# Appendix

# A Definition of Parameters

Originally, the coefficients $Q_{2i+1,n}$ and $R_{2i+1,n}$ are

$$\begin{aligned} Q_{2i+1,n} &= \sum_{k=1}^{n-i} \Omega_k \left(2i+1\right)^2 a_{2i+1,n-k} + \sum_{k=0}^{n-i-1} \varepsilon\omega_k \left(2i+1\right) b_{2i+1,n-k-1} \\ &\quad - \sum_{k_1+k_2+k_3=n-1} \varepsilon\omega_{k_1} \left\{ \sum_{j=0}^{k_3} \sum_{i_1=0}^{k_2} \sum_{i_2=0}^{j} \sum_{i_3=0}^{k_3-j} S \right\}, \end{aligned} \tag{A.1}$$

$$R_{2i+1,n} = \begin{cases} R^*_{2i+1,n} + \Omega_{n-1} b_{1,0} \cos\theta, & \text{if} \qquad i = 0, \\ R^*_{2i+1,n}, & \text{else}, \end{cases} \tag{A.2}$$

where $k_3 = n - 1 - k_1 - k_2$,

$$\begin{aligned} R^*_{2i+1,n} &= \sum_{k=1}^{n-i} \Omega_k \left(2i+1\right)^2 b_{2i+1,n-k} + \sum_{k=0}^{n-i-1} \varepsilon\omega_k \left(2i+1\right) a_{2i+1,n-k-1} \\ &\quad - \sum_{k_1+k_2+k_3=n-1} \varepsilon\omega_{k_1} \left\{ \sum_{j=0}^{k_3} \sum_{i_1=0}^{k_2} \sum_{i_2=0}^{j} \sum_{i_3=0}^{k_3-j} C \right\} \end{aligned} \tag{A.3}$$

and $S$, $C$ are depending on $i_1, i_2, i_3, j, k_2, k_3$ and defined by

$$\begin{aligned} S\left(i_1,i_2,i_3,j,k_2,k_3\right) &= S_1 \sin\left[2\left(i_1+i_2+i_3+1\right)+1\right]\theta \\ &\quad + S_2 \sin\left[2\left(i_1+i_2-i_3\right)+1\right]\theta \\ &\quad + S_3 \sin\left[2\left(i_1-i_2+i_3\right)+1\right]\theta \\ &\quad + S_4 \sin\left[2\left(i_1-i_2-i_3-1\right)+1\right]\theta, \end{aligned} \tag{A.4}$$

$$\begin{aligned} C\left(i_1,i_2,i_3,j,k_2,k_3\right) &= C_1 \cos\left[2\left(i_1+i_2+i_3+1\right)+1\right]\theta \\ &\quad + C_2 \cos\left[\left(2\left(i_1+i_2-i_3\right)+1\right)\right]\theta \\ &\quad + C_3 \cos\left[2\left(i_1-i_2+i_3\right)+1\right]\theta \\ &\quad + C_4 \cos\left[2\left(i_1-i_2-i_3-1\right)+1\right]\theta. \end{aligned} \tag{A.5}$$

Note that $S_i$ and $C_i$ are depending on $i_1, i_2, i_3, j, k_2, k_3$, for $i = 1, 2, 3, 4$ and defined by

