Chapter 3

Single Asset American Style Option Pricing Problems

An American option is an option which gives holder a right to exercise prior the expiration date. Merton [59] assumed that the stock price dynamics satisfies the geometric Brownian motion and proposed a parabolic FBP for valuation the vanilla American options. The final payoff ψ of the American option is given as $\max(x-K,0)$ for call or $\max(K-x,0)$ for put.

In this chapter, we shall consider the generalized American option pricing problems. Namely, the stock price dynamics is measured by the time-homogeneous diffusions and the final payoff function is given as a positive real-valued function in $C^3([0,\infty))$. By using the Feynman-Kac formula, the American option pricing problems can be modelled as a FBP. Under the given assumptions, we shall show that the free boundary and the solution of the FBP are increasing functions. The main contribution of this chapter is to provide a rigorous verification of the concavity of the free boundary. Consequently, we obtain that the optimal exercise boundary is strictly concave function in the American call pricing problem. Following the spirit of [12], we use this information to provide an asymptotic formula for $s(t)$ of the vanilla

American call when the remaining time is close to zero. We obtain that

$$s(\tau) \sim K + 2K\tau^{1/2}\left[\log\frac{\sigma^2}{4(r-q)\sqrt{\pi\tau}}\right]^{1/2}, \qquad \text{for } q > r,$$
$$s(\tau) \sim K + 2K\tau^{1/2}\left[\log\frac{\sigma^2}{8\sqrt{\pi}r\tau}\right]^{1/2}, \qquad \text{for } q = r,$$

as the remaining time τ is close to zero. Here r, q and σ are the constant interest rate, dividends rate and the constant volatility. These asymptotic formulas was also given in [18] by using different technique. Here, we provide a simple way to obtain the formulas by using the concavity of $s(t)$.

3.1 Free Boundary Problems Arising from Pricing of American Options

Let $X(t)$ be the stock price, which is a solution of the following time-homogeneous diffusion

$$dX(t) = (r-q)X(t)dt + \sigma X(t)dB(t), \text{ and } X(s) = x,$$

where $B(t)$ is a Brownian motion, and $\psi(x)$ be a bounded function. Denote

$$v(x,t) = \mathrm{E}\left(e^{-\int_t^T r(\xi)d\xi}\psi(X(T))|X(t) = x\right),$$

where $r(x)$ is the interest rate, then $v(x,\tau)$ is the price of a European option with the final payoff $\psi(x)$.

By using the Feynman-Kac formula, the European option's price v satisfies the following parabolic equation

$$\mathcal{L}v = 0 \text{ with } v(x,T) = \psi(x),$$

where $\mathcal{L} = \mathcal{L}_0 + \frac{\partial}{\partial \tau}$ and

$$\mathcal{L}_0 = \frac{\sigma^2 x^2}{2}\frac{\partial^2}{\partial x^2} + (r-q)x\frac{\partial}{\partial x} - r. \tag{3.1}$$

For the American option pricing problems, since American option can be exercised at any time $t < T$, this problem is formulated as an FBP. Let $u(x,\tau)$ be the value of an American option, then $u(x,\tau)$ must satisfy following inequality under no-arbitrage condition

$$u(x,\tau) \geq \psi(x),\ 0 \leq \tau \leq T.$$

Now, we separate the domain $\{(x,t)|0 \leq x < \infty,\ 0 \leq t \leq T\}$ into two parts: (i) a continuation region $\mathcal{C} = \{(x,t)|u(x,t) > \psi(x)\}$, and (ii) a stopping region $\mathcal{S} = \{(x,t)|u(x,t) = \psi(x)\}$. Given $\tau \in (0,T)$, we consider the following two cases. (1) $\psi(x)$ is a strictly increasing function and (2) ψ is a strictly decreasing function. When $\psi(x)$ is a strictly increasing function, there is a time-depended function $s(t)$ such that $u(x,t) > \psi(x)$ for $0 < x < s(t)$ (see [11]). Hence, the continuation region is described as $\mathcal{C} = \{(x,t)|0 < x < s(t),\ 0 < t < T\}$.

