6 Discussion and Open Problems

1. The homotopy perturbation method is more effective and simple than other traditional perturbation methods.

2. We give the modification of the homotopy perturbation method to find the approximation of (1.1) for any $\varepsilon > 0$.

3. From the comparisons in Section 5, we can say our numerical results may be accurate not only for the small parameter ε, but also for very large parameter.

4. Furthermore, by the proposed method, the 8-th order approximations are more accurate than high order approximations from other perturbation methods.

5. In addition, this powerful method can deal with the following forced van der Pol equation and van der Pol - Duffing equation by constructing a new homotopy. We remark that this homotopy is a little different from the homotopy discussed in Section 4. The further research will focus on studying the partial differential equation.

Hereafter, we give two examples for illustrations.

Example 1 Consider the forced van der Pol equation

$$\frac{d^2y}{dt^2} + y + \varepsilon\left(y^2 - 1\right)\frac{dy}{dt} = F\cos\omega t, \quad \text{for small } \varepsilon > 0. \tag{6.1}$$

First, we let $\tau = \omega t$, then (6.1) becomes

$$\omega^2\frac{d^2y}{d\tau^2} + y + \varepsilon\omega\left(y^2 - 1\right)\frac{dy}{d\tau} = F\cos\tau. \tag{6.2}$$

Construct an adaptable homotopy of (6.1) as follows:

$$\begin{aligned}&(1-p)\left(\omega^2 y'' + y - \omega^2 u_0'' - u_0\right)\\ &\qquad + p\left[\omega^2 y'' + y + \varepsilon\omega\left(y^2 - 1\right)y' - pF\cos\tau\right] = 0, \quad p \in [0,1]\,.\end{aligned} \tag{6.3}$$

Assume the approximate solution y of (6.3), frequency ω and amplitude A can be expressed in the power series of p as

$$y = y_0(\tau) + y_1(\tau)p + y_2(\tau)p^2 + y_3(\tau)p^3 + O\left(p^4\right), \quad (6.4)$$

$$\omega = \omega_0 + \omega_1 p + \omega_2 p^2 + \omega_3 p^3 + O\left(p^4\right), \quad (6.5)$$

$$A = A_0 + A_1 p + A_2 p^2 + A_3 p^3 + O\left(p^4\right). \quad (6.6)$$

Substituting (6.4), (6.5) into (6.3) and equating the coefficients of the same power of p, we have

$$\omega_0^2 y_0'' + y_0 = \omega_0^2 u_0'' + u_0, \quad (6.7)$$

$$\omega_0^2 y_1'' + y_1 = 2\omega_0\omega_1 u_0'' - u_0 - 2\omega_0\omega_1 y_0'' - \omega_0^2 u_0'' - \varepsilon\omega_0 y_0'\left(y_0^2 - 1\right), \quad (6.8)$$

$$\begin{aligned}\omega_0^2 y_2'' + y_2 = {} & F\cos\tau + \varepsilon\omega_0 y_1' + \varepsilon\omega_1 y_0' - 2\omega_0\omega_1 u_0'' \\ & + 2\omega_0\omega_2 u_0'' - 2\omega_0\omega_1 y_1'' - 2\omega_0\omega_2 y_0'' \\ & - 2\varepsilon\omega_0 y_0 y_1 y_0' + \omega_1^2 u_0'' - \omega_1^2 y_0'' - \varepsilon\omega_0 y_0^2 y_1' - \varepsilon\omega_1 y_0^2 y_0', \end{aligned} \quad (6.9)$$

$$\begin{aligned}\omega_0^2 y_3'' + y_3 = {} & \left(2\omega_0\omega_3 + 2\omega_1\omega_2\right) u_0'' - \left(2\omega_0\omega_2 + \omega_1^2\right) u_0'' - \left(2\omega_0\omega_2 + \omega_1^2\right) y_1'' \\ & - 2\omega_0\omega_1 y_2'' - 2\left(\omega_0\omega_3 + \omega_1\omega_2\right) y_0'' - 2\varepsilon y_0 y_1\left(\omega_0 y_1' + \omega_1 y_0'\right) \\ & - \varepsilon\omega_0 y_0'\left(2y_0 y_2 + y_1^2\right) - \varepsilon\left(y_0^2 - 1\right)\left(\omega_0 y_2' + \omega_1 y_1' + \omega_2 y_0'\right) \end{aligned} \quad (6.10)$$

and so on.