$$\begin{aligned}
S_1 &= \frac{1}{4}\,[a_{2i_1+1,k_2}\,(a_{2i_2+1,j}b_{2i_3+1,k_3-j} + b_{2i_2+1,j}a_{2i_3+1,k_3-j}) \\
&\quad + b_{2i_1+1,k_2}\,(a_{2i_2+1,j}a_{2i_3+1,k_3-j} - b_{2i_2+1,j}b_{2i_3+1,k_3-j})\,]\,, \qquad \text{(A.6)} \\
S_2 &= \frac{1}{4}\,[a_{2i_1+1,k_2}\,(a_{2i_2+1,j}b_{2i_3+1,k_3-j} - b_{2i_2+1,j}a_{2i_3+1,k_3-j}) \\
&\quad - b_{2i_1+1,k_2}\,(a_{2i_2+1,j}a_{2i_3+1,k_3-j} + b_{2i_2+1,j}b_{2i_3+1,k_3-j})\,]\,, \qquad \text{(A.7)} \\
S_3 &= \frac{1}{4}\,[a_{2i_1+1,k_2}\,(-a_{2i_2+1,j}b_{2i_3+1,k_3-j} + b_{2i_2+1,j}a_{2i_3+1,k_3-j}) \\
&\quad - b_{2i_1+1,k_2}\,(a_{2i_2+1,j}a_{2i_3+1,k_3-j} + b_{2i_2+1,j}b_{2i_3+1,k_3-j})\,]\,, \qquad \text{(A.8)} \\
S_4 &= \frac{1}{4}\,[-a_{2i_1+1,k_2}\,(a_{2i_2+1,j}b_{2i_3+1,k_3-j} + b_{2i_2+1,j}a_{2i_3+1,k_3-j}) \\
&\quad + b_{2i_1+1,k_2}\,(a_{2i_2+1,j}a_{2i_3+1,k_3-j} - b_{2i_2+1,j}b_{2i_3+1,k_3-j})\,]\,, \qquad \text{(A.9)}
\end{aligned}$$

$$\begin{aligned}
C_1 &= \frac{1}{4}\,[a_{2i_1+1,k_2}\,(-a_{2i_2+1,j}a_{2i_3+1,k_3-j} + b_{2i_2+1,j}b_{2i_3+1,k_3-j}) \\
&\quad + b_{2i_1+1,k_2}\,(a_{2i_2+1,j}b_{2i_3+1,k_3-j} + b_{2i_2+1,j}a_{2i_3+1,k_3-j})\,]\,, \qquad \text{(A.10)} \\
C_2 &= \frac{1}{4}\,[a_{2i_1+1,k_2}\,(a_{2i_2+1,j}a_{2i_3+1,k_3-j} + b_{2i_2+1,j}b_{2i_3+1,k_3-j}) \\
&\quad + b_{2i_1+1,k_2}\,(a_{2i_2+1,j}b_{2i_3+1,k_3-j} - b_{2i_2+1,j}a_{2i_3+1,k_3-j})\,)\,, \qquad \text{(A.11)} \\
C_3 &= \frac{1}{4}\,[a_{2i_1+1,k_2}a_{2i_2+1,j}a_{2i_3+1,k_3-j} + a_{2i_1+1,k_2}b_{2i_2+1,j}b_{2i_3+1,k_3-j} \\
&\quad + b_{2i_1+1,k_2}\,(-a_{2i_2+1,j}b_{2i_3+1,k_3-j} + b_{2i_2+1,j}a_{2i_3+1,k_3-j})\,] \qquad \text{(A.12)}
\end{aligned}$$

and

$$\begin{aligned}
C_4 &= \frac{1}{4}\,[a_{2i_1+1,k_2}\,(-a_{2i_2+1,j}a_{2i_3+1,k_3-j} + b_{2i_2+1,j}b_{2i_3+1,k_3-j}) \\
&\quad - b_{2i_1+1,k_2}\,(a_{2i_2+1,j}b_{2i_3+1,k_3-j} + b_{2i_2+1,j}a_{2i_3+1,k_3-j})\,]\,. \qquad \text{(A.13)}
\end{aligned}$$

Then by the change of variable, $(A.1)$ and $(A.3)$ are reduced to (4.45) and (4.46) respectively, with the variables defined as follows:

