)-N\left(r,\frac{1}{f-1}\right)+N(r,f)-\overline{N}(r,f)\\&=N^{*}(r)+S(r,f).\end{aligned}$$

Let z_0 be a simple pole of f. Then $F(z_0)\neq 0,\infty$. And near z_0 we can write

$$f(z)=\frac{R}{z-z_0}+a_0+O(z-z_0),\quad (R\neq 0).$$

Then

$$F = \frac{R^{2k}}{(k!)^2} + \frac{(k+1)R^{2k-1}(2a_0-1)}{(k!)^2}(z-z_0) + O(z-z_0)^2,$$

$$F' = \frac{(k+1)R^{2k-1}(2a_0-1)}{(k!)^2} + O(z-z_0),$$

$$\alpha' = \frac{1}{R} - \frac{2a_0-1}{R^2}(z-z_0) + O(z-z_0)^2,$$

and

$$\alpha'' = -\frac{2a_0-1}{R^2} + O(z-z_0).$$

So

$$G(z_0) = \frac{F'(z_0)}{F(z_0)} + (k+1)\frac{\alpha''(z_0)}{\alpha'(z_0)} = 0.$$

Hence the simple pole of f is the zeros of G. And G is a meromorphic function satisfying

$$\begin{aligned} T(r,G) &\le T\left(r, \frac{F'}{F}\right) + T\left(r, \frac{\alpha''}{\alpha'}\right) \\ &= m\left(r, \frac{F'}{F}\right) + N\left(r, \frac{F'}{F}\right) + S(r,f) \\ &= \overline{N}(r,F) + S(r,f). \end{aligned}$$

We distinguish the following two case.

Case 1. Suppose that $G \not\equiv 0$.Then

$$\begin{aligned} \overline{N}(r,f) &\le N\left(r, \frac{1}{G}\right) \\ &\le T(r,G) + O(1) \\ &\le \overline{N}(r,F) + S(r,f) \\ &\le N^*(r) + S(r,f). \end{aligned} \tag{4.13}$$

From (4.9), we have

$$g - 1 = \frac{f - f^{(k)}}{f^{(k)}}.$$

Hence

$$N\left(r, \frac{1}{g-1}\right) = N\left(r, \frac{1}{f^{(k)} - f}\right) - N\left(r, \frac{1}{f^{(k)}}\right),$$

and from (4.12) we get

$$\begin{aligned} N^*(r) + N\left(r, \frac{1}{f^{(k)}-1}\right) &= N\left(r, \frac{1}{g-1}\right) \\ &\leq T\left(r, \frac{1}{g}\right) + O(1) \\ &\leq N\left(r, \frac{f^{(k)}}{f}\right) + m\left(r, \frac{f^{(k)}}{f}\right) + O(1) \\ &= k\overline{N}(r,f) + S(r,f). \end{aligned}$$

Combining this with (4.13), we have

$$N\left(r, \frac{1}{f^{(k)}-1}\right) \leq (k-1)\overline{N}(r,f) + S(r,f). \tag{4.14}$$

From (4.14) and (4.11) we have

$$\begin{aligned} m\left(r, \frac{1}{f^{(k)}-1}\right) &= T(r, f^{(k)}) - N\left(r, \frac{1}{f^{(k)}-1}\right) + O(1) \\ &\geq (k+1)\overline{N}(r,f) - (k-1)\overline{N}(r,f) + S(r,f) \\ &= 2\overline{N}(r,f) + S(r,f). \end{aligned} \tag{4.15}$$

Note that

$$\begin{aligned} m\left(r, \frac{1}{f^{(k)}}\right) + m\left(r, \frac{1}{f^{(k)}-1}\right) &\leq m\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f) \\ &\leq T(r, f^{(k+1)}) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f), \end{aligned}$$

so from (4.15) we have

$$m\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k+1)}) - N\left(r, \frac{1}{f^{(k+1)}}\right) - 2\overline{N}(r,f) + S(r,f). \tag{4.16}$$