Let the initial approximation of (6.2) be

$$u_0(\tau) = A_0 \cos\tau. \quad (6.11)$$

Substituting (6.11) into (6.7), we get

$$\omega_0^2 y_0'' + y_0 = \left(1 - \omega_0^2\right) A_0 \cos\tau. \quad (6.12)$$

To eliminate the secular terms of (6.12), we put

$$\omega_0 = 1. \quad (6.13)$$

By solving (6.12) with the initial conditions $y_0(0) = A_0$ and $y_0'(0) = 0$, we get

$$y_0(\tau) = A_0 \cos\tau. \quad (6.14)$$

Substituting (6.11) and (6.14) into (6.8), we put

$$A_0 = 2 \tag{6.15}$$

to eliminate the secular terms. Substituting (6.13) and (6.15) into (6.8), we get

$$y_1'' + y_1 = 2\varepsilon \sin 3\tau. \tag{6.16}$$

Solving (6.16) with $y_1(0) = A_1$ and $y_1'(0) = 0$, we obtain

$$y_1(\tau) = \frac{3}{4}\varepsilon \sin\tau + b_1 \cos\tau - \frac{1}{4}\varepsilon \sin 3\tau, \tag{6.17}$$

where $b_1\,(= A_1)$ will be determined later in (6.9).

Similarly, we have the secular terms in (6.9) which should be vanished as

$$\begin{aligned} 2\varepsilon A_1 + \frac{3}{2}\varepsilon\omega_1 &= 0, && (6.18)\\ F + 4\omega_1 + 2A_1\omega_1 + \frac{1}{4}\varepsilon^2 &= 0. && (6.19) \end{aligned}$$

By solving (6.18) and (6.19), we find

$$\begin{aligned} A_1 &= -\frac{3}{4}\omega_1, && (6.20)\\ \omega_1 &= \frac{4}{3} - \frac{1}{6}\sqrt{24F + 6\varepsilon^2 + 64}. && (6.21) \end{aligned}$$

Substituting (6.20) into (6.9), we get

$$y_2'' + y_2 = -\frac{9}{4}\varepsilon\omega_1 \sin 3\tau - \frac{5}{2}\varepsilon\omega_1 \sin 3\tau - \frac{3}{2}\varepsilon^2 \cos 3\tau + \frac{5}{4}\varepsilon^2 \cos 5\tau. \tag{6.22}$$

Solving (6.22) with the initial conditions $y_2(0) = A_2$ and $y_2'(0) = 0$, we obtain

$$y_2(\tau) = -\frac{57}{32}\varepsilon\omega_1 \sin\tau + \frac{19}{32}\varepsilon\omega_1 \sin 3\tau + b_2 \cos\tau + \frac{3}{16}\varepsilon^2 \cos 3\tau - \frac{5}{96}\varepsilon^2 \cos 5\tau, \tag{6.23}$$

where $b_2 \left(= A_2 - \frac{13}{96}\varepsilon^2\right)$ will be determined later in (6.10).

Doing the same process, there exist the secular terms in (6.10) which should be eliminated as

$$\begin{aligned} 2\varepsilon b_2 + \frac{3}{2}\varepsilon\omega_2 + \frac{11}{32}\varepsilon^3 - \frac{111}{32}\varepsilon\omega_1^2 &= 0, && (6.24)\\ 4\omega_2 + 2b_2\omega_1 - \frac{3}{2}\omega_1\omega_2 + 2\omega_1^2 - \frac{3}{4}\omega_1^3 + \frac{1}{32}\varepsilon^2\omega_1 &= 0. && (6.25) \end{aligned}$$

