5 Numerical Comparison

First, we compare the frequency and amplitude of (1.1) with (3.4) and the results of Urabe [18] for small ε in Table 1 and Table 2. In Figure 4, Table 1 and Table 2, we find that the 4-th order approximation of modified homotopy perturbation method is almost equal to the 5-th order approximation of Poincaré-Lindstedt Method. That is, our proposed method is more effective than the Poincaré-Lindstedt Method. In addition, by comparing with the result of the numerical results [1], we know our proposed method is the best accurate. (Note that: M.H.= Modified homotopy perturbation method (4.58), P.L.= Poincaré-Lindstedt method (3.4), Sho.= Shohat transformation (3.5). and the purely numerical results [1] are obtained by Zonneveld and Strasberg.)

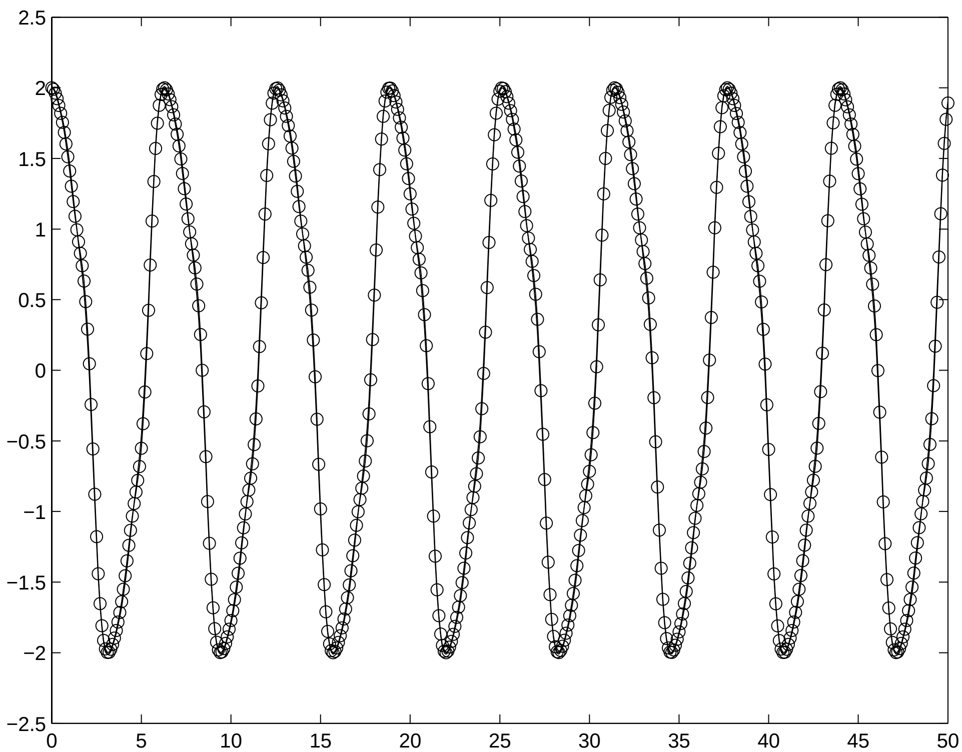

Figure 4 For $\varepsilon = 1$, the approximate solution u of (4.14) obtained by

1) Modified homotopy perturbation method denoted by "o-";

2) Poincaré-Lindstedt method denoted by "–".

Table 1 Values for period $T(\varepsilon)$				
ε	(1) M.H.	(2) P.L.	(3) Urabe's result	(4) Numerical result
0.2	6.298 876 7	6.298 876 7	6.300	
0.4	6.345 743 8	6.345 743 8	6.347	
0.6	6.423 098 5	6.423 098 5	6.422	
0.8	6.529 567 7	6.529 567 7	6.531	
1	6.662 735 7	6.662 735 7	6.687	6.6628686

(1) using 4 terms in the expansion by M.H.;

(2) using 5 terms in the expansion by P.L.;

(3) Urabe's result;

(4) the numerical result is given by [1].

Table 2 Values for amplitude $A(\varepsilon)$			
ε	(1) M.H.	(2) P.L.	(3) Urabe's result
0.2	2.000 395 1	2.000 413 7	2.000
0.4	2.001 332 7	2.001 618 8	2.000
0.6	2.002 150 9	2.003 507 9	2.004
0.8	2.001 995 3	2.005 901 5	2.006
1	2.000 123 3	2.008 548 5	2.009

(1) using 4 terms in the expansion by M.H.;

(2) using 5 terms in the expansion by P.L.;

(3) Urabe's result.

In Table 3, we also have the better result than the result of Shohat transformation for large $\varepsilon > 0$.

Table 3 Values for period $T(\varepsilon)$			
ε	(1) Numerical result	(2) M.H.	(3) Shohat Transformation
1	6.6628686	6. 662 7(−0.00016)	6.844 7(0.181 83)
2	7.62987448	7.519 2(−0.110 67)	8. 110 6(0.480 73)
3	8.85909550	8.594 5(−0.264 60)	9.592 2(0.733 1)
4	10.20352369	9.760 7(−0.442 82)	11.156(0.952 48)
5	11.61223067	10.973(−0.639 23)	12.760(1.147 8)
10	19.07836957	17.288(−1.790 4)	20.98(1.901 6)
(1) the numerical result is given by [1]; (2) using 4 terms in the expansion by M.H.; (3) using 4 terms in the expansion by Sho.			