It follows from (4.11) that

$$T(r, f^{(k+1)}) \leq (k+2)\overline{N}(r,f) + S(r,f). \tag{4.17}$$

By Theorem 2.18 we know

$$N\left(r, \frac{1}{f^{(k+1)}}\right) > (k-\varepsilon)\overline{N}(r,f) + S(r,f),$$

where ε is any given positive number. Combining this with (4.16) and (4.17) we get

$$m\left(r,\frac{1}{f^{(k)}}\right)<\varepsilon\overline{N}(r,f)+S(r,f). \tag{4.18}$$

Hence

$$m\left(r,\frac{1}{f-1}\right)\leq m\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)<\varepsilon\overline{N}(r,f)+S(r,f).$$

This and (4.14) imply

$$\begin{aligned}T(r,f)&=m\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{f-1}\right)\\&=m\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{f^{(k)}-1}\right)\\&<(k-1+\varepsilon)\overline{N}(r,f)+S(r,f).\end{aligned} \tag{4.19}$$

Since

$$T(r,f)\geq N\left(r,\frac{1}{f}\right)+O(1)=N\left(r,\frac{1}{f^{(k)}}\right)+O(1)=T(r,f^{(k)})-m\left(r,\frac{1}{f^{(k)}}\right)+O(1),$$

we derive from (4.11) and (4.18)

$$T(r,f)>(k+1)\overline{N}(r,f)-\varepsilon\overline{N}(r,f)+S(r,f).$$

We obtain from this and (4.19) that

$$\overline{N}(r,f)=S(r,f),$$

so

$$T(r,f)<S(r,f),$$

which is a contradiction.

Case 2. Suppose that $G\equiv 0$. Then $F(\alpha')^{k+1}\equiv \mathit{const}$. Let z_0 be a simple pole of f, then near z_0 we can write

$$f(z)=\frac{R}{z-z_0}+O(1),\quad (R\neq 0).$$

By computation, we get

$$\alpha'(z_0) = \frac{1}{R}, \qquad F(z_0) = \frac{R^{2k}}{(k!)^2}.$$

Hence

$$F(z_0)(\alpha'(z_0))^{2k} = \frac{1}{(k!)^2}. \tag{4.20}$$

If

$$F(z)(\alpha'(z))^{2k} \neq \frac{1}{(k!)^2},$$

from (4.20) we have

$$\begin{aligned}\overline{N}(r,f) &\leq N\left(r, \frac{1}{F(z)(\alpha'(z))^{2k} - \frac{1}{(k!)^2}}\right) \\ &\leq T(r, F(z)(\alpha'(z))^{2k}) + O(1) \\ &= T(r, (\alpha'(z)^{k-1}) + O(1) = S(r,f).\end{aligned}$$

Combining this with (4.11), we obtain

$$T(r, f^{(k)}) = S(r,f),$$

which is a contradiction. So

$$F(z)(\alpha'(z))^{2k} = \frac{1}{(k!)^2}.$$

Hence $\alpha' \equiv const$, $F \equiv const$ and

$$N^*(r) = 0, \qquad N(r,f) - \overline{N}(r,f) = 0.$$

Set

$$P = \frac{f^{(k)} - f}{f^{(k)}(f^{(k)} - 1)}, \qquad Q = \frac{(P')^{k+1}}{P^k}.$$

It is clear that P and Q are entire functions. Let z_0 be a simple pole of f. Then near z_0 we can write

$$f(z) = \frac{R}{z - z_0} + O(1), \qquad (R \neq 0).$$

By computation, we get

$$P(z) = \frac{(-1)^k(z-z_0)^{k+1}}{k!R}\left[1-\frac{(-1)^k}{k!}(z-z_0)^k+O(z-z_0)^{k+1}\right],$$

$$Q(z) = \frac{(-1)^k(k+1)^{k+1}}{k!R}\left[1-\frac{(-1)^k(k+1)}{k!}(z-z_0)^k+O(z-z_0)^{k+1}\right].$$