By solving (6.24) and (6.25), we find

$$b_2 = \frac{111}{64}\omega_1^2 - \frac{11}{64}\varepsilon^2 - \frac{3}{4}\omega_2, \tag{6.26}$$

$$\omega_2 = \frac{64\omega_1^2 + 87\omega_1^3 - 10\varepsilon^2\omega_1}{96\omega_1 - 128}. \tag{6.27}$$

Hence, when $p = 1$, the 3-rd order approximations of (6.2) can be expressed in the power series of ε as follows:

$$\begin{aligned} y(\tau) = {} & \left(\frac{45}{64}F + 2\frac{\sqrt{2}}{\sqrt{3F+8}} - \frac{21}{32}F\frac{\sqrt{2}}{\sqrt{3F+8}} + 1\right)\cos\tau \\ & +\varepsilon\left[\left(\frac{19}{32}\sqrt{6F+16} - \frac{13}{8}\right)\sin\tau + \left(\frac{13}{24} - \frac{19}{96}\sqrt{6F+16}\right)\sin 3\tau\right] \\ & +\varepsilon^2\left[\frac{3}{16}\cos 3\tau - \frac{5}{96}\cos 5\tau \right. \\ & \left. + \frac{1008F + 189F^2 - 848\sqrt{6F+16} - 183F\sqrt{6F+16} + 1344}{256\,(3F+8)^2}\right]\cos\tau \\ & +O\left(\varepsilon^3\right), \end{aligned} \tag{6.28}$$

$$\begin{aligned} A = {} & \left(1 + \frac{45}{64}F + 2\frac{\sqrt{2}}{\sqrt{3F+8}} - \frac{21}{32}F\frac{\sqrt{2}}{\sqrt{3F+8}}\right) \\ & +\varepsilon^2\left(\frac{8016F + 1503F^2 - 2544\sqrt{6F+16} - 549F\sqrt{6F+16} + 10\,688}{36\,864F + 6912F^2 + 49\,152}\right) \\ & +O\left(\varepsilon^3\right), \end{aligned} \tag{6.29}$$

and

$$\begin{aligned} \omega = {} & \left(\frac{29}{48}F - \frac{148}{9}\frac{\sqrt{2}}{\sqrt{3F+8}} - \frac{1}{3}\sqrt{2}\sqrt{3F+8} - \frac{103}{24}F\frac{\sqrt{2}}{\sqrt{3F+8}} + \frac{95}{9}\right) \\ & +\varepsilon^2\frac{24\sqrt{3F+8} - 87F\sqrt{2} - 112\sqrt{2} + 9F\sqrt{3F+8}}{512\sqrt{3F+8} + 192F\sqrt{3F+8}} + O\left(\varepsilon^3\right). \end{aligned} \tag{6.30}$$

Remark. When $F = 0$, the above approximations agree with the results of van der Pol equation in Section 4.

Example 2 Consider the van der Pol - Duffing equation, for $\varepsilon > 0$,

$$\frac{d^2y}{dt^2} + y + \varepsilon\left(y^2 - 1\right)\frac{dy}{dt} + \mu y^3 = 0, \quad \mu \geq 0. \tag{6.31}$$

Let $\tau = \omega t$, then (6.31) becomes

$$\omega^2 \frac{d^2y}{d\tau^2} + y + \varepsilon\omega\left(y^2 - 1\right)\frac{dy}{d\tau} + \mu y^3 = 0. \tag{6.32}$$

Construct an adaptable homotopy of (6.32) as

$$\begin{aligned} &(1-p)\left(\omega^2 y'' + y - \omega^2 u_0'' - u_0\right) \\ &\quad + p\left[\omega^2 y'' + y + \varepsilon\omega\left(y^2 - 1\right)y' + p\mu y^3\right] = 0, \quad p \in [0,1]. \end{aligned} \tag{6.33}$$

