4 Modified Homotopy Perturbation Method

Among many perturbation methods [1], [2], [3], they are usually based on a small parameter so that the approximations can be expanded in a series of small parameter. This assumption greatly restricts its applications. Here after, we introduce some basic concept of homotopy perturbation technique [5], [6], [8] which does not depend on a small parameter in the equation.

First, we consider the nonlinear oscillator

$$A(u) - f(r) = 0, \quad r \in \Lambda \tag{4.1}$$

with boundary conditions

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma,$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known function and Γ is the boundary of the domain Λ. Generally, A can be divided into two parts L and N, where L is linear, while N is nonlinear. i.e.

$$A(u) = L(u) + N(u).$$

We construct a homotopy $H(v;p)\colon \Lambda \times [0,1] \to \mathbb{R}$, which satisfies

$$\begin{aligned} H(v;p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[A(v) - f(r)\right] = 0, \\ p \in [0,1], \quad r \in \Lambda \end{aligned} \tag{4.2a}$$

or

$$H(v;p) = L(v) - L(u_0) + pL(u_0) + p\left[N(v) - f(r)\right] = 0, \tag{4.3a}$$

where u_0 is an initial approximation of (4.1) which satisfies the boundary condition. $(4.2a)$ and $(4.3a)$ are both called the perturbation equation with an embedding parameter p and it can be solved by the traditional perturbation techniques. Furthermore, from $(4.2a)$ and $(4.3a)$, we have

$$H(v;0) = L(v) - L(u_0), \tag{4.4}$$

$$H(v;1) = A(v) - f(r). \tag{4.5}$$

The changing process of p from zero to unity is just that $v(r,p)$ changes from u_0 to $u(r)$. In topology, this is called the deformation, and (4.4) and (4.5) are homotopic each other. By using the embedding parameter p as a "small parameter", the approximation of (4.1) can be expressed as a power series of p, i.e.

$$v = \sum_{i=0}^{\infty} v_i p^i.$$

When $p = 1$, it results the approximate solution of (4.1) given by

$$u = \lim_{p \to 1} v = \sum_{i=0}^{\infty} v_i.$$

To apply the homotopy perturbation method in (1.1), we construct a homotopy of (1.1) as follows:

$$(1-p)\left(\frac{d^2 v}{dt^2} + v - \frac{d^2 u_0(t)}{dt^2} - u_0(t)\right) + p\left(\frac{d^2 v}{dt^2} + v + \varepsilon\left(v^2 - 1\right)\frac{dv}{dt}\right) = 0, \quad p \in [0,1]. \tag{4.6}$$

Before finding the approximate solution of (4.6), we observe the deformation of the solution of (4.6) by changing the parameter p from 0 to 1, as in Figure 3:

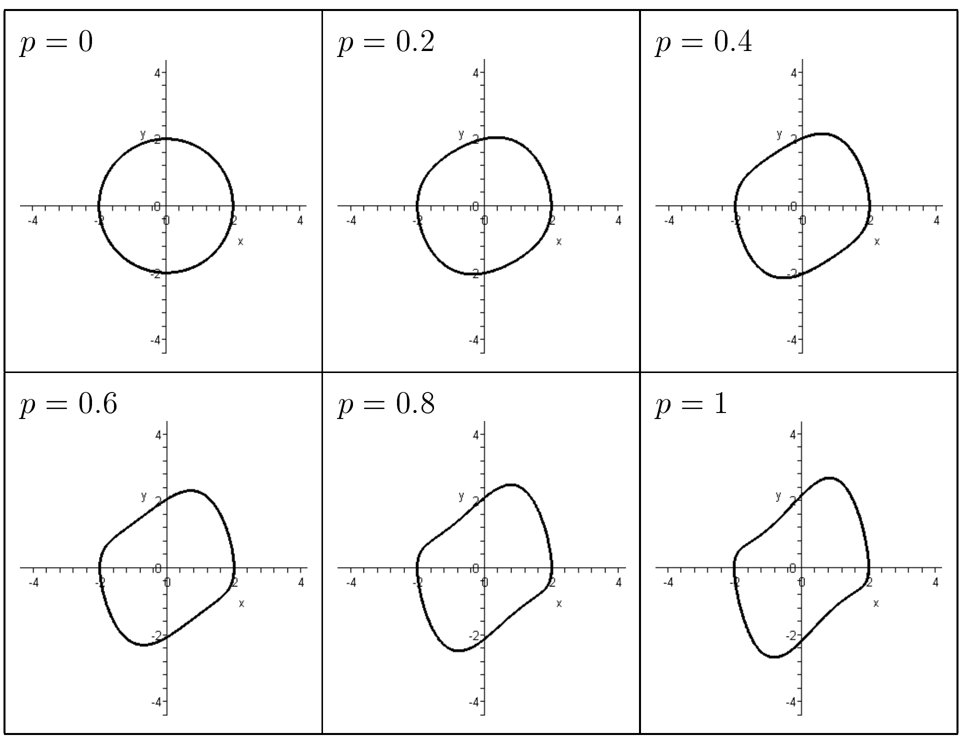

Figure 3 Phase plane of (4.6) with $\varepsilon = 1, v(0) = 2, v'(0) = 0$ and $u_0(t) = 2\cos t$.