Clearly, z_0 is the zero of order $k+1$ of $P(z)$ and the zero of order $k-1$ of $Q'(z)$, but not the zero of $Q(z)$. Hence

$$\begin{aligned}(k-1)\overline{N}(r,f) &\le N\left(r,\frac{Q}{Q'}\right)\\ &\le T\left(r,\frac{Q}{Q'}\right)+O(1)\\ &\le \overline{N}\left(r,\frac{1}{Q}\right)+S(r,f),\end{aligned}$$

and

$$\overline{N}\left(r,\frac{1}{Q}\right)+\overline{N}(r,f)\le \overline{N}\left(r,\frac{1}{P'}\right)$$

Therefore

$$kN(r,f)\le \overline{N}\left(r,\frac{1}{P'}\right)+S(r,f). \tag{4.21}$$

Note that P is an entire function, by Theorem 2.19 we get

$$N\left(r,\frac{1}{P'}\right)\le N\left(r,\frac{1}{P}\right)+S(r,f).$$

So

$$N_0\left(r,\frac{1}{P'}\right)\le \overline{N}\left(r,\frac{1}{P}\right)+S(r,f),$$

where $N_0\left(r,\frac{1}{P'}\right)$ is the counting function of the zeros of P' which are not the multiple zeros of P. Since the simple pole of f is the zero of order $k+1$ of P and the zeros of P only appear at the pole of f, hence

$$\begin{aligned}\overline{N}\left(r,\frac{1}{P'}\right) &\le \overline{N}(r,f)+N_0\left(r,\frac{1}{P'}\right)\\ &\le \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{P}\right)+S(r,f)\\ &\le 2\overline{N}(r,f)+S(r,f).\end{aligned}$$

From (4.21), we have

$$(k-2)\overline{N}(r,f) = S(r,f).$$

If $k \geq 3$, then

$$\overline{N}(r,f) = S(r,f).$$

Combining this with (4.11), we derive

$$T(r,f^{(k)}) = S(r,f).$$

This is a contradiction. Now let us consider the case $k=2$. If $k=2$, then

$$\begin{aligned} P &= \frac{f''-f}{f''(f''-1)} \\ &= \frac{1}{R}(z-z_0)^3\left[1-\frac{1}{2}(z-z_0)^2+O(z-z_0)^3\right]. \end{aligned}$$

Let

$$\omega = \left(\frac{P'}{P}\right)^2 + 3\left(\frac{P'}{P}\right)' + 9, \tag{4.22}$$

then ω is an entire function and $\omega(z_0)=0$.

If $\omega \not\equiv 0$, we have

$$\begin{aligned} \overline{N}(r,f) \leq N\left(r,\frac{1}{\omega}\right) &\leq T(r,\omega)+O(1) \\ &\leq m(r,\omega)+O(1) = S(r,f). \end{aligned}$$

Again from (4.11), we have

$$T(r,f^{(k)}) = S(r,f),$$

which is a contradiction. Therefore $\omega \equiv 0$.

Note that P is an entire function, and P has zeros of order 3 only at the poles of f. We can assume that

$$u^3 = P,$$

where u is an entire function, then

$$\frac{P'}{P} = 3\frac{u'}{u}.$$

Clearly from (4.22), we know that u satisfies the equation

$$u'' + u = 0, \tag{4.23}$$

so

$$u = c_1 e^{iz} + c_2 e^{-iz},$$

where c_1 and c_2 are constant. Since the pole of f are the zeros of u, hence $c_1 \neq 0$, $c_2 \neq 0$ and

$$T(r, u) = N\left(r, \frac{1}{u}\right) + S(r, f) = N(r, f) + S(r, f), \tag{4.24}$$

$$m\left(r, \frac{1}{u}\right) = S(r, f). \tag{4.25}$$

According to the definition of u, we have

$$f'' - f = f''(f'' - 1)u^3.$$

Hence

$$f = f''\left[1 - (f'' - 1)u^3\right],$$

$$f - 1 = (f'' - 1)\left(1 - f''u^3\right).$$

Taking this into (4.6), we get

$$e^{\alpha} = \frac{1 - (f'' - 1)u^3}{1 - f''u^3}.$$

Hence

$$f'' = \frac{1}{u^3} - \frac{1}{e^{\alpha} - 1}. \tag{4.26}$$

Notice that $m\left(r, \frac{1}{e^{\alpha}-1}\right) = S(r, f)$. From (4.25) and (4.26) we obtain

$$m(r, f'') = S(r, f). \tag{4.27}$$

According to the definition of F, we know

$$F = \frac{(f)^3(f-1)^3 f''(f''-1)}{(f''-f)^4} = \left(\frac{f(f-1)}{(f''-f)u}\right)^3.$$

