3 Some Traditional Perturbation Results

In the last two decades, many approximate perturbation methods have been applied to (1.1). In the following, we illustrate some approximate results by using different perturbation methods such as (1) Poincaré-Lindstedt Method, (2) Shohat transformation and (3) the method of multi-time scale.

(1) Poincaré-Lindstedt Method [14]

The essence of the method is to introduce a transformation of the independent variable $\theta = \omega(\varepsilon)\, t$. (1.1) is reduced to

$$\omega^2 \frac{d^2 u}{d\theta^2} + u = \omega\varepsilon \left(1 - u^2\right) \frac{du}{d\theta}. \tag{3.1}$$

Let u and ω be expanded in power series of ε. By substituting these power series into (3.1) and equating the coefficients of the same power of ε, we can get a system of second order linear differential equations. Then a suitable i-th coefficient of ω can be determined to eliminate the secular terms in each equation.

The approximate expansions of $A(\varepsilon)$, $\omega(\varepsilon)$ and u of (3.1) are given by Dadfar, Geer and Andersen [1], [3] as follows:

$$\begin{aligned} A(\varepsilon) &= 2 + \frac{1}{96}\varepsilon^2 - \frac{1033}{552960}\varepsilon^4 + \frac{1019689}{55738368000}\varepsilon^6 + \frac{9835512276689}{157315969843200000}\varepsilon^8 \\ &\quad - \frac{58533181813182818069}{73261417892098867200000000}\varepsilon^{10} + O\left(\varepsilon^{12}\right), \end{aligned} \tag{3.2}$$

$$\begin{aligned} \omega(\varepsilon) &= 1 - \frac{1}{16}\varepsilon^2 + \frac{17}{3072}\varepsilon^4 + \frac{35}{884736}\varepsilon^6 - \frac{678899}{5096079360}\varepsilon^8 + \frac{28160413}{2293235712000}\varepsilon^{10} \\ &\quad + \frac{16729607288111}{3698530556313600000}\varepsilon^{12} + O\left(\varepsilon^{14}\right) \end{aligned} \tag{3.3}$$

and

$$
\begin{aligned}
u(\theta,\varepsilon) &= 2\cos\theta + \left(\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta\right)\varepsilon + \left(-\frac{1}{8}\cos\theta + \frac{3}{16}\cos 3\theta - \frac{5}{96}\cos 5\theta\right)\varepsilon^2 \\
&\quad + \left(-\frac{7}{256}\sin\theta + \frac{21}{256}\sin 3\theta - \frac{35}{576}\sin 5\theta + \frac{7}{576}\sin 7\theta\right)\varepsilon^3 \\
&\quad + \left(\frac{73}{12288}\cos\theta - \frac{47}{1536}\cos 3\theta + \frac{1085}{27648}\cos 5\theta \right. \\
&\qquad \left. - \frac{2149}{110592}\cos 7\theta + \frac{61}{20480}\cos 9\theta\right)\varepsilon^4 + O\left(\varepsilon^5\right), \qquad (3.4)
\end{aligned}
$$

for small $\varepsilon > 0$, respectively.

In order to improve Poincaré-Lindstedt method, we present Shohat transformation which is available not only for small ε but for all $\varepsilon > 0$.

(2) Shohat Transformation [15]

By introducing a new expansion parameter $\lambda = \lambda(\varepsilon)$, defined by the transformation

$$
\lambda(\varepsilon) = \frac{\varepsilon}{1+\varepsilon}, \quad \text{for} \quad \varepsilon \geq 0,
$$

where $\lambda \in [0, 1)$, we obtain the solution

$$
\begin{aligned}
u &= 2\cos\theta + \left(\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta\right)\lambda \\
&\quad + \left(\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta - \frac{1}{8}\cos\theta + \frac{3}{16}\cos 3\theta - \frac{5}{96}\cos 5\theta\right)\lambda^2 \\
&\quad + O\left(\lambda^3\right) \qquad (3.5)
\end{aligned}
$$

and the asymptotic expansion for the product

$$
\varepsilon\omega = \lambda + \lambda^2 + \frac{15}{16}\lambda^3 + \frac{13}{16}\lambda^4 + O\left(\lambda^5\right), \qquad (3.6)
$$

which has a finite nonzero limit as $\varepsilon \to \infty$ with $\omega \to 0$. Furthermore, by the improvement [2], another transformation is introduced by

$$
\lambda\left(\varepsilon^2\right) = \frac{\varepsilon^2}{1+\varepsilon^2}, \quad \text{for} \quad \varepsilon > 0
$$

and an improved result is given by

$$(\varepsilon\omega)^2 = \lambda + \frac{7}{8}\lambda^2 + \frac{1175}{1536}\lambda^3 + \frac{296693}{442368}\lambda^4 + O\left(\lambda^5\right), \tag{3.7}$$

which is compared in Section 5.

From (3.4) and (3.5), we have that both Poincaré-Lindstedt Method and Shohat transformation yield a constant value for the amplitude. In addition, we introduce a popular perturbation method which give the variation of the amplitude.

(3) Method of Multi-time Scale [14]

Assume the a solution of (1.1) is given by an asymptotic representation of the form

$$U(t,\varepsilon) = \sum_{n=0}^{m} \varepsilon^n U_n(t_0, t_1, \ldots, t_m) + O\left(\varepsilon^{m+1}\right),$$

where

$$t_n = \varepsilon^n t.$$

By determining a first-order uniform expansion, we obtain

$$a = \frac{2A}{\left[A^2 + \left(4 - A^2\right)e^{-\varepsilon t}\right]^{1/2}},$$

where $U(0) = A$ and $\dot{U}(0) = 0$. Hence, the solution of (1.1) is given by

$$U(t,\varepsilon) = \frac{2A}{\left[A^2 + \left(4 - A^2\right)e^{-\varepsilon t}\right]^{1/2}} \cos t + O(\varepsilon).$$

Note that as $t \to \infty$, the amplitude $a \to 2$.

Besides, because $t_1 = \varepsilon t$, the result is agreement with that of the method of averaging [14].