When $\psi(x)$ is a strictly decreasing function, there is a time-depended function $s(t)$ such that $u(x,t) > \psi(x)$ for $s(t) < x < \infty$. Hence, the continuation region is described as $\mathcal{C} = \{(x,t)|s(t) < x < \infty,\ 0 < t < T\}$.

And then, by using the no arbitrage condition, the price of the American option satisfies the high contact condition, that is $u_x(x,t) = \psi'(x)$.

To investigate the American option's price u with the optimal exercise boundary s, we consider the following one phase one-dimensional FBP for linear parabolic equations (3.1):

$$
\left.\begin{aligned}
&\mathcal{L}u = 0 && 0 < x < s(t),\ 0 < t < \infty, && (3.2)\\
&u(x,t) > \psi(x) && 0 < x < s(t),\ 0 < t < \infty, && (3.3)\\
&u(0,t) = \psi(0) && 0 < t < \infty, && (3.4)\\
&u(x,0) = \psi(x) && 0 \le x \le s(0), \qquad (\mathbf{P}) && (3.5)\\
&u(s(t),t) = \psi(s(t)) && 0 \le t < \infty, && (3.6)\\
&\frac{\partial u}{\partial x}(s(t),t) = \psi'(s(t)) && 0 \le t < \infty, && (3.7)
\end{aligned}\right.
$$

Here, $\mathcal{L}$ is rewritten as

$$\mathcal{L} = \mathcal{L}_0 - \frac{\partial}{\partial t}$$

and $\mathcal{L}_0$ is defined as

$$\mathcal{L}_0 \equiv a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x} + c(x).$$

We assume that coefficients $a, b, c \in C^{2+\alpha}([0,\infty))$ for some $\alpha \in (0,1)$ with $a(x) \ge a_0 > 0$ for $0 \le x < \infty$ and $c(x) \le 0$ for $0 \le x < \infty$ and $\psi \in C^3([0,\infty))$ is a positive real-valued function with

$$\mathcal{L}_0\psi(x) \begin{cases} > 0 & \text{for } 0 \le x < d, \\ < 0 & \text{for } d < x < \infty, \end{cases} \tag{3.8}$$

for some $d > 0$. We shall discuss some properties for the solution (s,u) of Problem (P).

This problem had been studied by Kotlow [40]. He showed that the problem (P) is well-posed and that $u(x,t)$ and $s(t)$ are both nondecreasing functions of t.

When $\mathcal{L}$ is given as

$$\mathcal{L}_0 \equiv \frac{\partial^2}{\partial x^2},$$

Problem (P) is called a Stefan problem. Friedman [21] showed that the free boundary for the Stefan problem is smooth and monotone increasing. Moreover, Friedman and Jensen [22] showed that the free boundary is a concave function by using the maximum principle to control the level curve.

In addition, we make the following assumptions which will be used later.

Assumptions

(A) $c(x)$ *is a nonincreaing function in* $[0, \infty)$.

(B) $\psi(x)$ *is a positive strictly increasing convex function in* $[0, \infty)$.

(C) $\frac{d}{dx}\mathcal{L}_0\psi(x) \leq 0$ *and* $c(x) + b'(x) \leq 0$ *in* $[0, \infty)$.

(D) $\limsup_{\xi \to \infty} \mathcal{L}_0\psi(\xi) < 0$.

3.2 Properties of the Solution

Let $\{s, u\}$ be the solution of (P) and denote C, namely continuation region, as

$$C = \{(x, t); 0 < x < s(t),\ 0 < t < \infty\}. \tag{3.9}$$

Now we define $\hat{u}$ on $\bar{Q}$ by

$$\hat{u}(x,t) = \begin{cases} u(x,t) & (x,t) \in C \\ \psi(x) & (x,t) \in \bar{Q} - C \end{cases}$$

where $\bar{Q} = [0, \infty) \times [0, \infty)$.

In this section, we will show that

1. $s(t)$ is a strictly increasing function with $s(0) = d$,
2. $u(x, \cdot)$ is a strictly increasing function in C,
3. $u_x(\cdot, t)$ is increasing function in C.

