Chapter 1

Introduction

An American option is a contract that give holder a right to exercise the option prior to the date of expiration. Since the additional right should not be worthless, the value of an American option can not be less than an equivalence European option. The extra cost is usually called the exercise premium. The holder holding an American option should decide a range of prices within which it is better to exercise the option than to hold it to expiry. The dividing price between exercise and non-exercise is called the optimal exercise price and the collection of these optimal exercise prices for all times constitutes a curve, which is commonly called the optimal exercise boundary. Because the optimal exercise boundary is not known a priori, the problem of determining the option price with the optimal exercise boundary is then a free boundary problem (FBP).

1.1 Motivations and Research Objectives

1. By observing the results of simulations, people has convinced that the optimal exercise boundary $s(t)$ is convex for the American put and is concave for the American call. The rigorous verifications of the convexity for $s(t)$ of the American put have been

proposed by Chen *et al.* [12] and Ekström [17] during 2004. The convexity of $s(t)$ for the American put provides a useful information for its approximation. When the remaining time is close to zero, Chen *et al.* [12] used this information to provide an asymptotic formula for the optimal exercise boundary of the American put. However, their methods do not work for verifying the concavity of $s(t)$ for the American call. Therefore, we shall propose a rigorous proof of the concavity of the optimal exercise boundary for the American call.

2. The exchange options can be applied to real options like insurance, banking products, other contracts that are modelled as an option. The performance incentive fee can be traded as an European exchange option (EEO) by comparing the rate of return of managed portfolio against the rate of the return of the standard [55]. The margin account, the exchange off and the standly commitment are also options to exchange one risky asset for another. There are a considerable number of studies [4, 27] that model the real option as an American exchange option (AEO) when early exercise is positive possibility. The optimal exercise ratio is interpreted as the critical investment trigger. Therefore, finding accurate approximate or numerical solutions for the optimal exercise ratio are important for the real option problems. We shall investigate the AEO pricing problems and propose an analytic approximation for the optimal exercise ratio.

3. Due to the difficulty of finding the closed form pricing formula of an American-style option, a considerable number of numerical methods or the simulation-based approaches are proposed to valuate the American-style options. For example, MacMillan [56] as well as Barone-Adesi and Whaley [3] both used the numerical methods to ap-

proximate the solution of the American-style option pricing problems. Longstaff and Schwartz [53] as well as Rogers [65] proposed a Monte Carlo method for pricing the American-style options. These methods can simulate the value of the American-style option very fast. However, all their methods do not provide the hedging portfolio of the option. The binomial method provides a way to obtain both the value and the hedging portfolio. We shall investigate a new approach based on the binomial tree method for American option pricing problems by using the nonlinear programming approach.

1.2 Major Results

1.2.1 Single Asset American Option Pricing Problems

We consider a one dimensional parabolic FBP and show that the solutions of this problem are strictly increasing under some given assumptions. Moreover, we propose a rigorous proof of that the free boundary $s(t)$ is a concave function. The American call pricing model derived by Merton [59] is a special case of this parabolic FBP. Nevertheless, the rigorous verification of the concavity for the optimal exercise boundary has been neglected in their works. By using our results, a rigorous verification of the concavity of the optimal exercise boundary is obtained. Finally, we apply the concavity of the optimal exercise boundary to derive its asymptotic formula.

1.2.2 American Exchange Option Pricing Problems

We model the AEO pricing problems as an FBP and show that the solution of this FBP has the following two properties. (1) The value of the AEO and the optimal exercise ratio are both strictly increasing functions of the remaining time. (2) The value of the alive AEO is an increasing function of the price of asset 1 and a decreasing function of the price of asset 2. To find the solution of this FBP, one possible method is to convert the FBP into an integral equation (IE) by using the Green's function. However, the exact solution of this IE is obtained difficulty for the finite time horizon AEO. For the finite time horizon AEO, we provide an asymptotic solution and a numerical solution for this IE. The numerical solution is obtained by extending the integral recursive method proposed by [36]. We find that our asymptotic solution is very close to the numerical solution as time near to maturity. For the infinite time horizon AEO, we obtain the exact value of both the optimal exercise ratio and the AEO.

1.2.3 Optimization Approaches for Pricing American Style Options

We proposes two mixed integer nonlinear programming (MINLP) models for valuating a European option and an American option. Precisely, we formulate the procedure of the binomial approach as an MINLP model for valuation a European option and for valuation an American option. In both models, we introduce a decision variable which is a 0-1 variable to represent the decision of an option holder.

These decision variables provide an optimal exercise strategy for buyers in the discrete setting.

The methods to solve a MINLP problem or a binomial method require dramatically more numerical computations. We show that the MINLP models can be solved by their nonlinear programming (NLP) relaxations. This provides a far more efficient way for obtaining the solution of this MINLP problem.

Moreover, when the observed market price is less than its fair price, we propose a self-finance model to minimize the expected loss for the writer. In the computational results, we find that the solution of minimum loss model provides an optimal strategy for constructing a lower expected loss portfolio.

1.3 Organization of the Dissertation

The dissertation comprises six chapters. Chapter 1 outlines the dissertation. Chapter 2 provides a literature review.

In Chapter 3, the American option pricing problems are modeled as an FBP. We shall provide a rigorous verification of the concavity for the free boundary of this F-BP. By applying the monotonicity and the concavity properties to the American call, we show that the optimal exercise boundary of the American options is a strictly decreasing concave function when the stock price is measured by the time-homogeneous diffusion. An asymptotic formulas for the optimal exercise boundary as time is near to expiration are provided in the last section.

Chapter 4 considers an FBP for the AEO pricing problems. We will deduce an

IE for the free boundary of this FBP and propose an asymptotic solution of this IE as the remaining time near to maturity. Moreover, the solutions of this FBP can be obtained when the expiration date tends to infinity. We also show that the value of an AEO is a decreasing function of the price of asset 2 and the remaining time, and is an increasing function of the price of asset 1. And then, we propose a recursive integration method to find the solution of the optimal exercise ratio numerically. By observing the numerical results, we found that our asymptotic solution is close to the numerical solution.

In Chapter 5, we give an optimization approach for pricing of European options and American options We will propose the MINLP models for pricing American options. We show that the MINLP models can be solved by their NLP relaxations which provide an efficient way for computing the value of an American option and its replication portfolio. The conclusions and future studies are given in Chapter 6.