Substituting (6.4), (6.5) into (6.33) and equating the coefficients of the same power of p, we have

$$\omega_0^2 y_0'' + y_0 = \omega_0^2 u_0'' + u_0, \tag{6.34}$$

$$\omega_0^2 y_1'' + y_1 = 2\omega_0\omega_1 u_0'' - u_0 - 2\omega_0\omega_1 y_0'' - \omega_0^2 u_0'' - \varepsilon\omega_0 y_0'\left(y_0^2 - 1\right), \tag{6.35}$$

$$\begin{aligned} \omega_0^2 y_2'' + y_2 = {} & \left(2\omega_0\omega_2 + \omega_1^2\right)u_0'' - 2\omega_0\omega_1 y_1'' - \mu y_0^3 - 2\omega_0\omega_1 u_0'' - 2\varepsilon\omega_0 y_0 y_1 y_0' \\ & - \left(2\omega_0\omega_2 + \omega_1^2\right)y_0'' - \varepsilon\left(\omega_0 y_1' + \omega_1 y_0'\right)\left({y_0}^2 - 1\right), \end{aligned} \tag{6.36}$$

$$\begin{aligned} \omega_0^2 y_3'' + y_3 = {} & \left(2\omega_0\omega_3 + 2\omega_1\omega_2\right)u_0'' - 3\mu y_0^2 y_1 - \left(2\omega_0\omega_2 + \omega_1^2\right)u_0'' \\ & - \left(2\omega_0\omega_2 + \omega_1^2\right)y_1'' - 2\omega_0\omega_1 y_2'' + \left(-2\omega_0\omega_3 - 2\omega_1\omega_2\right)y_0'' \\ & - 2\varepsilon y_0 y_1\left(\omega_0 y_1' + \omega_1 y_0'\right) - \varepsilon\omega_0 y_0'\left(2y_0 y_2 + y_1^2\right) \\ & - \varepsilon\left(y_0^2 - 1\right)\left(\omega_0 y_2' + \omega_1 y_1' + \omega_2 y_0'\right) \end{aligned} \tag{6.37}$$

and so on.

Let the initial approximation of (6.32) be

$$u_0(\tau) = A_0 \cos\tau. \tag{6.38}$$

Substituting (6.38) into (6.34), we have

$$\omega_0^2 y_0'' + y_0 = \left(1 - \omega_0^2\right)A_0 \cos\tau. \tag{6.39}$$

To eliminate the secular terms of (6.39), we put

$$\omega_0 = 1. \tag{6.40}$$

Solving (6.39) with $y_0(0) = A_0$ and $y_0'(0) = 0$, we obtain

$$y_0(\tau) = A_0 \cos \tau. \tag{6.41}$$

Substituting (6.38) and (6.41) into (6.35), (6.35) becomes

$$y_1'' + y_1 = \frac{1}{4}\varepsilon A_0^3 \sin 3\tau + \frac{1}{4}\varepsilon A_0 \left(A_0^2 - 4\right) \sin \tau. \tag{6.42}$$

To eliminate the secular terms of (6.39), we let

$$A_0 = 2. \tag{6.43}$$

By solving (6.42) with $y_1(0) = A_1$ and $y_1'(0) = 0$, we get

$$y_1(\tau) = \frac{3}{4}\varepsilon \sin \tau - \frac{1}{4}\varepsilon \sin 3\tau + b_1 \cos \tau, \tag{6.44}$$

where $b_1 \,(= A_1)$ will be determined later in (6.36).

Similarly, there are the secular terms in (6.36) which must be eliminated as

$$2\varepsilon b_1 + \frac{3}{2}\varepsilon\omega_1 = 0, \tag{6.45}$$

$$4\omega_1 - 6\mu + 2b_1\omega_1 + \frac{1}{4}\varepsilon^2 = 0. \tag{6.46}$$

By solving (6.45) and (6.46), we have

$$b_1 = -\frac{3}{4}\omega_1, \tag{6.47}$$

$$\omega_1 = \frac{4}{3} - \frac{1}{6}\sqrt{6\varepsilon^2 - 144\mu + 64}, \tag{6.48}$$

where $0 \leq \mu < \frac{1}{24}\varepsilon^2 + \frac{4}{9}$. Substituting (6.47) into (6.36), we get

$$y_2'' + y_2 = -\frac{9}{4}\varepsilon\omega_1 \sin 3\tau - 2\mu \cos 3\tau - \frac{5}{2}\varepsilon\omega_1 \sin 3\tau - \frac{3}{2}\varepsilon^2 \cos 3\tau + \frac{5}{4}\varepsilon^2 \cos 5\tau. \tag{6.49}$$