Let

$$v\left(t,p\right)=\sum_{i=0}^{\infty}v_i(t)p^i \tag{4.7}$$

and the initial approximate solution of (1.1) be

$$u_0\left(t\right)=A\cos\omega\left(\varepsilon\right)t, \tag{4.8}$$

where $\omega\left(\varepsilon\right)$ is unknown and satisfies $\omega\left(0\right)=1$. By substituting (4.7) into (4.6) and equating the coefficients of the same power of p, we first have

$$v_0\left(t\right)=u_0\left(t\right)=A\cos\omega t. \tag{4.9}$$

Substituting (4.8) and (4.9) into the coefficient of p^1, we have

$$v_1''+v_1=\frac{1}{4}A^3\varepsilon\omega\sin 3\omega t+A\left(\omega^2-1\right)\cos\omega t+\frac{1}{4}\varepsilon\omega A\left(A^2-4\right)\sin\omega t.$$

To eliminate the secular terms, we take $\omega=1$, $A=2$ and obtain the solution

$$v_1\left(t\right)=\frac{3}{4}\varepsilon\sin t-\frac{1}{4}\varepsilon\sin 3t. \tag{4.10}$$

Substituting (4.8), (4.9) and (4.10) into the coefficient of p^2, we have

$$\begin{aligned} v_2''+v_2 &= \frac{1}{4}\varepsilon^2\cos t-\frac{3}{2}\varepsilon^2\cos 3t+\frac{5}{4}\varepsilon^2\cos 5t, \\ v_2(0) &= 0, \\ v_2'(0) &= 0 \end{aligned}$$

and the solution is found by

$$v_2\left(t\right)=\frac{1}{8}t\varepsilon^2\sin t-\frac{13}{96}\varepsilon^2\cos t+\frac{3}{16}\varepsilon^2\cos 3t-\frac{5}{96}\varepsilon^2\cos 5t.$$

Hence, the approximation of (1.1) is

$$\begin{aligned} U &\approx 2\cos t+\left(\frac{3}{4}\varepsilon\sin t-\frac{1}{4}\varepsilon\sin 3t\right) \\ &\quad +\left(\frac{1}{8}t\varepsilon^2\sin t-\frac{13}{96}\varepsilon^2\cos t+\frac{3}{16}\varepsilon^2\cos 3t-\frac{5}{96}\varepsilon^2\cos 5t\right), \end{aligned} \tag{4.11}$$

which breaks down for $t\geq O\left(\varepsilon^{-2}\right)$ because of the existence of the secular term as $t^n\sin t$ or $t^n\cos t$.

In order to obtain an uniform expansion for the solution of (1.1), we give the modification of this homotopy perturbation method for (1.1). First, we let

$$\theta = \omega(\varepsilon)\, t, \tag{4.12}$$

where $\omega(\varepsilon) = 2\pi/T(\varepsilon)$ and $T(\varepsilon)$ are the (unknown) frequency and period of the limit cycle respectively. Note that $\omega(0) = 1$. Note that we let

$$u(\theta;\varepsilon) = U(t;\varepsilon), \tag{4.13}$$

then u is periodic in θ with period 2π. (1.1) becomes

$$\omega^2 \frac{d^2u}{d\theta^2} + u = \omega\varepsilon\left(1-u^2\right)\frac{du}{d\theta}. \tag{4.14}$$

In addition, we impose the phase condition that u have a maximum at $\theta = 0$, that is,

$$\frac{d}{d\theta}u(0;\varepsilon) = 0, \qquad u(0;\varepsilon) = A(\varepsilon) > 0. \tag{4.15}$$

Construct a homotopy $H(v;p)\colon \Lambda \times [0,1] \to \mathbb{R}$, which satisfies

$$H(v;p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[L(v) + \omega\varepsilon\left(v^2-1\right)\frac{dv}{d\theta}\right] = 0, \tag{4.16}$$

where $p \in [0,1]$, $L(u) = \omega^2 \dfrac{d^2u}{d\theta^2} + u$ and u_0 is an initial approximation of (4.14), satisfying the boundary conditions. Now we are going to find the solution $v(\theta;\varepsilon;p)$, $\omega(\varepsilon;p)$ and $A(\varepsilon;p)$ in formal series of power p.

$$v(\theta;\varepsilon;p) = \sum_{i=0}^{\infty} v_i(\theta;\varepsilon)\, p^i, \tag{4.17}$$

$$\omega(\varepsilon;p) = \sum_{i=0}^{\infty} \omega_i(\varepsilon)\, p^i, \tag{4.18}$$

$$A(\varepsilon;p) = \sum_{i=0}^{\infty} A_i(\varepsilon)\, p^i. \tag{4.19}$$

From (4.16), we see that when $p=1$, (4.16) is reduced to (4.14), so we have

$$u(\theta;\varepsilon) = \lim_{p\to 1} v(\theta;\varepsilon;p) = \sum_{i=0}^{\infty} v_i(\theta;\varepsilon), \tag{4.20}$$

$$\omega(\varepsilon) = \lim_{p\to 1} \omega(\varepsilon;p) = \sum_{i=0}^{\infty} \omega_i(\varepsilon), \tag{4.21}$$

$$A(\varepsilon) = \lim_{p\to 1} A(\varepsilon;p) = \sum_{i=0}^{\infty} A_i(\varepsilon). \tag{4.22}$$