$$
\begin{aligned}
S_5 &= \begin{cases} S_9\left(i_1, i_2, j, k, k_2, k_3\right), & \text{if } k = i, \\ 0, & \text{else,} \end{cases} \\
S_6 &= \begin{cases} S_{10}\left(i_1, i_2, j, k, k_2, k_3\right), & \text{if } k = i, \\ 0, & \text{else,} \end{cases} \\
S_7 &= \begin{cases} S_{11}\left(i_1, i_2, j, k, k_2, k_3\right), & \text{if } k = i, \\ 0, & \text{else,} \end{cases} \\
S_8 &= \begin{cases} S_{12}\left(i_1, i_2, j, k, k_2, k_3\right), & \text{if } k = i, \\ 0, & \text{else,} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
C_5 &= \begin{cases} C_9\left(i_1, i_2, j, k, k_2, k_3\right), & \text{if } k = i, \\ 0, & \text{else,} \end{cases} \\
C_6 &= \begin{cases} C_{10}\left(i_1, i_2, j, k, k_2, k_3\right), & \text{if } k = i, \\ 0, & \text{else,} \end{cases} \\
C_7 &= \begin{cases} C_{11}\left(i_1, i_2, j, k, k_2, k_3\right), & \text{if } k = i, \\ 0, & \text{else,} \end{cases} \\
C_8 &= \begin{cases} C_{12}\left(i_1, i_2, j, k, k_2, k_3\right), & \text{if } k = i, \\ 0, & \text{else,} \end{cases}
\end{aligned}
$$

where $S_i$ and $C_i$ are depending on $i_1, i_2, j, k, k_2, k_3$, for $i = 9, 10, 11, 12$ and defined by

$$
\begin{aligned}
S_9 &= S_1\left(i_1, i_2, k - i_1 - i_2 - 1, j, k_2, k_3\right), \\
S_{10} &= \begin{cases} S_2|_{i_3 = i_1 + i_2 - k}, & \text{if } \left(i_1 + i_2 - k_3 + j\right)\left(i_1 + i_2\right) \geq 0, \\ S_2^*, & \text{else,} \end{cases} \\
S_{11} &= \begin{cases} S_3^*, & \text{if } \left(i_1 - i_2 + k_3 - j\right)\left(i_1 - i_2\right) \geq 0, \\ S_3^{**}, & \text{else,} \end{cases} \\
S_{12} &= \begin{cases} S_4^*, & \text{if } \left(i_1 - i_2 - 1\right)\left(i_1 - i_2 - k_3 + j - 1\right) \geq 0, \\ S_4^{**}, & \text{else,} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
C_9 &= C_1\left(i_1, i_2, k-i_1-i_2-1, j, k_2, k_3\right), \\
C_{10} &= \begin{cases} C_2|_{i_3=i_1+i_2-k}, & \text{if } \left(i_1+i_2-k_3+j\right)\left(i_1+i_2\right) \geq 0, \\ C_2^*, & \text{else,} \end{cases} \\
C_{11} &= \begin{cases} C_3^*, & \text{if } \left(i_1-i_2+k_3-j\right)\left(i_1-i_2\right) \geq 0, \\ C_3^{**}, & \text{else,} \end{cases} \\
C_{12} &= \begin{cases} C_4^*, & \text{if } \left(i_1-i_2-1\right)\left(i_1-i_2-k_3+j-1\right) \geq 0, \\ C_4^{**}, & \text{else,} \end{cases}
\end{aligned}
$$

where $S_l^*, C_l^*, S_m^{**}, C_m^{**}$, for $l = 2,3,4$ and $m = 3,4$, are depending on $i_1, i_2, j, k, k_2, k_3$