However for large values of $\varepsilon >> 1$, we have the following approximation [4] for exact period:

$$T_{ex} \approx 2\varepsilon \int_{2/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{1}{v} - 3v \right) dv = 1.614\varepsilon, \quad \text{for } \varepsilon >> 1 \tag{5.1}$$

and the approximate period obtained by modified Poincaré-Lindstedt methods [7]:

$$T_H = 2\pi \sqrt{1 + \frac{1}{8}} = \frac{\pi}{\sqrt{2}} \varepsilon \sqrt{1 + \frac{8}{\varepsilon^2}} \approx \frac{\pi}{\sqrt{2}} \varepsilon,$$

where $\lim_{\varepsilon \to \infty} \frac{T_{ex}}{T_H} = 0.727$ with the maximal relative error for all $\varepsilon > 0$, is less than 37.5%. In order to obtain the ratio between the exact period T_{ex} and approximate period $T = 2\pi/\omega$, we have

$$\begin{aligned} \lim_{\varepsilon \to \infty} \frac{T_{ex}}{T} &= \lim_{\varepsilon \to \infty} \frac{1.614}{2\pi} (\varepsilon\omega) \\ &= \lim_{\varepsilon \to \infty} 0.256\,88\,(\varepsilon\omega)\,. \end{aligned} \tag{5.2}$$

From (4.62), we get expansion for $\varepsilon\omega$ as follows:

$$\varepsilon\omega \approx \sum_{i=0}^{1} (\varepsilon\omega_i) = \lambda + \lambda^2 + \lambda^3 + \lambda^4 - \frac{1}{16}\lambda^3 - \frac{3}{16}\lambda^4 + O\left(\lambda^5\right).$$

Then, we have

$$\lim_{\varepsilon\to\infty}\frac{T_{ex}}{T_1}=\lim_{\varepsilon\to\infty}0.256\,88\,(\varepsilon\omega)=0.256\,88\,(3.\,75)=0.963\,3,$$

where the maximal relative error for all $0<\varepsilon<\infty$ is less than 3.7%. Furthermore, to obtain higher order approximations, we construct a Matlab program which can evaluate the various coefficients discussed in Section 4. Thus the n-th order expansions of $\omega(\varepsilon)$, $A(\varepsilon)$ and $u(\theta,\varepsilon)$ can be computed effectively. First, in Table 4, we compare the results obtained from modified homotopy perturbation method with the results from [18].

Table 4 Comparison for period T and amplitude A, when $\varepsilon=10$.

	M.H.	Urabe's result from (5.3)	Cartwright's result from (5.4)	Dorodnicyn's result (5.5)	
				Original	Corrected
T	18.743	19.1550(0.412)	19.393(0.65)	18.831(0.088)	19.184(0.441)
A	2.036 3	2.0145(−0.021 8)	2.036(−0.000 3)	2.0138(−0.022 5)	

Note that: In Table 4, the values in the first column are obtained from using 8 terms in the expansion and the other values are computed from the asymptotic expressions in [18] as follows:

(1) Urabe's series for period

$$\begin{aligned}T&=(3-2\log 2)\,\varepsilon+7.014321\varepsilon^{-1/3}\\&\quad-\frac{1}{3}\frac{\log\varepsilon}{\varepsilon}-1.3246\varepsilon^{-1}+O\left(\varepsilon^{-4/3}\right).\end{aligned}\tag{5.3}$$

(2) Cartwright's series for period

$$T=(3-2\log 2)\,\varepsilon+7.014\varepsilon^{-1/3}+o\left(\varepsilon^{-4/3}\right).\tag{5.4}$$

(3) Dorodnicyn's series for period

$$\begin{aligned}T&=(3-2\log 2)\,\varepsilon+7.014321\varepsilon^{-1/3}-\frac{22}{9}\frac{\log\varepsilon}{\varepsilon}\\&\quad+0.00087\varepsilon^{-1}+O\left(\varepsilon^{-4/3}\right).\end{aligned}\tag{5.5}$$

Secondly, in Table 5, we find that the 8-th order approximation by the proposed method is more closer to the numerical result than others.

Table 5 Values for period $T(\varepsilon)$

ε	(1) Numerical result	(2) M.H.	(3) Andersen's result	(4) Buonomo's result	(5)Urabe's result
1	6.6628686	6.6490	6.663286859	6.63286859	6.687
2	7.62987448	7.5535	7.629874480	7.62987447	7.6310
3	8.85909550	8.7271	8.85909604	8.85909549	8.8613
4	10.20352369	10.033	10.203911	10.20352376	10.2072
5	11.61223067	11.412	11.61892	11.61224061	11.6055
10	19.07836957	18.743	19.446	19.09846721	19.1550
20	34.6823253	33.981	36.631	35.0603638	
100	162.83707	157.79	179.38	169.122629	

(1) the numerical result is given by [1];

(2) using 8 terms in the expansion by M.H.;

(3) Andersen's result using 82 terms in the expansion [1];

(4) Buonomo's result using 203 terms in the expansion [2];

(5) Urabe's result [18].