In order to prove these results, we need some preliminaries.

Definition 3.1. *Given $t \in [0, \infty)$, the t-section of C is defined to be*

$$C_t = \{x \in \mathbb{R} | 0 < x < s(t)\}. \tag{3.10}$$

Clearly, we have

$$C = \bigcup_{t<\infty} (C_t \times \{t\})$$

and

$$s(t) = \sup\{x | x \in C_t\}. \tag{3.11}$$

The following properties of u and $s(t)$ have been proved by Kotlow [40].

Lemma 3.2. *Let $\{s, u\}$ be a solution of (P). They have the following properties:*

(a) $u_t > 0$ *in* C.

(b) $s(0) = d < \infty$ *and* $s(t) > d$ *for* $0 < t < \infty$.

(c) $s(t)$ *is a nondecreasing function.*

(d) There exists a $s^\infty \in (d, \infty)$ *such that* $s(t) \to s^\infty$ *uniformly as* $t \to \infty$ *if Assumption (D) holds.*

Remark 3.3. *If $d = \infty$, then the continuation region $C = (0, \infty) \times (0, \infty)$.*

Let

$$\mathcal{M}_0 = a(x)\frac{\partial^2}{\partial x^2} + (b(x) + a'(x))\frac{\partial}{\partial x} + (c(x) + b'(x)). \tag{3.12}$$

be an elliptic operator. We define a parabolic operator $\mathcal{M}$ as

$$\mathcal{M} = \mathcal{M}_0 - \frac{\partial}{\partial t}.$$

Lemma 3.4. *Let $\{s, u\}$ be a solution of (P). At points $(s(t), t)$, $t > 0$, u satisfies $u_t(s(t), t) = 0$.*

Proof. By (3.6), we have $u(s(t), t) = \psi(s(t))$. Differentiating $u(s(t), t) = \psi(s(t))$ with respect to t, we have

$$u_x(s(t), t)s'(t) + u_t(s(t), t) = \psi'(s(t))s'(t).$$

Since $u_x(s(t), t) = \psi'(s(t))$ by (3.7), we have $u_t(s(t), t) = 0$. □

Lemma 3.5. *Let $\{s, u\}$ be a solution of (P) and $w = \hat{u} - \psi$ in $\bar{Q}$. Then $w(\cdot, t)$ has a local maximum in $(0, s(t))$. Moreover, this local maximum can not lie in $(d, s(t))$.*

Proof. By (3.3) and (3.6), we have $w(0, t) = w(s(t), t) = 0$ and $w(x, t) > 0$ on C. This implies that there exists a $d' \in (0, s(t_0))$ for some $t_0 > 0$, which may depend on t_0, such that $w(d', t_0)$ is a local maximum. Now, we claim that $d' \notin (d, s(t_0))$. Suppose that $d' \in (d, s(t_0))$ is a local maximum of $w(x, t_0)$. We define

$$\begin{aligned}
\Omega_{t_0} &= \{(x, t) \in C | d \leq x \leq s(t),\ t \leq t_0\}, \\
\partial_p \Omega_{t_0} &= \{(x, t) \in \partial\Omega_{t_0} | t < t_0\}.
\end{aligned}$$

and apply the differential operator $\mathcal{L}$ to w. By (3.8) and (3.2), w satisfies the parabolic differential equation

$$\mathcal{L}w = (\mathcal{L}_0 - \frac{\partial}{\partial t})u - \mathcal{L}_0\psi(x) = -\mathcal{L}_0\psi(x) > 0 \quad \text{on } \Omega_{t_0}.$$

Now, we apply the maximum principle to this equation. Since $(d', t_0) \in \Omega_{t_0} - \partial_p \Omega_{t_0}$ is a nonnegative maximum over Ω_{t_0}, it implies that w is a constant function on Ω_{t_0}. This contradicts to that $w_t = u_t > 0$. So $d' \leq d$. $\square$

Theorem 3.6. *Let $\{s, u\}$ be a solution of (P). Then*

(a) $s(t)$ is a strictly increasing function.

(b) $u_x(x, t) > 0$ for $(x, t) \in C$ if (A), (B) hold.