Substituting (4.17),(4.18) into (4.16) and equating the coefficients of terms with the equal power in p, we obtain

$$\omega_0^2 v_n''(\theta;\varepsilon) + v_n(\theta;\varepsilon) = F_n(\theta;\varepsilon), \quad v_n(0;\varepsilon) = A_n(\varepsilon), \quad v_n'(0;\varepsilon) = 0, \tag{4.23}$$

where $v' = \dfrac{dv}{d\theta}$,

$$F_0 = \Omega_0 u_0'' + u_0, \tag{4.24}$$

$$F_1 = -\Omega_1 v_0'' + \Omega_1 u_0'' - \Omega_0 u_0'' - u_0 + \varepsilon\omega_0 v_0' (1 - f_0), \tag{4.25}$$

$$\vdots$$

$$\begin{aligned} F_n = & -\sum_{k=1}^{n} \Omega_k v_{n-k}'' + \Omega_n u_0'' - \Omega_{n-1} u_0'' \\ & + \sum_{k=0}^{n-1} \varepsilon\omega_k v_{n-k-1}' - \sum_{k_1+k_2+k_3=n-1} \varepsilon\omega_{k_1} v_{k_2}' f_{k_3}, \quad \text{for} \quad n \geq 2, \end{aligned} \tag{4.26}$$

and

$$\Omega_k = \sum_{i=0}^{k} \omega_i \omega_{k-i}, \tag{4.27}$$

$$f_k = \sum_{j=0}^{k} v_j v_{k-j}. \tag{4.28}$$

Let the initial approximate solution of (4.14) be

$$u_0(\theta;\varepsilon) = A_0(\varepsilon)\cos\theta, \tag{4.29}$$

where A_0 is to be determined in the equation (4.23) with $n=1$ and $A_0(0) = A(0)$. From (4.23) with $n=0$ we obtain

$$\omega_0^2 v_0'' + v_0 = A_0\left(1 - \omega_0^2\right)\cos\theta.$$

Since we want v_0 to be periodic, we required that the coefficients of $\sin\theta$ and $\cos\theta$ in (4.24) vanish. Thus we see that

$$\omega_0 = 1 \tag{4.30}$$

and

$$v_0(\theta;\varepsilon) = u_0(\theta;\varepsilon) = A_0\cos\theta. \tag{4.31}$$

By substituting (4.29),(4.31) into (4.23) with $n = 1$, we obtain

$$v_1'' + v_1 = \frac{1}{4}\varepsilon A_0\left(A_0^2 - 4\right)\sin\theta + \frac{1}{4}\varepsilon A_0^3\sin 3\theta. \tag{4.32}$$

To eliminate the secular terms we require

$$A_0 = 2. \tag{4.33}$$

Hence, the solution is

$$v_1(\theta;\varepsilon) = \frac{3}{4}\varepsilon\sin\theta - \frac{1}{4}\varepsilon\sin 3\theta + b_{1,1}\cos\theta, \tag{4.34}$$

where $b_{1,1}$ is equal to A_1that is to be determined in the equation (4.23) with $n = 2$. Continuing this procedure we find that

$$\omega_1 = \frac{8 \pm \sqrt{64 + 6\varepsilon^2}}{6}, \tag{4.35}$$

$$b_{1,1} = -\frac{3}{4}\omega_1. \tag{4.36}$$

Because of (4.15) and (4.20), we take $b_{1,1} = A_1 > 0$, that is,

$$\omega_1 = \frac{8 - \sqrt{64 + 6\varepsilon^2}}{6}. \tag{4.37}$$

Then we get

$$\begin{aligned} v_2(\theta;\varepsilon) &= \left(\frac{19}{64}\sqrt{6\varepsilon^2 + 64} - \frac{19}{8}\right)\varepsilon\sin\theta + \left(\frac{19}{24} - \frac{19}{192}\sqrt{6\varepsilon^2 + 64}\right)\varepsilon\sin 3\theta \\ &\quad + b_{1,2}\cos\theta + \frac{3}{16}\varepsilon^2\cos 3\theta - \frac{5}{96}\varepsilon^2\cos 5\theta, \end{aligned} \tag{4.38}$$

where $b_{1,2}\left(= A_2 - \frac{13}{96}\varepsilon^2\right)$ will be determined later in the equation (4.23) with $n = 3$. From the operational analysis, there are some important observations as follows:

1. Developing the products on the right hand side of (4.23), we obtain an interesting property. Starting from the equation v_0, (4.26) can be reduced to the simplest form composed of $\sin(2i+1)\theta$ and $\cos(2i+1)\theta$, for $i = 0, 1, \ldots, n$. Furthermore, the maximum index of the coefficient of the angle θ of cos is one more than the sin's if n is even, whereas the situation is contrary if n is odd. Besides, the solution v_n corresponding to (4.26) also has the same property, that is, for $n \geq 0$ the solution v_n is of the form:

$$v_{2k} = \sum_{i=0}^{2k-1} a_{2i+1,2k} \sin(2i+1)\theta + \sum_{i=0}^{2k} b_{2i+1,2k} \cos(2i+1)\theta$$

or

$$v_{2k+1} = \sum_{i=0}^{2k+1} a_{2i+1,2k+1} \sin(2i+1)\theta + \sum_{i=0}^{2k} b_{2i+1,2k+1} \cos(2i+1)\theta.$$

2. By substituting the obtained solutions $v_0, v_1, \ldots, v_{n-1}$ into the equation (4.23) of v_n, we focus on finding out the coefficients of $\sin\theta$ and $\cos\theta$ to eliminate the secular terms. Thus we obtain a system of equations for ω_{n-1} and $b_{1,n-1}$. Moreover, when $n \geq 3$, it becomes a system of two linear equations

$$\begin{aligned} c_1\omega_{n-1} + d_1 b_{1,n-1} + e_1 &= 0, \\ c_2\omega_{n-1} + d_2 b_{1,n-1} + e_2 &= 0, \end{aligned}$$

where c_1, c_2, d_1, d_2, e_1 and e_2 are constants which are defined by (4.49)-(4.54). Note that this system is solvable with the solutions