Since $F \equiv const$, so

$$\frac{f(f-1)}{(f''-f)u} = c,$$

where $c \neq 0$ is a constant. Hence

$$f'' - f = \frac{f(f-1)}{cu},$$

which gives

$$f'' = f\left(1 + \frac{f-1}{cu}\right),$$

$$f'' - 1 = (f-1)\left(1 + \frac{f}{cu}\right).$$

From (4.6), we have

$$e^{\alpha} = \frac{(1 + \frac{f}{cu})}{(1 + \frac{f-1}{cu})}.$$

Hence

$$f = 1 - cu + \frac{1}{e^{\alpha} - 1}.$$

Note that $\alpha' \equiv const$, let $\alpha' = d$. From the above equality we obtain

$$f' = -cu' - d\left[\frac{1}{e^{\alpha}-1} + \frac{1}{(e^{\alpha}-1)^2}\right],$$

$$f'' = -cu'' + d^2\left[\frac{1}{e^{\alpha}-1} + \frac{3}{(e^{\alpha}-1)^2} + \frac{2}{(e^{\alpha}-1)^3}\right].$$

From this and (4.23), we have

$$u = \frac{1}{c}f'' - \frac{d^2}{c}\left[\frac{1}{e^{\alpha}-1} + \frac{3}{(e^{\alpha}-1)^2} + \frac{2}{(e^{\alpha}-1)^3}\right].$$

Since $m\left(r, \frac{1}{e^{\alpha}-1}\right) = S(r,f)$, we derive from (4.27) and the above equality

$$m(r,u) = S(r,f).$$

Hence

$$T(r,u) = m(r,u) = S(r,f).$$

Taking this into (4.24), we have $N(r,f) = S(r,f)$. Again from (4.11), we have

$$T(r, f^{(k)}) = S(r,f),$$

which is a contradiction, so $f \equiv f^{(k)}$. ❑

Theorem 4.3 *[1] Let f be a non-constant meromorphic function, a and b be two distinct finite non-zero values. If f and $f^{(k)}$ share values a and b CM, then $f \equiv f^{(k)}$.*

Proof. Without loss generality, we assume that $a\ (\neq 0, 1)$, $b = 1$ are CM shared values of f and $f^{(k)}$. Otherwise, consider $\frac{f}{b}$. Suppose that $f \not\equiv f^{(k)}$. Under the hypothesis of Theorem 4.3, we know

$$\frac{(f-a)(f^{(k)}-1)}{(f-1)(f^{(k)}-a)} = e^{\alpha}, \tag{4.28}$$

where α is an entire function. From (4.28) we have

$$T(r, e^{\alpha}) = O(T(r, f)), \quad (r \notin E),$$

Similar to the proof of Theorem 4.2, it is easy to prove

$$N(r, f) = \overline{N}(r, f) + S(r, f). \tag{4.29}$$

Set

$$g = \frac{f-a}{f^{(k)}-a}, \tag{4.30}$$

then g is an entire function and

$$m\left(r, \frac{1}{g}\right) = m\left(r, \frac{f^{(k)}-a}{f-a}\right) \leq m\left(r, \frac{f^{(k)}}{f-a}\right) + m\left(r, \frac{a}{f-a}\right) \leq m\left(r, \frac{1}{f-a}\right) + S(r, f).$$

From (4.28) and (4.30), we have

$$f^{(k)} = \frac{1-a}{g} + \frac{a-1}{e^{\alpha}-1}\left(1 - \frac{1}{g}\right) + a.$$

Hence

$$m(r, f^{(k)}) \leq 2m\left(r, \frac{1}{g}\right) + m\left(r, \frac{1}{e^{\alpha}-1}\right) + O(1).$$