(c) $u_x(x, t) < \psi'(x)$ for $(x, t) \in C_d$, where $C_d = \{(x, t \in C | x > d)\}$, if (A), (B), (C) hold.

Proof. For (a), we only need to show that $s(t_2) \neq s(t_1)$, for $t_2 < t_1$ by (c) in Lemma 3.2. Suppose that there is an interval $[t_1, t_2]$ such that $s(t) = s(t_1)$ for all $t \in [t_1, t_2]$, then $u_x(s(t), t)$ is a constant function for all $t \in (t_1, t_2)$. Since $\mathcal{L}u = 0$ in $(0, s(t_1)) \times (t_1, t_2)$ and $u(s(t), t) = \psi(s(t_1))$ for all $t \in [t_1, t_2]$, we have $u \in C^\infty([0, s(t_1)) \times (t_1, t_2))$. Since $u_t(s(t_1), t_1) = 0$ for $t \in (t_1, t_2)$, $u_t > 0$ in $(0, s(t_1)) \times (t_1, t_2)$ and $\mathcal{L}u_t = 0$ in $(0, s(t_1)) \times (t_1, t_2)$, we have $u_{tx}(s(t), t) < 0$ for $t \in (t_1, t_2)$ by applying the boundary point form of the maximum principle. And then, we have $u_{xt}(s(t), t) = u_{tx}(s(t), t) < 0$ which implies that $u_x(s(t), t)$ is strictly decreasing for $t \in (t_1, t_2)$. On the other hand, $s(t)$ is a nondecreasing function in $[t_1, t_2]$ and $\psi(x)$ is a strictly increasing convex function. This implies that $\psi'(s(t))$ is a nondecreasing function in $[t_1, t_2]$. Hence, we obtain that $u_x(s(t), t) = \psi'(s(t))$ is a nondecreasing function in $[t_1, t_2]$ and there is a contradiction. So $s(t)$ is a strictly increasing function.

We now show that (b) $u_x \geq 0$. We first consider that

$$\begin{aligned}\frac{\partial}{\partial t}u_x &= \frac{\partial}{\partial x}\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(\mathcal{L}_0 u)\\ &= \frac{\partial}{\partial x}(a(x)u_{xx} + b(x)u_x + c(x)u)\\ &= a(x)\frac{\partial^2}{\partial x^2}u_x + (b(x) + a'(x))\frac{\partial}{\partial x}u_x + (c(x) + b'(x))u_x + c'(x)u\end{aligned}$$

Consequently, u_x satisfies the parabolic differential equation

$$\mathcal{M}u_x = -c'(x)u.$$

Let $w(x,t) = \hat{u}(x,t) - \psi(x)$ on $\bar{Q}$. Then $w(0,t) = 0$ and $w(x,t) > 0$ on C by (3.3) and (3.4). So we have

$$w_x(0,t) = \lim_{h \to 0^+} \frac{w(h,t) - w(0,t)}{h} \geq 0.$$

This implies that

$$u_x(0,t) \geq \psi'(0) > 0 \tag{3.13}$$

for $0 < t < \infty$. We apply the maximum principle to this equation. Since $u_x(s(t),t) = \psi'(s(t)) > 0$ for $t \geq 0$ by (3.7), $u_x(0,t) > 0$ for $0 < t < \infty$ by (3.13), $u_x(x,0) = \psi'(x) > 0$ for $0 < x < s(0)$ by (3.5) and $c'(x) \leq 0$ by assumption (A), this shows $u_x(x,t) > 0$ for $0 < x < s(t)$ and $0 < t < \infty$.