$$\omega_{n-1} = \frac{d_1 e_2 - d_2 e_1}{c_1 d_2 - c_2 d_1}, \tag{4.39}$$

$$b_{1,n-1} = \frac{c_2 e_1 - c_1 e_2}{c_1 d_2 - c_2 d_1}. \tag{4.40}$$

Therefore, the periodic solution v_n can be written in the general form

$$v_n(\theta; \varepsilon) = \sum_{i=0}^{n} a_{2i+1,n}(\varepsilon) \sin(2i+1)\theta + \sum_{i=0}^{n} b_{2i+1,n}(\varepsilon) \cos(2i+1)\theta, \tag{4.41}$$

where $a_{2n+1,n} = 0$ if n is even and $b_{2n+1,n} = 0$ if n is odd. From the phase condition (4.15), we have

$$v_n(0;\varepsilon) = \sum_{i=0}^{n} b_{2i+1,n} = A_n, \tag{4.42}$$

$$v_n'(0;\varepsilon) = \sum_{i=0}^{n} (2i+1)\, a_{2i+1,n} = 0. \tag{4.43}$$

Besides, we observe that $F_n(\theta;\varepsilon)$ can be expanded in the form

$$F_n(\theta;\varepsilon) = \sum_{i=0}^{n} Q_{2i+1,n}(\varepsilon)\sin(2i+1)\theta + \sum_{i=0}^{n} R_{2i+1,n}(\varepsilon)\cos(2i+1)\theta, \quad \text{for } n \geq 0, \tag{4.44}$$

where for $i = 0, 1, 2, \ldots, n$,

$$Q_{2i+1,n} = \sum_{k=1}^{n-i} \Omega_k (2i+1)^2 a_{2i+1,n-k} + \sum_{k=0}^{n-i-1} \varepsilon\omega_k (2i+1)\, b_{2i+1,n-k-1} - \sum_{k_1+k_2+k_3=n-1} \left[\sum_{j=0}^{k_3} \sum_{i_1=0}^{k_2} \sum_{i_2=0}^{j} \varepsilon\omega_{k_1} G_1(i_1, i_2, i, k_2, k_3) \right], \tag{4.45}$$

$$R_{2i+1,n} = \begin{cases} R^*_{2i+1,n} + \Omega_{n-1} b_{1,0}\cos\theta, & \text{if } i = 0, \\ R^*_{2i+1,n}, & \text{else,} \end{cases} \tag{4.46}$$

here $k_3 = n - 1 - k_1 - k_2$,

$$R^*_{2i+1,n} = \sum_{k=1}^{n-i} \Omega_k (2i+1)^2 b_{2i+1,n-k} + \sum_{k=0}^{n-i-1} \varepsilon\omega_k (2i+1)\, a_{2i+1,n-k-1} - \sum_{k_1+k_2+k_3=n-1} \left[\sum_{j=0}^{k_3} \sum_{i_1=0}^{k_2} \sum_{i_2=0}^{j} \varepsilon\omega_{k_1} G_2(i_1, i_2, i, k_2, k_3) \right].$$

Moreover, $G_1$and G_2 depending on i_1, i_2, i, k_2, k_3 are defined by

$$G_1 = \sum_{k=i_1+i_2+1}^{i_1+i_2+k_3-j+1} S_5 + \sum_{k=0}^{\max(i_1+i_2, |i_1+i_2-k_3+j|)} S_6 + \sum_{k=0}^{\max(|i_1-i_2+k_3-j|, |i_1-i_2|)} S_7 + \sum_{k=0}^{\max(|i_1-i_2-1|, i_1-i_2-k_3+j-1)} S_8,$$

$$G_2 = \sum_{k=i_1+i_2+1}^{i_1+i_2+k_3-j+1} C_5 + \sum_{k=0}^{\max(i_1+i_2, |i_1+i_2-k_3+j|)} C_6 + \sum_{k=0}^{\max(|i_1-i_2+k_3-j|, |i_1-i_2|)} C_7 + \sum_{k=0}^{\max(|i_1-i_2-1|, i_1-i_2-k_3+j-1)} C_8.$$

Since S_i and C_i, for $i = 5, 6, 7, 8$ and so complicated that are given in the Appendix A. From (4.31), (4.30), (4.33), (4.34), (4.37), (4.36),(4.38), we obtain

$$\omega_0 = 1, \quad a_{1,0} = 0, \quad b_{1,0} = 2, \quad A_0 = 2,$$
$$a_{1,1} = \tfrac{3}{4}\varepsilon, \quad a_{3,1} = -\tfrac{1}{4}\varepsilon, \quad b_{3,1} = 0,$$

$$\omega_1 = \frac{8 - \sqrt{64 + 6\varepsilon^2}}{6}, \quad b_{1,1} = A_1 = \frac{-8 + \sqrt{64 + 6\varepsilon^2}}{8},$$
$$a_{1,2} = -\frac{57}{32}\varepsilon\omega_1, \quad a_{3,2} = \frac{19}{32}\varepsilon\omega_1, \quad a_{5,2} = 0,$$
$$b_{1,2} = A_2 - \frac{13}{96}\varepsilon^2, \quad b_{3,2} = \frac{3}{16}\varepsilon^2, \quad b_{5,2} = -\frac{5}{96}\varepsilon^2,$$

where $b_{1,2}\left(= A_2 - \dfrac{13}{96}\varepsilon^2\right)$ will be determined later in the equation (4.23) with $n = 3$.