By second fundamental theorem for 0, 1 and ∞, we get

$$\begin{aligned} T(r, e^{\alpha}) &\leq N\left(r, \frac{1}{e^{\alpha}}\right) + N\left(r, \frac{1}{e^{\alpha}-1}\right) + N(r, e^{\alpha}) + S(r, e^{\alpha}) \\ &\leq N\left(r, \frac{1}{e^{\alpha}-1}\right) + S(r, f) \\ &\leq T\left(r, \frac{1}{e^{\alpha}-1}\right) + S(r, f) \\ &\leq T(r, e^{\alpha}) + S(r, f). \end{aligned}$$

So

$$N\left(r,\frac{1}{e^{\alpha}-1}\right)=T(r,e^{\alpha})+S(r,f)=T\left(r,\frac{1}{e^{\alpha}-1}\right)+S(r,f),$$

then

$$m\left(r,\frac{1}{e^{\alpha}-1}\right)=S(r,f).$$

We have

$$m(r,f^{(k)})\leq 2m\left(r,\frac{1}{g}\right)+S(r,f)\leq 2m\left(r,\frac{1}{f-a}\right)+S(r,f).$$

It leads to

$$\begin{aligned}N(r,f^{(k)})&\leq T(r,f^{(k)})\\&=m(r,f^{(k)})+N(r,f^{(k)})\\&\leq 2m\left(r,\frac{1}{f-a}\right)+N(r,f^{(k)})+S(r,f).\end{aligned}$$

From (4.29), we have

$$(k+1)\overline{N}(r,f)\leq T(r,f^{(k)})\leq (k+1)\overline{N}(r,f)+2m\left(r,\frac{1}{f-a}\right)+S(r,f). \quad (4.31)$$

Set

$$\begin{aligned}N^{*}(r)&=N\left(r,\frac{1}{f^{(k)}-f}\right)-N\left(r,\frac{1}{f-a}\right)-N\left(r,\frac{1}{f-1}\right)\\&=N\left(r,\frac{1}{f^{(k)}-f}\right)-N\left(r,\frac{1}{f^{(k)}-a}\right)-N\left(r,\frac{1}{f^{(k)}-1}\right). \quad (4.32)\end{aligned}$$

Noting that

$$\begin{aligned}m\left(r,\frac{1}{f-a}\right)+m\left(r,\frac{1}{f-1}\right)&=m\left(r,\frac{1}{f-a}+\frac{1}{f-1}\right)+O(1)\\&\leq m\left(r,\frac{f'}{f-a}+\frac{f'}{f-1}\right)+m\left(r,\frac{1}{f'}\right)+O(1)\\&=m\left(r,\frac{1}{f'}\right)+S(r,f)\end{aligned}$$

and

$$\begin{aligned}
&m\left(r,\frac{1}{f^{(k)}}\right)+m\left(r,\frac{1}{f^{(k)}-a}\right)+m\left(r,\frac{1}{f^{(k)}-1}\right)\\
&=m\left(r,\frac{1}{f^{(k)}}+\frac{1}{f^{(k)}-a}+\frac{1}{f^{(k)}-1}\right)+O(1)\\
&\leq\ \left(r,\frac{f^{(k+1)}}{f^{(k)}}+\frac{f^{(k+1)}}{f^{(k)}-a}+\frac{f^{(k+1)}}{f^{(k)}-1}\right)+m\left(r,\frac{1}{f^{(k+1)}}\right)+O(1)\\
&=m\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f),
\end{aligned}$$

this and (4.32) imply

$$\begin{aligned}
N^*(r)+2T(r,f)&=N^*(r)+T\left(r,\frac{1}{f-a}\right)+T\left(r,\frac{1}{f-1}\right)+O(1)\\
&=N\left(r,\frac{1}{f^{(k)}-f}\right)+m\left(r,\frac{1}{f-a}\right)+m\left(r,\frac{1}{f-1}\right)+O(1)\\
&\leq T(r,f^{(k)}-f)+m\left(r,\frac{1}{f'}\right)+S(r,f)\\
&=m\left(r,f\left(\frac{f^{(k)}}{f}-1\right)\right)+N(r,f^{(k)}-f)+m\left(r,\frac{1}{f'}\right)+S(r,f)\\
&\leq m(r,f)+k\overline{N}(r,f)+N(r,f)+m\left(r,\frac{1}{f'}\right)+S(r,f)\\
&=T(r,f)+k\overline{N}(r,f)+m\left(r,\frac{1}{f'}\right)+S(r,f)
\end{aligned}$$

