# 2 Existence and Uniqueness of Stable Limit Cycle

From the phase plane analysis, we see that the trajectory approaches a limit cycle as $t \to \infty$, for different $\varepsilon > 0$ from Figure 1 and Figure 2. Two program file 11 and 12 written in Maple language are given in Appendix B.

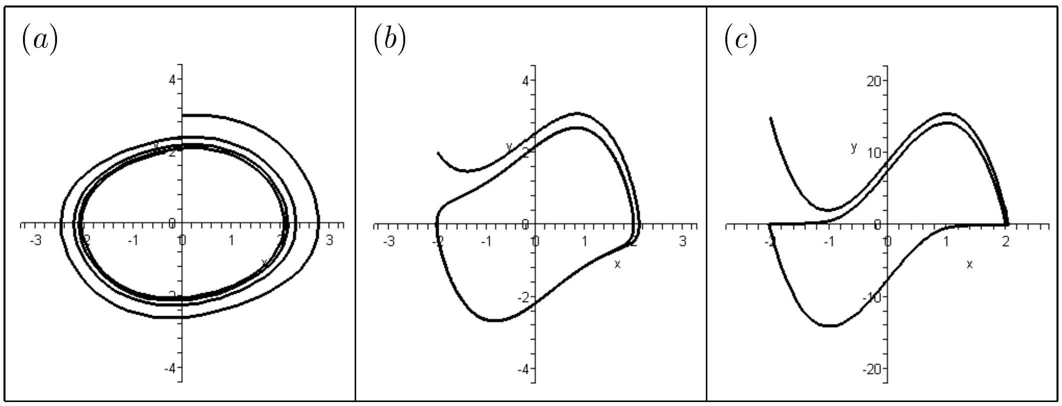


**Figure 1** Phase plane $(1.2a) : (a)\, \varepsilon = 0.1; (b)\varepsilon = 1; (c)\varepsilon = 10.$

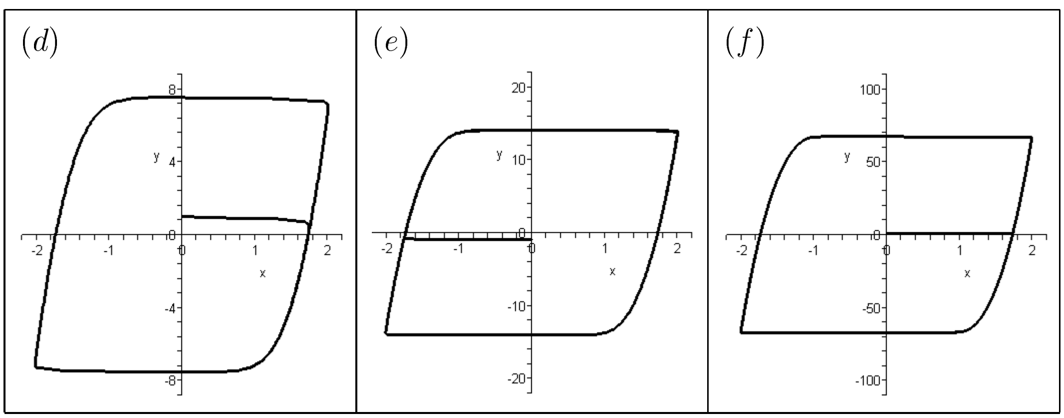


**Figure 2** Liénard plane $(1.2b) : (c)\, \varepsilon = 10; (d)\, \varepsilon = 20; (e)\varepsilon = 100.$

Before proving the existence and uniqueness of stable limit cycle of (1.1), we list some preliminary theorems for references.

**Theorem 1** *[12] Consider the equation of Liénard*

$$\ddot{x} + f(x)\,\dot{x} + x = 0 \tag{2.3}$$

*with $f(x)$ Lipschitz-continuous on $\mathbb{R}$. Assume that if*

$(a)$ *$F(x) = \int_0^x f(s)\,ds$ is an odd function and $F(x) \to \infty$ as $x \to \infty$.*

$(b)$ *There exists a constant $\beta > 0$ such that,*

*$F(x) > 0$ and is monotonically increasing $x \in (\beta, \infty)$.*

$(c)$ *There exists a constant $\alpha > 0$ such that for $0 < x < \alpha$, $F(x) < 0$,*

*then* (2.3) *has at least one periodic solution. Moreover if $\alpha = \beta$, then there exists only one limit cycle.*

**Theorem 2** *[15]Consider the two-dimensional system*

$$\begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y) \end{cases} \tag{2.4}$$

*with $T$-periodic solution $x = \phi(t)$ and $y = \psi(t)$. If*

$$\int_0^T \left[P_x(\phi(t), \psi(t)) + Q_y(\phi(t), \psi(t))\right] dt < 0, \tag{2.5}$$

*then the periodic solution of* (2.4) *is stable.*

**Corollary 3** (1.1) *has the only one stable limit cycle, for $\varepsilon > 0$.*

***Proof***. (1) Let

$$f(x) = \varepsilon\left(x^2 - 1\right),$$

then

$$F(x) = \int_0^x \varepsilon\left(x^2 - 1\right) ds = \varepsilon\left(\frac{1}{3}x^3 - x\right).$$

With $\alpha = \beta = \sqrt{3}$, the conditions $(a)$-$(c)$ of Theorem 1 are satisfied. Then, by Theorem 1 (1.1) has the only one limit cycle.

(2) Let

$$\begin{cases} P(x, y) = x, \\ Q(x, y) = \varepsilon\left(1 - x^2\right) y. \end{cases} \tag{2.6}$$

From experiences [13], the first-order approximation of the periodic solution of (1.1) by

$$\phi(t) = 2\cos t \quad \text{and} \quad \psi(t) = -2\sin t$$

with the period $2\pi$. From (2.5) and (2.6), we have

$$\int_0^{2\pi} \varepsilon \left(1 - (2\cos t)^2\right) dt = -2\pi < 0, \quad \text{for } \varepsilon > 0.$$

Therefore, the periodic solution of (1.1) is stable by Theorem 2. ❑