Continuing this process, by eliminating the secular terms $Q_{1,n}$ and $R_{1,n}$, we get a linear system of two equation, for $n \geq 3$

$$c_1\omega_{n-1} + d_1 b_{1,n-1} + e_1 = 0, \tag{4.47}$$
$$c_2\omega_{n-1} + d_2 b_{1,n-1} + e_2 = 0, \tag{4.48}$$

where c_1, c_2, d_1, d_2, e_1 and e_2 are computed from (4.45), (4.46) by

$$c_1 = 2\omega_0 a_{1,1} - \varepsilon b_{1,0} + \frac{1}{4}\varepsilon b_{1,0}^3 = \frac{3}{2}\varepsilon, \tag{4.49}$$
$$d_1 = \varepsilon\omega_0\left(\frac{3}{4}b_{1,0}^2 - 1\right) = 2\varepsilon, \tag{4.50}$$
$$c_2 = 2\omega_0\left(b_{1,0} + b_{1,1}\right) = 2 + \frac{1}{4}\sqrt{6\varepsilon^2 + 64}, \tag{4.51}$$
$$d_2 = 2\omega_0\omega_1 = \frac{8}{3} - \frac{1}{3}\sqrt{6\varepsilon^2 + 64} \tag{4.52}$$

and

$$
\begin{aligned}
e_1 &= \sum_{k=1}^{n-1} \Omega_k a_{1,n-k} - \sum_{k=0}^{n-1} \varepsilon\omega_k b_{1,n-k-1} - c_1\omega_{n-1} - d_1 b_{1,n-1} \\
&\quad - \sum_{k_1=0}^{n-1} \left[\sum_{k_2=0}^{n-1-k_1} \sum_{j=0}^{k_3} \sum_{i_1=0}^{k_2} \sum_{i_2=0}^{j} \varepsilon\omega_{k_1} G_1(i_1, i_2, 0, k_2, k_3) \right], \qquad (4.53) \\
e_2 &= \sum_{k=1}^{n-1} \Omega_k b_{1,n-k} + \Omega_{n-1} b_{1,0} + \sum_{k=0}^{n-1} \varepsilon\omega_k a_{1,n-k-1} - c_2\omega_{n-1} - d_2 b_{1,n-1} \\
&\quad - \sum_{k_1=0}^{n-1} \left[\sum_{k_2=0}^{n-1-k_1} \sum_{j=0}^{k_3} \sum_{i_1=0}^{k_2} \sum_{i_2=0}^{j} \varepsilon\omega_{k_1} G_2(i_1, i_2, 0, k_2, k_3) \right] \qquad (4.54)
\end{aligned}
$$

Remark. Equation (4.47) and (4.48) is solvable, because the determinant

$$c_1 d_2 - c_2 d_1 = -\varepsilon\sqrt{6\varepsilon^2 + 64} \neq 0, \quad \text{for} \quad \varepsilon > 0.$$

From the relation (4.42) between A_{n-1} and b_{n-1}, we also obtain

$$A_{n-1} = \sum_{i=0}^{n-1} b_{2i+1,n-1}. \qquad (4.55)$$

For the moment, $\omega_i, a_{2i+1,i}, b_{2i+1,i}$ and A_i are known for $i = 0, 1, 2, \ldots, n-1$. Substituting them into (4.45),(4.46), the coefficients $Q_{2i+1,n}$ and $R_{2i+1,n}$ can also be formulated. Thus, the solution v_n of (4.23) can be found with the coefficients

$$a_{2i+1,n}(\varepsilon) = \begin{cases} \sum\limits_{k=1}^{n} \dfrac{2k+1}{(2k+1)^2 - 1} Q_{2k+1,n}, & \text{if} \quad i = 0, \\ -\dfrac{1}{(2i+1)^2 - 1} Q_{2i+1,n}, & \text{if} \quad 1 \le i \le n \end{cases} \qquad (4.56)$$

and

$$b_{2i+1,n}(\varepsilon) = \begin{cases} A_n + \sum\limits_{k=1}^{n} \dfrac{1}{(2k+1)^2 - 1} R_{2k+1,n}, & \text{if} \quad i = 0, \\ -\dfrac{1}{(2i+1)^2 - 1} R_{2i+1,n}, & \text{if} \quad 1 \le i \le n. \end{cases} \qquad (4.57)$$

Note that $b_{1,n}$ will be determined in the equation (4.23) of v_{n+1}.

Finally, the n-order approximate solution is obtained by

$$u(\theta,\varepsilon)=\sum_{i=0}^{n} v_i(\theta,\varepsilon). \tag{4.58}$$

Following the above procedure, we propose an algorithm which is given as follows.

Algorithm 4

Input: $n, \{\omega_k\}_{k=0}^{1}, \{A_k\}_{k=0}^{1}, \{Q_{2i+1,k}\}_{i=0}^{k}, \{R_{2i+1,k}\}_{i=0}^{k}$, *for* $k=0$ *to* 2, $\{a_{2i+1,k}\}_{i=0}^{k}$, *for* $k=0,1, b_{1,0}$ *and* $\{b_{2i+1,1}\}_{i=0}^{1}$

Output: $\omega_n, A_n, \{a_{2i+1,n}\}_{i=0}^{n}$ *and* $\{b_{2i+1,n}\}_{i=0}^{n}$

for $m=3$ to $n+1$

do $a_{1,m-1}\leftarrow 0$

$b_{1,m-1}\leftarrow 0$

$k\leftarrow 1$

while $k\leq m-1$

do $a_{1,m-1}\leftarrow a_{1,m-1}+\dfrac{2k+1}{(2k+1)^2-1}Q_{2k+1,m-1}$

$a_{2k+1,m-1}\leftarrow -\dfrac{2k+1}{(2k+1)^2-1}Q_{2k+1,m-1}$

$b_{2k+1,m-1}\leftarrow -\dfrac{2k+1}{(2k+1)^2-1}R_{2k+1,m-1}$

$k\leftarrow k+1$

$Q_{1,m}\leftarrow 0$

$R_{1,m}\leftarrow 0$

$c_1\leftarrow 2\omega_0 a_{1,1}-\varepsilon b_{1,0}+\dfrac{1}{4}\varepsilon b_{1,0}^3$