and

$$\begin{aligned}
&N^*(r)+2T(r,f^{(k)})\\
&=N^*(r)+T\left(r,\frac{1}{f^{(k)}-a}\right)+T\left(r,\frac{1}{f^{(k)}-1}\right)+O(1)\\
&=N\left(r,\frac{1}{f^{(k)}-f}\right)+m\left(r,\frac{1}{f^{(k)}-a}\right)+m\left(r,\frac{1}{f^{(k)}-1}\right)+O(1)\\
&\leq T(r,f^{(k)}-f)+m\left(r,\frac{1}{f^{(k+1)}}\right)-m\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)\\
&\leq T(r,f)+k\overline{N}(r,f)+T(r,f^{(k+1)})-N\left(r,\frac{1}{f^{(k+1)}}\right)-m\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)\\
&\leq T(r,f)+(k+1)\overline{N}(r,f)+T(r,f^{(k)})-N\left(r,\frac{1}{f^{(k+1)}}\right)-m\left(r,\frac{1}{f^{(k)}}\right)+S(r,f).
\end{aligned}$$

Therefore

$$N^*(r) + T(r, f) \leq k\overline{N}(r, f) + m\left(r, \frac{1}{f'}\right) + S(r, f) \tag{4.33}$$

and

$$N^*(r)+T(r, f^{(k)}) \leq T(r, f)+(k+1)\overline{N}(r, f)-N\left(r, \frac{1}{f^{(k+1)}}\right)-m\left(r, \frac{1}{f^{(k)}}\right)+S(r, f). \tag{4.34}$$

Since

$$m\left(r, \frac{1}{f'}\right) \leq m\left(r, \frac{f^{(k)}}{f'}\right) + m\left(r, \frac{1}{f^{(k)}}\right) + O(1) = m\left(r, \frac{1}{f^{(k)}}\right) + S(r, f),$$

combining this with (4.33) and (4.34) we have

$$2N^*(r) + T(r, f^{(k)}) \leq (2k+1)\overline{N}(r, f) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f). \tag{4.35}$$

By Theorem 2.18 we know

$$N(r, \frac{1}{f^{(k+1)}}) > (k-\varepsilon)\overline{N}(r, f) + S(r, f),$$

where ε is any given positive number. With (4.35) we derive

$$2N^*(r) + T(r, f^{(k)}) \leq (k+1+\varepsilon)\overline{N}(r, f) + S(r, f).$$

From (4.31) we have

$$T(r, f^{(k)}) \leq (k+1+\varepsilon)\overline{N}(r, f) + S(r, f) \tag{4.36}$$

and

$$N^*(r) \leq \frac{\varepsilon}{2}\overline{N}(r, f) + S(r, f). \tag{4.37}$$

Now set

$$F = \frac{[(f-a)(f-1)]^{k+1}[(f^{(k)}-a)(f^{(k)}-1)]^{k-1}}{(f^{(k)}-f)^{2k}},$$

$$G = \begin{cases} \frac{F'}{F} + (k+1)\frac{\alpha''}{\alpha'}, & \text{if } k \geq 2 \\ \frac{F'}{F} + 2\frac{\alpha''}{\alpha'} - 2, & \text{if } k = 1. \end{cases}$$

If z^* is a pole of order p of f, when $p = 1$, z^* is not pole or zero of F, and when $p > 1$, z^* is a pole of F. So F has no zeros and

$$\begin{aligned}\overline{N}(r, F) &\leq N\left(r, \frac{1}{f^{(k)} - f}\right) - N\left(r, \frac{1}{f}\right) - N\left(r, \frac{1}{f-1}\right) + N(r, f) - \overline{N}(r, f) \\ &= N^*(r) + S(r, f).\end{aligned}$$