Finally, we show that (c) $u_x(x,t) < \psi'(x)$ for $d < x < s(t)$ and $0 < t < \infty$. Let $w(x,t) = \hat{u}(x,t) - \psi(x)$ on $\bar{Q}$. Since $w(s(t),t) = w_x(s(t),t) = 0$ by (3.6) and (3.7), $w(x,t) > 0$ on C by (3.3), and the continuity of w and w_x, we have that, for any $t > 0$, there exists a $\delta > 0$ such that $w_x(x,t) < 0$ for $s(t) - \delta < x < s(t)$ and $t > 0$. Suppose that there is a $x' \in (d, s(t) - \delta)$ with $w_x(x',t) \geq 0$, then there exists a local maximum in $[x', s(t)) \subseteq (d, s(t))$. This contradicts to Lemma 3.5. So we obtain that there is no $x \in (d, s(t))$ with $w_x(x,t) \geq 0$. Hence, we have $w_x(x,t) < 0$ for $(x,t) \in C_d$. This implies that $u_x(x,t) < \psi'(x)$ for $(x,t) \in C_d$. □

3.3 Concavity of the Free Boundary

Suppose that (A), (B) and (C) hold. We will show that $s(t)$ in (3.9) is a concave function; consequently the continuation region of C in (3.9) is a convex set.

In order to prove our main theorem, we need a lemma.

Lemma 3.7. *Let $\{s, u\}$ be a solution of (P) and define $w = u - \psi$ on $\bar{C}_d$, where $\bar{C}_d = \{(x,t) \in \mathbb{R}^2 | d < x \leq s(t), 0 < t < \infty\}$. At points $(s(t), t)$, $t > 0$, w has the following properties*

$$
\begin{aligned}
&w(s(t),t) = 0,\ \ w_t(s(t),t) = 0,\ \ w_x(s(t),t) = 0,\\
&w_{xx}(s(t),t) = -\frac{1}{a(s(t))}(\mathcal{L}_0\psi(s(t))),\\
&w_{tx}(s(t),t) = \frac{1}{a(s(t))}(\mathcal{L}_0\psi(s(t)))s'(t),\\
&w_{tt}(s(t),t) = -\frac{1}{a(s(t))}(\mathcal{L}_0\psi(s(t)))(s')^2(t).
\end{aligned}
$$

Proof. Since $u(s(t),t) = \psi(s(t))$, $u_x(s(t),t) = \psi'(s(t))$ and $u_t(s(t),t) = 0$, we obtain the first three equalities. By (3.8) and (3.2), we have

$$aw_{xx} + bw_x + cw - w_t = -\mathcal{L}_0\psi.$$

Since $w = 0$, $w_x = 0$ and $w_t = 0$ at $(s(t),t)$, we have

$$w_{xx} = -\frac{1}{a}(\mathcal{L}_0\psi).$$

Differentiating the equality

$$w_x(s(t), t) = 0$$

with respect to t, we have

$$w_{xx}s'(t) + w_{tx} = 0.$$

So $w_{tx} = -w_{xx}s'(t) = \frac{1}{a}(\mathcal{L}_0\psi(x))s'(t)$. Next, w_{tt} can be found by differentiating

$$w_t(s(t), t) = 0.$$

We obtain

$$w_{xt}(s(t), t)s'(t) + w_{tt}(s(t), t) = 0.$$

So $w_{tt} = -w_{xt}(s(t), t)s'(t) = -\frac{1}{a}(\mathcal{L}_0\psi(x))(s')^2(t)$. □

Now, we can state and proof our mean theorem of this chpater.

Theorem 3.8. *Let (A), (B), (C) and (D) hold and $\{s, u\}$ be a solution of (P). Then $s(t)$ is a concave function.*

Proof. We have known that $s(t)$ is a strictly increasing function. Suppose that there is a point t_0 with $s'(t_0) = m > 0$ and $s''(t_0) > 0$. This implies that there exists an $\varepsilon > 0$ such that $s'(t) > m$ for $t \in (t_0, t_0 + \varepsilon)$. We consider the line

$$y(t) = m(t - t_0) + s(t_0)$$

for some $t > 0$. So $y(t_0) = s(t_0)$. Since $s(t)$ is bounded above by (d) of Lemma 3.2 and $m > 0$, there must exist another point $t_1 > t_0$ such that $y(t_1) = s(t_1)$. Let $f(t) = w(y(t), t)$ for some $t > t_3$ where $t_3 = \inf\{t | (y(t), t) \in C_d\}$. Since $w_t(s(t), t) = 0$

and $w_x(s(t),t) = 0$, we have

$$\begin{aligned} f'(t_i) &= mw_x(y(t_i),t_i) + w_t(y(t_i),t_i) \\ &= mw_x(s(t_i),t_i) + w_t(s(t_i),t_i) = 0, \ i = 1,2. \end{aligned}$$