$$
\begin{aligned}
S_2^* &= \begin{cases} S_{2,1}^*, & \text{if } \left|i_1+i_2-k_3+j\right| \leq \ i_1+i_2, \\ S_{2,2}^*, & \text{else,} \end{cases} \\
S_3^* &= \begin{cases} S_3|_{i_3=k-i_1+i_2}, & \text{if } \left|i_1-i_2\right| \leq \left|i_1-i_2+k_3-j\right|, \\ -\,S_3|_{i_3=k+1-i_1+i_2}, & \text{else,} \end{cases} \\
S_3^{**} &= \begin{cases} S_{3,1}^{**}, & \text{if } \left|i_1-i_2\right| \leq \left|i_1-i_2+k_3-j\right|, \\ S_{3,2}^{**}, & \text{else,} \end{cases} \\
S_4^* &= \begin{cases} S_4|_{i_3=i_1-i_2-k-1}, & \text{if } \left|i_1-i_2-k_3+j-1\right| \leq \left|i_1-i_2-1\right|, \\ -\,S_4|_{i_3=i_1-i_2-k-2}, & \text{else,} \end{cases} \\
S_4^{**} &= \begin{cases} S_{4,1}^{**}, & \text{if } \left|i_1-i_2-k_3+j-1\right| \leq \left|i_1-i_2-1\right|, \\ S_{4,2}^{**}, & \text{else,} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
C_2^* &= \begin{cases} C_{2,1}^*, & \text{if } |i_1+i_2-k_3+j| \leq\ i_1+i_2, \\ C_{2,2}^*, & \text{else,} \end{cases} \\
C_3^* &= \begin{cases} C_3|_{i_3=k-i_1+i_2}\,, & \text{if } |i_1-i_2| \leq |i_1-i_2+k_3-j|\,, \\ C_3|_{i_3=k+1-i_1+i_2}\,, & \text{else,} \end{cases} \\
C_3^{**} &= \begin{cases} C_{3,1}^{**}, & \text{if } |i_1-i_2| \leq |i_1-i_2+k_3-j|\,, \\ C_{3,2}^{**}, & \text{else,} \end{cases} \\
C_4^* &= \begin{cases} C_4|_{i_3=i_1-i_2-k-1}\,, & \text{if } |i_1-i_2-k_3+j-1| \leq |i_1-i_2-1|\,, \\ C_4|_{i_3=i_1-i_2-k-2}\,, & \text{else,} \end{cases} \\
C_4^{**} &= \begin{cases} C_{4,1}^{**}, & \text{if } |i_1-i_2-k_3+j-1| \leq |i_1-i_2-1|\,, \\ C_{4,2}^{**}, & \text{else.} \end{cases}
\end{aligned}
$$

Note that $S^*_{2,1}, C^*_{2,1}, S^*_{2,1}, C^*_{2,2}$ and $S^{**}_{l,m}, C^{**}_{l,m}$, for $l = 3, 4$ and $m = 1, 2$, are depending on $i_1, i_2, j, k, k_2, k_3$

$$S^*_{2,1} = \begin{cases} S_2|_{i_3=i_1+i_2-k} - S_2|_{i_3=i_1+i_2-k-1}\,, & \text{if } 0 \le k \le |i_1+i_2-k_3+j|\,, \\ S_2|_{i_3=i_1+i_2-k}\,, & \text{if } |i_1+i_2-k_3+j| < k \le i_1+i_2, \\ 0, & \text{else,} \end{cases}$$

$$S^*_{2,2} = \begin{cases} S_2|_{i_3=i_1+i_2-k} - S_2|_{i_3=i_1+i_2-k-1}\,, & \text{if } 0 \le k \le i_1+i_2, \\ -S_2|_{i_3=i_1+i_2-k-1}\,, & \text{if } i_1+i_2 < k \le |i_1+i_2-k_3+j|\,, \\ 0, & \text{else,} \end{cases}$$

$$S^{**}_{3,1} = \begin{cases} S_3|_{i_3=k-i_1+i_2} - S_3|_{i_3=k+1-i_1+i_2}\,, & \text{if } 0 \le k \le i_1-i_2, \\ S_3|_{i_3=k-i_1+i_2}\,, & \text{if } i_1-i_2 < k \le i_1-i_2+k_3-j, \\ 0, & \text{else,} \end{cases}$$

$$S^{**}_{3,2} = \begin{cases} S_3|_{i_3=k-i_1+i_2} - S_3|_{i_3=k+1-i_1+i_2}\,, & \text{if } 0 \le k \le i_1-i_2+k_3-j, \\ -S_3|_{i_3=k+1-i_1+i_2}\,, & \text{if } i_1-i_2+k_3-j < k \le i_1-i_2, \\ 0, & \text{else,} \end{cases}$$

$$S^{**}_{4,1} = \begin{cases} S_4|_{i_3=i_1-i_2-k-1} - S_4|_{i_3=i_1-i_2-k-2}\,, & \text{if } 0 \le k \le i_1-i_2-k_3+j-1, \\ S_4|_{i_3=i_1-i_2-k-1}\,, & \text{if } i_1-i_2-k_3+j-1 < k \le i_1-i_2-1, \\ 0, & \text{else,} \end{cases}$$