Let z_0 be a simple pole of f. Then $F(z_0) \neq 0, \infty$. And near z_0 we can write

$$f(z) = \frac{R}{z - z_0} + a_0 + O(z - z_0), \quad (R \neq 0).$$

If $k \geq 2$, we have

$$F = \frac{R^{2k}}{(k!)^2} + \frac{(k+1)R^{2k-1}(2a_0 - a - 1)}{(k!)^2}(z - z_0) + O(z - z_0)^2,$$

$$F' = \frac{(k+1)R^{2k-1}(2a_0 - a - 1)}{(k!)^2} + O(z - z_0),$$

$$\alpha' = (a-1)\left[-\frac{1}{R} + \frac{2a_0 - a - 1}{R^2}(z - z_0) + O(z - z_0)^2\right],$$

and

$$\alpha'' = (a-1)\left[\frac{2a_0 - 1}{R^2} + O(z - z_0)\right],$$

so

$$G(z_0) = \frac{F'(z_0)}{F(z_0)} + (k+1)\frac{\alpha''(z_0)}{\alpha'(z_0)} = 0.$$

If $k = 1$, we have

$$F = R^2 + 2R(2a_0 - a - 1 - R)(z - z_0) + O(z - z_0)^2,$$

$$F' = 2R(2a_0 - a - 1 - R) + O(z - z_0),$$

$$\alpha' = (a-1)\left[-\frac{1}{R} + \frac{2a_0 - a - 1 - 2R}{R^2}(z - z_0) + O(z - z_0)^2\right],$$

and

$$\alpha'' = (a-1)\left[\frac{2a_0 - 1 - 2R}{R^2} + O(z - z_0)\right],$$

so

$$G(z_0) = \frac{F'(z_0)}{F(z_0)} + 2\frac{\alpha''(z_0)}{\alpha'(z_0)} - 2 = 0.$$

Hence the simple pole of f is the zeros of G. And G is a meromorphic function satisfying

$$\begin{aligned} T(r, G) &\leq T\left(r, \frac{F'}{F}\right) + T\left(r, \frac{\alpha''}{\alpha'}\right) \\ &= m\left(r, \frac{F'}{F}\right) + N\left(r, \frac{F'}{F}\right) + S(r, f) \\ &= \overline{N}(r, F) + S(r, f). \end{aligned}$$

We distinguish the following two case.

Case 1. Suppose that $G \not\equiv 0$. Then

$$\overline{N}(r, f) \leq N\left(r, \frac{1}{G}\right) \leq T(r, G) + O(1) \leq N^*(r) + S(r, f).$$

This and (4.37) lead to

$$\overline{N}(r, f) \leq \frac{\varepsilon}{2}\overline{N}(r, f) + S(r, f).$$

Hence

$$\overline{N}(r, f) = S(r, f),$$

and from (4.36) we derive

$$T(r, f^{(k)}) < S(r, f).$$

This is a contradiction.

Case 2. Suppose that $G \equiv 0$.

We discuss the following two subcases.

Subcase 1. If $k = 1$, from $G \equiv 0$ we know

$$\frac{f'}{f-a} + \frac{f'}{f-1} - \frac{f''-f'}{f'-f} + \frac{\alpha''}{\alpha'} - 1 = 0.$$

Taking integrate of this, we have

$$\alpha' \frac{(f-a)(f-1)}{f'-f} = ce^z, \tag{4.38}$$

where c $(\neq 0)$ is a constant. Let z_0 be a simple pole of f. Then near z_0 we can write

$$f(z) = \frac{R}{z - z_0} + O(1), \quad (R \neq 0).$$

From (4.38), we get

$$ce^{z_0} = a - 1.$$

Since $N(r, f) - \overline{N}(r, f) = S(r, f)$, we have

$$\begin{aligned} N(r, f) &\leq N\left(r, \frac{1}{ce^z - (a-1)}\right) \\ &= T(r, e^z) + O(1) \\ &= \frac{r}{\pi} + O(1). \end{aligned} \tag{4.39}$$