$d_1 \leftarrow \varepsilon\omega_0 \left(\frac{3}{4}b_{1,0}^2 - 1\right)$

$e_1 \leftarrow 0$

$c_2 \leftarrow 2\omega_0 \left(b_{1,0} + b_{1,1}\right)$

$d_2 \leftarrow 2\omega_0\omega_1$

$e_2 \leftarrow 0$

for $\quad k_1 \leftarrow 0$ to $m-1$

do $\quad e_1 \leftarrow e_1 + \Omega\left(k_1\right) a_{1,m-k_1} - \varepsilon\omega_{k_1} b_{1,m-k_1}$

$e_2 \leftarrow e_2 + \Omega\left(k_1\right) b_{1,m-k_1} + \varepsilon\omega_{k_1} a_{1,m-k_1}$

$e_1 \leftarrow e_1 - \varepsilon\omega_0 b_{1,m-1}$

$e_2 \leftarrow e_2 + \varepsilon\omega_0 a_{1,m-1}$

for $\quad k_1 \leftarrow 0$ to $m-1$

for $\quad k_2 \leftarrow 0$ to $m-1-k_1$

do $\quad k_3 \leftarrow m-1-k_1-k_2$

for $\quad j \leftarrow 0$ to k_3

for $\quad i_1 \leftarrow 0$ to k_2

for $\quad i_2 \leftarrow 0$ to j

for $\quad i_3 \leftarrow 0$ to $k_3 - j$

do $\quad$ if $\quad i_1 + i_2 + i_3 + 1 = 0$

then $\quad e_1 \leftarrow e_1 - \varepsilon\omega_{k_1} S_1\left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

$e_2 \leftarrow e_2 - \varepsilon\omega_{k_1} C_1\left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

if $\quad i_1 + i_2 - i_3 = 0$

then $\quad e_1 \leftarrow e_1 - \varepsilon \omega_{k_1} S_2 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

$\quad e_2 \leftarrow e_2 - \varepsilon \omega_{k_1} C_2 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

if $\quad i_1 - i_2 + i_3 = 0$

then $\quad e_1 \leftarrow e_1 - \varepsilon \omega_{k_1} S_3 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

$\quad e_2 \leftarrow e_2 - \varepsilon \omega_{k_1} C_3 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

if $\quad i_1 - i_2 - i_3 - 1 = 0$

then $\quad e_1 \leftarrow e_1 - \varepsilon \omega_{k_1} S_4 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

$\quad e_2 \leftarrow e_2 - \varepsilon \omega_{k_1} C_4 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

if $\quad i_1 + i_2 + i_3 + 1 = -1$

then $\quad e_1 \leftarrow e_1 + \varepsilon \omega_{k_1} S_1 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

$\quad e_2 \leftarrow e_2 - \varepsilon \omega_{k_1} C_1 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

if $\quad i_1 + i_2 - i_3 = -1$

then $\quad e_1 \leftarrow e_1 + \varepsilon \omega_{k_1} S_2 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

$\quad e_2 \leftarrow e_2 - \varepsilon \omega_{k_1} C_2 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

if $\quad i_1 - i_2 + i_3 = -1$

then $\quad e_1 \leftarrow e_1 + \varepsilon \omega_{k_1} S_3 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

$\quad e_2 \leftarrow e_2 - \varepsilon \omega_{k_1} C_3 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

if $\quad i_1 - i_2 - i_3 - 1 = -1$

then $\quad e_1 \leftarrow e_1 + \varepsilon \omega_{k_1} S_4 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

$$e_2 \leftarrow e_2 - \varepsilon \omega_{k_1} C_4 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$$

$$\omega_{m-1} \leftarrow \frac{d_1 e_2 - d_2 e_1}{c_1 d_2 - c_2 d_1}$$

$$b_{1,m-1} \leftarrow \frac{c_2 e_1 - c_1 e_2}{c_1 d_2 - c_2 d_1}$$

$A_{m-1} \leftarrow b_{1,0}$

$k \leftarrow 1$

while $k \leq m-1$

do $A_{m-1} \leftarrow A_{m-1} + b_{1,k}$

$k \leftarrow k+1$

for $i \leftarrow 1$ to m

$Q_{2i+1,m} \leftarrow 0$

$R_{2i+1,m} \leftarrow 0$

for $k_1 \leftarrow 0$ to $m-1$

for $k_2 \leftarrow 0$ to $m-1-k_1$

do $k_3 \leftarrow m-1-k_1-k_2$

for $j \leftarrow 0$ to k_3

for $i_1 \leftarrow 0$ to k_2

for $i_2 \leftarrow 0$ to j

for $i_3 \leftarrow 0$ to $k_3 - j$

do if $i_1 + i_2 + i_3 + 1 = i$

then $Q_{2i+1,m} \leftarrow Q_{2i+1,m} - \varepsilon \omega_{k_1} S_1 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

$$R_{2i+1,m} \leftarrow R_{2i+1,m} - \varepsilon\omega_{k_1} C_1 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$$

if $\quad i_1 + i_2 - i_3 = i$

then $\quad Q_{2i+1,m} \leftarrow Q_{2i+1,m} - \varepsilon\omega_{k_1} S_2 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