$$S^{**}_{4,2} = \begin{cases} S_4|_{i_3=i_1-i_2-k-1} - S_4|_{i_3=i_1-i_2-k-2}\,, & \text{if } 0 \le k \le i_1-i_2-1, \\ -S_4|_{i_3=i_1-i_2-k-2}\,, & \text{if } i_1-i_2-1 < k \le i_1-i_2-k_3+j-1, \\ 0, & \text{else} \end{cases}$$

and

$$C^{*}_{2,1} = \begin{cases} C_2|_{i_3=i_1+i_2-k} + C_2|_{i_3=i_1+i_2-k-1}\,, & \text{if } 0 \le k \le |i_1+i_2-k_3+j|\,, \\ C_2|_{i_3=i_1+i_2-k}\,, & \text{if } |i_1+i_2-k_3+j| < k \le i_1+i_2, \\ 0, & \text{else,} \end{cases}$$

$$C^{*}_{2,2} = \begin{cases} C_2|_{i_3=i_1+i_2-k} + C_2|_{i_3=i_1+i_2-k-1}\,, & \text{if } 0 \le k \le i_1+i_2, \\ C_2|_{i_3=i_1+i_2-k-1}\,, & \text{if } i_1+i_2 < k \le |i_1+i_2-k_3+j|\,, \\ 0, & \text{else,} \end{cases}$$

$$C^{**}_{3,1} = \begin{cases} C_3|_{i_3=k-i_1+i_2} + C_3|_{i_3=k+1-i_1+i_2}\,, & \text{if } 0 \le k \le i_1-i_2, \\ C_3|_{i_3=k-i_1+i_2}\,, & \text{if } i_1-i_2 < k \le i_1-i_2+k_3-j, \\ 0, & \text{else,} \end{cases}$$

$$C^{**}_{3,2} = \begin{cases} C_3|_{i_3=k-i_1+i_2} + C_3|_{i_3=k+1-i_1+i_2}\,, & \text{if } 0 \le k \le i_1-i_2+k_3-j, \\ C_3|_{i_3=k+1-i_1+i_2}\,, & \text{if } i_1-i_2+k_3-j < k \le i_1-i_2, \\ 0, & \text{else,} \end{cases}$$

$$C^{**}_{4,1} = \begin{cases} C_4|_{i_3=i_1-i_2-k-1} + C_4|_{i_3=i_1-i_2-k-2}\,, & \text{if } 0 \le k \le i_1-i_2-k_3+j-1, \\ C_4|_{i_3=i_1-i_2-k-1}\,, & \text{if } i_1-i_2-k_3+j-1 < k \le i_1-i_2-1, \\ 0, & \text{else,} \end{cases}$$

$$C^{**}_{4,2} = \begin{cases} C_4|_{i_3=i_1-i_2-k-1} + C_4|_{i_3=i_1-i_2-k-2}\,, & \text{if } 0 \le k \le i_1-i_2-1, \\ C_4|_{i_3=i_1-i_2-k-2}\,, & \text{if } i_1-i_2-1 < k \le i_1-i_2-k_3+j-1, \\ 0, & \text{else.} \end{cases}$$

# B Source Code in Matlab

**File01 omega.m**

```
function omega = omega(k,w)
omega=0;
for j=1:k+1
    if j<=length(w)
        i=k-j+2;
        if i<=length(w)
            omega=omega+w(j)*w(i);
        else
        end
     else
    end
end
```

**File02 Q1.m**

```
function Q1=Q1(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w)
Q1=epsilon*w(k1+1)/4*(2*i1+1)*...
(a(i1+1,k2+1)*a(i2+1,j+1)*b(i3+1,k3-j+1)+...
a(i1+1,k2+1)*b(i2+1,j+1)*a(i3+1,k3-j+1)+...
b(i1+1,k2+1)*a(i2+1,j+1)*a(i3+1,k3-j+1)-...
b(i1+1,k2+1)*b(i2+1,j+1)*b(i3+1,k3-j+1));
```

