1 Introduction

In this thesis, we shall consider van der Pol equation

$$\ddot{U} + U = \varepsilon\,(1 - U^2)\,\dot{U}, \quad \varepsilon > 0, \tag{1.1}$$

where $\dot{U} = \dfrac{dU}{dt}$. This equation is a mathematical model of self-sustained oscillations of a triode electric circuit with a cubic current-voltage characteristic in [16], [17]. Because of the nonlinear damping term in $\dot{U}$, it shows that if the voltage is large ($|U| > 1$), the damping is positive and decreases the energy of the system. While the voltage is small ($|U| < 1$), the damping is negative and increases the energy. That is, this phenomenon of oscillation just describes the periodic behavior of the voltage U. This characteristic can be observed numerically from the following phase plane analysis. Note that (1.1) can be written in the system of two first order differential equations as (1) phase plane or (2) Liénard plane [10], [16], that is,

$$(1) \quad \begin{cases} \dot{U} = V, \\ \dot{V} = -U + \varepsilon\,(1 - U^2)\,V \end{cases} \qquad \text{for small } \varepsilon > 0, \tag{1.2a}$$

or

$$(2) \quad \begin{cases} \dot{U} = V + \varepsilon\left(U - \dfrac{1}{3}U^3\right), \\ \dot{V} = -U, \end{cases} \qquad \text{for all } \varepsilon > 0. \tag{1.2b}$$

Besides, we see that the orbit approaches a closed curve for a long time. On the other hand, the analytical study on the existence and uniqueness of the limit cycle has been discussed by many authors [12], [15]. In general, to find the approximation of the limit cycle, many perturbation methods are used by [1], [3], [2], [14] and [16] for small parameter ε. However, for large parameter ε, some suitable transformation must be employed, e.g. Shohat transformation [2], [15]. In contrast to traditional perturbation methods, there is a different approach-homotopy perturbation method [5], [6], [8]. This method has been used to deal with Duffing equation with nonlinearity of high order [6], [8], Lighthill equation [5], [13] and other nonlinear equations [9], [11], [14]. From their results, it shows that this method is effective and simple. However, we find that it is not really applicable to van der Pol equation (1.1), because the secular terms can not be eliminated in the computation.

In this thesis, we shall propose a modified homotopy perturbation method to obtain the approximate solution of (1.1). Instead of (1.1), we consider (4.14) and construct its homotopy with an embedding parameter $p \in [0, 1]$. We also devise an algorithm to get the approximate amplitude and frequency of the limit cycle of (1.1). Some numerical results are obtained with good accuracy by the above algorithm. In addition, this powerful method also can deal with the forced van der Pol equation and van der Pol - Duffing equation.

The content of this thesis is organized as follows. In Section 2, we first introduce the existence and uniqueness of the stable limit cycle of (1.1). In Section 3, several traditional perturbation methods are illustrated for the comparisons in Section 5. In Section 4, we first introduce the basic idea of the homotopy perturbation method and indicate that this method is useful but not applicable to (1.1). Then, we modify this homotopy method by introducing the transformations (4.12) and (4.13). Besides, the computing algorithm is given. In Section 5, by the previous algorithm with Matlab 7, we get the approximate frequency and amplitude and give some tables for comparison between our results with other results in Section 3. Then we can find that our results are more accurate than the results obtained in Section 3. Finally, we give a discussion about the advantage of the proposed method. Besides, we also obtain some approximate results for the solutions of the forced van der Pol equation and van der Pol - Duffing equation by this powerful method.