$$R_{2i+1,m} \leftarrow R_{2i+1,m} - \varepsilon\omega_{k_1} C_2 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$$

if $\quad i_1 - i_2 + i_3 = i$

then $\quad Q_{2i+1,m} \leftarrow Q_{2i+1,m} - \varepsilon\omega_{k_1} S_3 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

$$R_{2i+1,m} \leftarrow R_{2i+1,m} - \varepsilon\omega_{k_1} C_3 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$$

if $\quad i_1 - i_2 - i_3 - 1 = i$

then $\quad Q_{2i+1,m} \leftarrow Q_{2i+1,m} - \varepsilon\omega_{k_1} S_4 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

$$R_{2i+1,m} \leftarrow R_{2i+1,m} - \varepsilon\omega_{k_1} C_4 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$$

if $\quad i_1 + i_2 + i_3 + 1 = -i - 1$

then $\quad Q_{2i+1,m} \leftarrow Q_{2i+1,m} + \varepsilon\omega_{k_1} S_1 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

$$R_{2i+1,m} \leftarrow R_{2i+1,m} - \varepsilon\omega_{k_1} C_1 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$$

if $\quad i_1 + i_2 - i_3 = -i - 1$

then $\quad Q_{2i+1,m} \leftarrow Q_{2i+1,m} + \varepsilon\omega_{k_1} S_2 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

$$R_{2i+1,m} \leftarrow R_{2i+1,m} - \varepsilon\omega_{k_1} C_2 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$$

if $\quad i_1 - i_2 + i_3 = -i - 1$

then $\quad Q_{2i+1,m} \leftarrow Q_{2i+1,m} + \varepsilon\omega_{k_1} S_3 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$

$$R_{2i+1,m} \leftarrow R_{2i+1,m} - \varepsilon\omega_{k_1} C_3 \left(k_1, k_2, k_3, i_1, i_2, i_3, j, a, b, \varepsilon, \omega\right)$$

if $\quad i_1 - i_2 - i_3 - 1 = -i - 1$

then $Q_{2i+1,m} \leftarrow Q_{2i+1,m} + \varepsilon\omega_{k_1} S_4(k_1,k_2,k_3,i_1,i_2,i_3,j,a,b,\varepsilon,\omega)$

$R_{2i+1,m} \leftarrow R_{2i+1,m} - \varepsilon\omega_{k_1} C_4(k_1,k_2,k_3,i_1,i_2,i_3,j,a,b,\varepsilon,\omega)$

Algorithm 5 *Function* $\Omega(k)$

do $\Omega \leftarrow 0$

for $i = 0$ to k

$\Omega \leftarrow \Omega + \omega_i \omega_{k-i}$

return Ω

Algorithm 6 *Function* $S_1(k_1,k_2,k_3,i_1,i_2,i_3,j,a,b,\varepsilon,\omega)$

return $$\frac{1}{4}\left[a_{2i_1+1,k_2}\left(a_{2i_2+1,j}b_{2i_3+1,k_3-j} + b_{2i_2+1,j}a_{2i_3+1,k_3-j}\right) + b_{2i_1+1,k_2}\left(a_{2i_2+1,j}a_{2i_3+1,k_3-j} - b_{2i_2+1,j}b_{2i_3+1,k_3-j}\right)\right]$$

Algorithm 7 *Function* $S_2(k_1,k_2,k_3,i_1,i_2,i_3,j,a,b,\varepsilon,\omega)$

return $$\frac{1}{4}\left[a_{2i_1+1,k_2}\left(a_{2i_2+1,j}b_{2i_3+1,k_3-j} - b_{2i_2+1,j}a_{2i_3+1,k_3-j}\right) - b_{2i_1+1,k_2}\left(a_{2i_2+1,j}a_{2i_3+1,k_3-j} + b_{2i_2+1,j}b_{2i_3+1,k_3-j}\right)\right]$$

Algorithm 8 *Function* $S_3(k_1,k_2,k_3,i_1,i_2,i_3,j,a,b,\varepsilon,\omega)$

return $$\frac{1}{4}\left[a_{2i_1+1,k_2}\left(-a_{2i_2+1,j}b_{2i_3+1,k_3-j} + b_{2i_2+1,j}a_{2i_3+1,k_3-j}\right) - b_{2i_1+1,k_2}\left(a_{2i_2+1,j}a_{2i_3+1,k_3-j} + b_{2i_2+1,j}b_{2i_3+1,k_3-j}\right)\right]$$

Algorithm 9 *Function* $S_4(k_1,k_2,k_3,i_1,i_2,i_3,j,a,b,\varepsilon,\omega)$

return $$\frac{1}{4}\left[-a_{2i_1+1,k_2}\left(a_{2i_2+1,j}b_{2i_3+1,k_3-j} + b_{2i_2+1,j}a_{2i_3+1,k_3-j}\right) + b_{2i_1+1,k_2}\left(a_{2i_2+1,j}a_{2i_3+1,k_3-j} - b_{2i_2+1,j}b_{2i_3+1,k_3-j}\right)\right]$$

Algorithm 10 *Function* $C_1\left(k_1,k_2,k_3,i_1,i_2,i_3,j,a,b,\varepsilon,\omega\right)$

$$\text{return}\quad \frac{1}{4}\left[a_{2i_1+1,k_2}\left(-a_{2i_2+1,j}a_{2i_3+1,k_3-j}+b_{2i_2+1,j}b_{2i_3+1,k_3-j}\right)\right.$$
$$\left.+b_{2i_1+1,k_2}\left(a_{2i_2+1,j}b_{2i_3+1,k_3-j}+b_{2i_2+1,j}a_{2i_3+1,k_3-j}\right)\right]$$