**File03 Q2.m**

```
function Q2=Q2(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w)
Q2=epsilon*w(k1+1)/4*(2*i1+1)*...
(a(i1+1,k2+1)*a(i2+1,j+1)*b(i3+1,k3-j+1)-...
a(i1+1,k2+1)*b(i2+1,j+1)*a(i3+1,k3-j+1)-...
b(i1+1,k2+1)*a(i2+1,j+1)*a(i3+1,k3-j+1)-...
b(i1+1,k2+1)*b(i2+1,j+1)*b(i3+1,k3-j+1));
```

**File04 Q3.m**

```
function Q3=Q3(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w)
Q3=epsilon*w(k1+1)/4*(2*i1+1)*...
(-a(i1+1,k2+1)*a(i2+1,j+1)*b(i3+1,k3-j+1)+...
a(i1+1,k2+1)*b(i2+1,j+1)*a(i3+1,k3-j+1)-...
b(i1+1,k2+1)*a(i2+1,j+1)*a(i3+1,k3-j+1)-...
b(i1+1,k2+1)*b(i2+1,j+1)*b(i3+1,k3-j+1));
```

**File05 Q4.m**

```
function Q4=Q4(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w)
Q4=epsilon*w(k1+1)/4*(2*i1+1)*...
(-a(i1+1,k2+1)*a(i2+1,j+1)*b(i3+1,k3-j+1)-...
a(i1+1,k2+1)*b(i2+1,j+1)*a(i3+1,k3-j+1)+...
b(i1+1,k2+1)*a(i2+1,j+1)*a(i3+1,k3-j+1)-...
b(i1+1,k2+1)*b(i2+1,j+1)*b(i3+1,k3-j+1));
```

**File06 R1.m**

```
function R1=R1(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w)
R1=epsilon*w(k1+1)/4*(2*i1+1)*(-a(i1+1,k2+1)*a(i2+1,j+1)*a(i3+1,k3-j+1)+...
a(i1+1,k2+1)*b(i2+1,j+1)*b(i3+1,k3-j+1)+...
b(i1+1,k2+1)*a(i2+1,j+1)*b(i3+1,k3-j+1)+...
b(i1+1,k2+1)*b(i2+1,j+1)*a(i3+1,k3-j+1));
```

**File07 R2.m**

```
function R2=R2(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w)
R2=epsilon*w(k1+1)/4*(2*i1+1)*(a(i1+1,k2+1)*a(i2+1,j+1)*a(i3+1,k3-j+1)+...
a(i1+1,k2+1)*b(i2+1,j+1)*b(i3+1,k3-j+1)+...
b(i1+1,k2+1)*a(i2+1,j+1)*b(i3+1,k3-j+1)-...
b(i1+1,k2+1)*b(i2+1,j+1)*a(i3+1,k3-j+1));
```

**File08 R3.m**

```
function R3=R3(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w)
R3=epsilon*w(k1+1)/4*(2*i1+1)*(a(i1+1,k2+1)*a(i2+1,j+1)*a(i3+1,k3-j+1)+...
a(i1+1,k2+1)*b(i2+1,j+1)*b(i3+1,k3-j+1)-...
b(i1+1,k2+1)*a(i2+1,j+1)*b(i3+1,k3-j+1)+...
b(i1+1,k2+1)*b(i2+1,j+1)*a(i3+1,k3-j+1));
```

**File09 R4.m**

```
function R4=R4(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w)
R4=epsilon*w(k1+1)/4*(2*i1+1)*(-a(i1+1,k2+1)*a(i2+1,j+1)*a(i3+1,k3-j+1)+...
a(i1+1,k2+1)*b(i2+1,j+1)*b(i3+1,k3-j+1)-...
b(i1+1,k2+1)*a(i2+1,j+1)*b(i3+1,k3-j+1)-...
b(i1+1,k2+1)*b(i2+1,j+1)*a(i3+1,k3-j+1));
```