Solving (6.49) with $y_2(0) = A_2$ and $y_2'(0) = 0$, we obtain

$$\begin{aligned} y_2(\tau) &= -\frac{57}{32}\varepsilon\omega_1 \sin\tau + \frac{19}{32}\varepsilon\omega_1 \sin 3\tau + b_2 \cos\tau \\ &\quad + \left(\frac{1}{4}\mu + \frac{3}{16}\varepsilon^2\right)\cos 3\tau - \frac{5}{96}\varepsilon^2 \cos 5\tau, \end{aligned} \tag{6.50}$$

where $b_2 \left(= A_2 - \frac{1}{4}\mu - \frac{13}{96}\varepsilon^2\right)$ which will be determined later in (6.37).

In the same process, we have the secular terms in (6.37) which should be vanished as

$$2\varepsilon b_2 - \frac{5}{4}\mu\varepsilon + \frac{3}{2}\varepsilon\omega_2 + \frac{11}{32}\varepsilon^3 - \frac{111}{32}\varepsilon\omega_1^2 = 0, \tag{6.51}$$

$$4\omega_2 + \frac{27}{4}\mu\omega_1 + 2b_2\omega_1 - \frac{3}{2}\omega_1\omega_2 + 2\omega_1^2 - \frac{3}{4}\omega_1^3 + \frac{1}{32}\varepsilon^2\omega_1 = 0. \tag{6.52}$$

By solving (6.51) and (6.52), we get

$$b_2 = \frac{5}{8}\mu - \frac{3}{4}\omega_2 - \frac{11}{64}\varepsilon^2 + \frac{111}{64}\omega_1^2, \tag{6.53}$$

$$\omega_2 = \frac{256\mu\omega_1 + 64\omega_1^2 + 87\omega_1^3 - 10\varepsilon^2\omega_1}{96\omega_1 - 128}. \tag{6.54}$$

Hence, when $p = 1$, we obtain the 3-rd order approximate solution of (6.31), for $0 \le \mu < \frac{1}{24}\varepsilon^2 + \frac{4}{9}$,

$$\begin{aligned} y(\tau) &= 2\cos\tau + \left(\frac{3}{4}\varepsilon\sin\tau + \frac{1}{4}\mu\cos 3\tau - \frac{1}{4}\varepsilon\sin 3\tau\right) + \frac{3}{16}\varepsilon^2\cos 3\tau - \frac{5}{96}\varepsilon^2\cos 5\tau \\ &\quad - \left(1 - \frac{1}{8}\sqrt{6\varepsilon^2 - 144\mu + 64}\right)\cos\tau - \varepsilon\left(\frac{19}{8} - \frac{19}{64}\sqrt{6\varepsilon^2 - 144\mu + 64}\right)\sin\tau \\ &\quad + \varepsilon\left(\frac{19}{24} - \frac{19}{192}\sqrt{6\varepsilon^2 - 144\mu + 64}\right)\sin 3\tau \\ &\quad + \left(\frac{21}{256}\varepsilon^2 - \frac{179}{32}\mu + \frac{1592\mu - 65\varepsilon^2}{32\sqrt{6\varepsilon^2 - 144\mu + 64}}\right)\cos\tau, \end{aligned} \tag{6.55}$$

the frequency

$$\omega = \frac{3}{64}\varepsilon^2 - \frac{23}{24}\mu + \frac{95}{9} - \frac{1}{6}\sqrt{6\varepsilon^2 - 144\mu + 64} + \frac{5880\mu - 249\varepsilon^2 - 4736}{72\sqrt{6\varepsilon^2 - 144\mu + 64}} \tag{6.56}$$

and the amplitude

$$A = 1 + \frac{167}{768}\varepsilon^2 - \frac{171}{32}\mu + \frac{1}{8}\sqrt{6\varepsilon^2 - 144\mu + 64} + \frac{1592\mu - 65\varepsilon^2}{32\sqrt{6\varepsilon^2 - 144\mu + 64}}. \tag{6.57}$$