We rewrite (4.38) as

$$(f - a)(f - 1) = \frac{c}{\alpha'} e^z f \left(\frac{f'}{f} - 1\right). \tag{4.40}$$

From this,

$$\begin{aligned} 2m(r, f) &= m(r, (f - a)(f - 1)) + S(r, f) \\ &\leq m(r, \frac{1}{\alpha'}) + m(r, e^z) + m(r, f) + m\left(r, \frac{f'}{f}\right) + S(r, f) \\ &= m(r, e^z) + m(r, f) + S(r, f), \end{aligned}$$

so

$$m(r, f) \leq m(r, e^z) + S(r, f) = \frac{r}{\pi} + S(r, f).$$

(4.39) and the above equality imply

$$T(r, f) \leq \frac{2r}{\pi} + S(r, f).$$

Since $T(r, e^\alpha) = O(T(r, f)), \quad (r \notin E)$, with the above equality we know that $\lambda(f) \leq 1$ and $\lambda(e^\alpha) \leq \lambda(f) \leq 1$. Since α is not constant, α must be a linear function and α' must be a constant. we get $\lambda(e^\alpha) = \lambda(f) = 1$. Let $\frac{\alpha'}{c} = d$, taking it into (4.40) we have

$$f' - a = (f - a)[1 + de^{-z}(f - 1)].$$

Thus

$$g = \frac{1}{1 + de^{-z}(f-1)} = \frac{f-a}{f'-a} \tag{4.41}$$

is an entire function with $\lambda(g) \leq 1$. Since $g(z_0) = 0$, we know

$$N(r, f) \leq N\left(r, \frac{1}{g}\right) = T(r, g) + O(1). \tag{4.42}$$

From (4.41), we get

$$f = \frac{e^z}{dg} - \frac{e^z}{d} + 1.$$

Substituting the above equality into (4.41), we have

$$g^2 = -\frac{1}{1 + ade^{-z}} + g\left(\frac{a-1}{a} + \frac{a+1}{a(1+ade^{-z})} - \frac{1}{1+ade^{-z}}\frac{g'}{g}\right). \tag{4.43}$$

Note that

$$m\left(r, \frac{1}{1+ade^{-z}}\right) = m\left(r, \frac{e^z}{e^z + ad}\right) = S(r, e^z) = S(r, f).$$

It follows from (4.43) that

$$\begin{aligned} 2m(r, g) &\leq m(r, g) + m\left(r, \frac{g'}{g}\right) + S(r, f) \\ &= m(r, g) + S(r, f). \end{aligned}$$

Hence

$$T(r, g) = m(r, g) = S(r, f).$$

From (4.42), we get

$$N(r, f) = S(r, f),$$

and so from (4.36)

$$T(r, f') = S(r, f),$$

which is a contradiction.

Subcase 2. If $k \geq 2$, similar to the proof of Theorem 4.2, we can get a contradiction.

This completes the proof of Theorem 4.3, so $f \equiv f^{(k)}$. ❑

Corollary 4.4 *Let f be a non-constant meromorphic function. If f and $f^{(k)}$ share distinct finite values a and b CM, then $f \equiv f^{(k)}$.*

P. Li gave the following example, which shows that the condition f and f^k have two shared values CM in corollary 4.4 is essential.

Example 4.5 *Let a_1 be any finite value, $a_2 = a_1 + \sqrt{2}i$. Let ω be a non-constant solution of the following Riccati differential equation*

$$\omega' = (\omega - a_1)(\omega - a_2). \tag{4.44}$$

Let

$$f = (\omega - a_1)(\omega - a_2) - \frac{1}{3}.$$

We get

$$f' = \omega'(2\omega - a_1 - a_2)$$

and

$$f'' = 6\omega' f, \tag{4.45}$$

$$f'' + \frac{1}{6} = 6\left(f + \frac{1}{6}\right)^2. \tag{4.46}$$

(4.44) implies that 0 is the Picard exceptional value of ω', and from (4.45) we know that 0 is a CM shared value of f and f''. (4.46) implies that $-\frac{1}{6}$ is the IM shared value of f and f'', but $f \neq f''$.