**File10 homovdp.m**

```
format long;
END = input('>> Which time do you want to terminate?
                \n>> Terminal time = ');
t=0:0.1:END;
epsilon = input('>> What epsilon do we want to choose? \n>> epsilon = ');
n=input('>> Please choose a number n to approximate.
            \n( Note that: n >= 2. )\n>> n = ');
w=zeros(1,n+1);                    % frequency
A=zeros(1,n+1);                    % amplitude
a=zeros(n+1);                      % coefficients of sine
b=zeros(n+1);                      % coefficients of cosine
Q=zeros(n+1);                      % coefficients of sine in the equation
R=zeros(n+1);                      % The coefficients of cosine in the equation
s=sin(t);                          % Constrcut an array s=[sint,sin3t,...,sin(2i+1)t]'
c=cos(t);                          % and c=[cost,cos3t,...,cos(2i+1)t]'
for i=1:n
   s=[s;sin((2*i+1)*t)];
    c=[c;cos((2*i+1)*t)];
end
w(1)=1; % Let w_0 = 1
A(1)=2; % Let b_1,0 = 2
b(1,1)=A(1); % A_0 = b_1,0 = 2
Q(2,1)=1/4*epsilon*b(1,1)^3;       % the coefficient of sin(3t) in EQ
a(1,2)=3/8*Q(2,1);                 % the coefficient of sin(t) in solution
a(2,2)=-1/8*Q(2,1);                % the coefficient of sin(3t)
w(2)=(8-sqrt(64+6*epsilon^2))/6;
A(2)=-3/4*w(2);
b(1,2)=A(2);
```

```
Q(2,2)=1/2*epsilon*(6*b(1,2)-5*w(2));    %the coefficient of sine in 2nd DE
R(2,2)=-3/2*epsilon^2;                   %the cofficient of cosine in 2nd DE
R(3,2)=5/4*epsilon^2;
for m=3:n+1
    for k=1:m-1
       a(1,m)=a(1,m)+(2*k+1)/((2*k+1)^2-1)*Q(k+1,m-1);
       a(k+1,m)=-1/((2*k+1)^2-1)*Q(k+1,m-1);
       b(k+1,m)=-1/((2*k+1)^2-1)*R(k+1,m-1);
    end
    c1=2*w(1)*a(1,2)-epsilon*b(1,1)+1/4*epsilon*b(1,1)^3;
    d1=epsilon*w(1)*(3/4*b(1,1)^2-1);
    c2=2*w(1)*(b(1,1)+b(1,2));
    d2=2*w(1)*w(2);
    e1=0;
    e2=0;
    for k=1:m-1
       e1=e1+omega(k,w)*a(1,m-k+1)-epsilon*w(k+1)*b(1,m-k);
       e2=e2+omega(k,w)*b(1,m-k+1)+epsilon*w(k+1)*a(1,m-k);
    end
    e1=e1-epsilon*w(1)*b(1,m);
    e2=e2+omega(m-1,w)*b(1,1)+epsilon*w(1)*a(1,m);
    for k1=0:m-1 % Q(1,n+1) and R(1,n+1),that is secular term
       for k2=0:m-1-k1
          k3=m-1-k1-k2;
             for j=0:k3
                for i1=0:k2
                   for i2=0:j
                      for i3=0:k3-j
                         if i1+i2+i3+1==0
                            e1=e1-Q1(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
```

```
   e2=e2-R1(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
end
if i1+i2-i3==0
   e1=e1-Q2(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
   e2=e2-R2(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
end
if i1-i2+i3==0
   e1=e1-Q3(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
   e2=e2-R3(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
end
if i1-i2-i3-1==0
   e1=e1-Q4(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
   e2=e2-R4(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
end
if i1+i2+i3+1==-1
   e1=e1+Q1(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
   e2=e2-R1(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
end
if i1+i2-i3==-1
   e1=e1+Q2(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
   e2=e2-R2(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
end
if i1-i2+i3==-1