Algorithm 11 *Function* $C_2\left(k_1,k_2,k_3,i_1,i_2,i_3,j,a,b,\varepsilon,\omega\right)$

$$\text{return}\quad \frac{1}{4}\left[a_{2i_1+1,k_2}\left(a_{2i_2+1,j}a_{2i_3+1,k_3-j}+b_{2i_2+1,j}b_{2i_3+1,k_3-j}\right)\right.$$
$$\left.+b_{2i_1+1,k_2}\left(a_{2i_2+1,j}b_{2i_3+1,k_3-j}-b_{2i_2+1,j}a_{2i_3+1,k_3-j}\right)\right)$$

Algorithm 12 *Function* $C_3\left(k_1,k_2,k_3,i_1,i_2,i_3,j,a,b,\varepsilon,\omega\right)$

$$\text{return}\quad \frac{1}{4}\left[a_{2i_1+1,k_2}\left(a_{2i_2+1,j}a_{2i_3+1,k_3-j}+b_{2i_2+1,j}b_{2i_3+1,k_3-j}\right)\right.$$
$$\left.+b_{2i_1+1,k_2}\left(-a_{2i_2+1,j}b_{2i_3+1,k_3-j}+b_{2i_2+1,j}a_{2i_3+1,k_3-j}\right)\right]$$

Algorithm 13 *Function* $C_4\left(k_1,k_2,k_3,i_1,i_2,i_3,j,a,b,\varepsilon,\omega\right)$

$$\text{return}\quad \frac{1}{4}\left[a_{2i_1+1,k_2}\left(-a_{2i_2+1,j}a_{2i_3+1,k_3-j}+b_{2i_2+1,j}b_{2i_3+1,k_3-j}\right)\right.$$
$$\left.-b_{2i_1+1,k_2}\left(a_{2i_2+1,j}b_{2i_3+1,k_3-j}+b_{2i_2+1,j}a_{2i_3+1,k_3-j}\right)\right]$$

Once the secular terms are found, we solve for the variables ω_{n-1} and b_{n-1}. Then, the remaining terms of (4.26) also can be obtained by the recursive relations. Especially, if we want to seek the n-th order approximation, then we have to compute the secular terms $Q_{1,n+1}$ and $R_{1,n+1}$ in the equation of v_{n+1}. For instance, if we want to obtain the 4-th order approximation, then we will have ω_3 and b_3 in the equation (4.23) with $n=4$.

In order to compare our results with other's, we expand (4.58) in the power series. For small $\varepsilon>0$, (4.58) can be expanded in powers of ε directly. For example, in the third-order approximation,

$$
\begin{aligned}
u(\theta,\varepsilon) &= \left(\frac{19}{64}\sqrt{6\varepsilon^2+64}-\frac{13}{8}\right)\varepsilon\sin\theta+\left(\frac{13}{24}-\frac{19}{192}\sqrt{6\varepsilon^2+64}\right)\varepsilon\sin 3\theta \\
&\quad+\left(1+\frac{21}{256}\varepsilon^2-\frac{31}{48}\sqrt{6\varepsilon^2+64}+\frac{4736+249\varepsilon^2}{96\sqrt{6\varepsilon^2+64}}\right)\cos\theta \\
&\quad+\frac{3}{16}\varepsilon^2\cos 3\theta-\frac{5}{96}\varepsilon^2\cos 5\theta \\
&= 2\cos\theta+\varepsilon\left(\frac{3}{4}\sin\theta-\frac{1}{4}\sin 3\theta\right) \\
&\quad+\varepsilon^2\left(-\frac{1}{8}\cos\theta+\frac{3}{16}\cos 3\theta-\frac{5}{96}\cos 5\theta\right)+O\left(\varepsilon^3\right) \qquad (4.59)
\end{aligned}
$$

with the amplitude A and the frequency ω given by

$$
\begin{aligned}
A &= 1+\frac{167}{768}\varepsilon^2-\frac{31}{48}\sqrt{6\varepsilon^2+64}+\frac{4736+249\varepsilon^2}{96\sqrt{6\varepsilon^2+64}} \\
&= 2+\frac{1}{96}\varepsilon^2+O\left(\varepsilon^4\right) \qquad (4.60)
\end{aligned}
$$

and

$$
\begin{aligned}
\omega &= \frac{95}{9}+\frac{3}{64}\varepsilon^2-\frac{1}{6}\sqrt{6\varepsilon^2+64}-\frac{4736+249\varepsilon^2}{72\sqrt{6\varepsilon^2+64}} \\
&= 1-\frac{1}{16}\varepsilon^2+O\left(\varepsilon^4\right). \qquad (4.61)
\end{aligned}
$$

On the other hand, when ε is large, we introduce the transformation defined by $\lambda=\dfrac{\varepsilon}{1+\varepsilon}$. Then we expand (4.58), (4.22) in power series of λ and obtain the asymptotic expansion for the product

$$
\begin{aligned}
\varepsilon\omega &= \lambda+\lambda^2+\frac{15}{16}\lambda^3+\frac{13}{16}\lambda^4+\frac{979}{1536}\lambda^5+\frac{1327}{3072}\lambda^6+\frac{3007\,223}{14\,155\,776}\lambda^7 \\
&\quad-\frac{18\,853}{884\,736}\lambda^8-\frac{23\,866\,480\,583}{81\,537\,269\,760}\lambda^9-\frac{1806\,021\,755}{2717\,908\,992}\lambda^{10}+O\left(\lambda^{11}\right), \qquad (4.62)
\end{aligned}
$$

which shall be compared in Section 5.