Specially, when ε is small, the above approximations can be expressed in the power series of ε, for $0 \leq \mu < \frac{4}{9}$,

$$
\begin{aligned}
y(\tau) &= \left[\left(1+\frac{2}{\sqrt{4-9\mu}}-\frac{179}{32}\mu+\frac{127}{16}\frac{\mu}{\sqrt{4-9\mu}}\right)\cos\tau+\frac{1}{4}\mu\cos 3\tau\right] \\
&+\varepsilon\left[\left(\frac{19}{16}\sqrt{4-9\mu}-\frac{13}{8}\right)(\sin\tau)+\left(\frac{13}{24}-\frac{19}{48}\sqrt{4-9\mu}\right)(\sin 3\tau)\right] \\
&+\varepsilon^2\left[\frac{3}{16}\cos 3\tau-\frac{5}{96}\cos 5\tau\right. \\
&\left.+\frac{189\mu\sqrt{4-9\mu}-84\sqrt{4-9\mu}-357\mu+424}{256\sqrt{4-9\mu}\,(9\mu-4)}\cos\tau\right]+O\left(\varepsilon^3\right), \quad (6.58)
\end{aligned}
$$

$$
\begin{aligned}
A &= 1-\frac{171}{32}\mu+\frac{2}{\sqrt{4-9\mu}}+\frac{127}{16}\frac{\mu}{\sqrt{4-9\mu}} \\
&+\varepsilon^2\left[\frac{1503\mu\sqrt{4-9\mu}-668\sqrt{4-9\mu}-1071\mu+1272}{768\sqrt{4-9\mu}\,(9\mu-4)}\right] \\
&+O\left(\varepsilon^3\right), \quad (6.59)
\end{aligned}
$$

and

$$
\begin{aligned}
\omega &= \left(\frac{245}{12}\frac{\mu}{\sqrt{4-9\mu}}-\frac{148}{9\sqrt{4-9\mu}}-\frac{2}{3}\sqrt{4-9\mu}-\frac{23}{24}\mu+\frac{95}{9}\right) \\
&+\varepsilon^2\left(\frac{243\mu^2-216\mu-56\sqrt{4-9\mu}+325\mu\sqrt{4-9\mu}+48}{5184\mu^2-4608\mu+1024}\right) \\
&+O\left(\varepsilon^3\right). \quad (6.60)
\end{aligned}
$$

When $\mu = \varepsilon$, we have

$$
\begin{aligned}
y(\tau) &= 2\cos\tau+\varepsilon\left(\frac{3}{4}\sin\tau+\frac{1}{4}\cos 3\tau-\frac{1}{4}\sin 3\tau-\frac{1}{2}\cos\tau\right) \\
&+\varepsilon^2\left(\frac{1597}{256}\cos\tau-\frac{171}{64}\sin\tau+\frac{3}{16}\cos 3\tau-\frac{5}{96}\cos 5\tau+\frac{57}{64}\sin 3\tau\right) \\
&+O\left(\varepsilon^3\right), \quad (6.61) \\
A &= 2-\frac{1}{4}\varepsilon+\frac{4895}{768}\varepsilon^2+O\left(\varepsilon^3\right), \quad (6.62) \\
\omega &= 1+\frac{3}{2}\varepsilon-\frac{107}{32}\varepsilon^2+O\left(\varepsilon^3\right). \quad (6.63)
\end{aligned}
$$

Remark. When $\mu = 0$, the above approximations agree with the results of van der Pol equation in Section 4.