   e1=e1+Q3(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
   e2=e2-R3(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
end
if i1-i2-i3-1==-1
   e1=e1+Q4(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
   e2=e2-R4(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
end
```

```
                end
            end
        end
    end
  end
end
w(m)=(d1*e2-d2*e1)/(c1*d2-c2*d1);
b(1,m)=(-e1-c1*w(m))/d1;
for j=1:m
  A(m)=A(m)+b(j,m); % A(k-1)=A_(k-2)=sum(b_1,0,b_1,1,...b(1,k-2))
end
for i=1:m % Q(2i+1,m) and R(2i+1,m), for i=1,2,...,m
  Q(i+1,m)=0;
  R(i+1,m)=0;
  for k=1:m-i
    Q(i+1,m)=Q(i+1,m)+omega(k,w)*(2*i+1)^2*a(i+1,m-k+1);
    R(i+1,m)=R(i+1,m)+omega(k,w)*(2*i+1)^2*b(i+1,m-k+1);
  end
  for k=0:m-1-i
    Q(i+1,m)=Q(i+1,m)-epsilon*w(k+1)*(2*i+1)*b(i+1,m-k);
    R(i+1,m)=R(i+1,m)+epsilon*w(k+1)*(2*i+1)*a(i+1,m-k);
  end
  for k1=0:m-1
    for k2=0:m-1-k1
      k3=m-1-k1-k2;
      for j=0:k3
        for i1=0:k2
          for i2=0:j
            for i3=0:k3-j
              if i1+i2+i3+1==i
```

```
   Q(i+1,m)=Q(i+1,m)-Q1(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
   R(i+1,m)=R(i+1,m)-R1(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
end
if i1+i2-i3==i
   Q(i+1,m)=Q(i+1,m)-Q2(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
   R(i+1,m)=R(i+1,m)-R2(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
end
if i1-i2+i3==i
   Q(i+1,m)=Q(i+1,m)-Q3(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
   R(i+1,m)=R(i+1,m)-R3(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
end
if i1-i2-i3-1==i
   Q(i+1,m)=Q(i+1,m)-Q4(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
   R(i+1,m)=R(i+1,m)-R4(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
end
if i1+i2+i3+1==-i-1
   Q(i+1,m)=Q(i+1,m)+Q1(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
   R(i+1,m)=R(i+1,m)-R1(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
end
if i1+i2-i3==-i-1
   Q(i+1,m)=Q(i+1,m)+Q2(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
   R(i+1,m)=R(i+1,m)-R2(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
end
if i1-i2+i3==-i-1
   Q(i+1,m)=Q(i+1,m)+Q3(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
   R(i+1,m)=R(i+1,m)-R3(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
end
if i1-i2-i3-1==-i-1
   Q(i+1,m)=Q(i+1,m)+Q4(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
   R(i+1,m)=R(i+1,m)-R4(k1,k2,k3,i1,i2,i3,j,a,b,epsilon,w);
```

```
                    end
                end
            end
        end
    end
  end
end
a1=sum(a');
b1=sum(b');
v=a1*s+b1*c;
W=sum(w); % the sum of w(1)=w_0, w(2)=w_1, ..., w(n+1)=w_n
AL=sum(A);
T=2*pi/W;
[T,AL]
plot(t,v,'o-')
hold on
```

**File11 Phase Plane.mv**

```
> k:=1:
> deq1:=diff(x(t),t)=y:
> deq2:=diff(y(t),t)=-x+k*(1-x^2)*y:
> with(DEtools):
> DEplot([deq1,deq2], [x,y], t=0..25, x=-2.5..2.5, y=-20..20,
                {[x(0)=-2, y(0)=15]}, stepsize=0.01, linecolor=black, thickness=2,
                 arrows=none, method=rkf45);
```

**File12 LienardPlane.mv**

```
> k:=100:
> deq1:=diff(x(t),t)=y+k*(x-1/3*x^3):
> deq2:=diff(y(t),t)=-x:
> with(DEtools):
> DEplot([deq1,deq2], [x,y], t=0..250, x=-2..2, y=-100..100,
                {[x(0)=0, y(0)=1]}, stepsize=0.1,
                linecolor=black,thickness=2, arrows=none,method=rkf45);
```