We also have $f(t_0) = w(y(t_0),t_0) = 0$, $f(t_1) = w(y(t_1),t_1) = 0$ and $(y(t),t) \in C_d$ for $t \in (t_0,t_1)$, which implies that $f(t) = w(y(t),t) > 0$ for $t \in (t_0,t_1)$. So there exists a local maximum of f between t_0 and t_1, namely $f(t_2)$ where $t_2 \in (t_0,t_1)$. This implies that $f(t_2) > 0$ and $f'(t_2) = 0$. Without loss of generality, we can assume that there is no local maximum between t_2 and t_1. Since $f(t_0) = f(t_1) = 0$, $f(t_2) > 0$, and $f'(t_i) = 0$, $i = 0,1,2$, we have

$$f'(t) < 0 \text{ for } t \in (t_2,t_1) \tag{3.14}$$

and $f'(t) > 0$ for some interval, say (t_4,t_2), where $t_0 \le t_4 < t_2$.

Since

$$\lim_{x \to s(t)} \frac{w_x(x,t)}{w_t(x,t)} = -s'(t)$$

by l'Hôpital's rule and $w < 0$ on C_d, let

$$v = \begin{cases} \dfrac{w_t}{w_x}, & (x,t) \in C_d, \\ -s'(t), & x = s(t), \end{cases}$$

which is well-defined on $\bar{C}_d$. Then we have that

$$\begin{aligned} f'(t) &= mw_x(y(t),t) + w_t(y(t),t) \\ &= w_x(y(t),t)(m + v(y(t),t)) \end{aligned} \tag{3.15}$$

for $t > t_3$. Applying the differential operator $\mathcal{L}$ to the equality $vw_x = w_t$, we obtain that v satisfies the following parabolic equation

$$av_{xx} + (b + \frac{2w_{xx}}{w_x})v_x - \frac{1}{w_x}(\mathcal{M}_0\psi')v - v_t = 0 \text{ on } \bar{C}_d. \tag{3.16}$$

Let Γ_α be the level curves on which $v = \alpha$. Since $w_x < 0$, $\mathcal{M}_0\psi' \leq 0$ and v satisfies the parabolic equation (3.16), the x-coordinate along Γ_α can not first increasing (decreasing) and then decreasing (increasing). This is because that there would be a region whose parabolic boundary is a part of Γ_α; consequently $v \equiv \alpha$ in this region and hence $v \equiv \alpha$ in C_d. Since Γ_α is understood to be continued as long it remains in C_d, for each α there is a $g_\alpha(t)$ such that

$$\Gamma_\alpha = \{(g_\alpha(t), t) | v(g_\alpha(t), t) = \alpha, t > 0\}.$$

Since $f'(t_i) = 0,\ i = 0, 1, 2$ and $f'(t) = w_x(y(t), t)(m + v(y(t), t))$, we have $v(y(t_i), t_i) = -m$, $i = 0, 1, 2$. This implies that $(y(t_i), t_i) \in \Gamma_{-m}$, $i = 0, 1, 2$. Now we consider the function $g_{-m}(t)$. Since $(y(t_i), t_i) \in \Gamma_{-m}$, that is $v(y(t_i), t_i) = -m$, $i = 0, 1, 2$, we have $y(t_i) = g_{-m}(t_i),\ i = 0, 1, 2$.

Since $g_{-m}(t)$ is continuous on (t_3, t_1), we only have the following two cases: (1) $y(t) < g_{-m}(t)$ for $t \in (t_2, t_1)$, and (2) $y(t) > g_{-m}(t)$ for $t \in (t_2, t_1)$.

We first consider case (1). We have $w_x < 0$ on C_d by (c) of Theorem 3.6. Since $f'(t) = w_x(y(t), t)(m + v(y(t), t)) < 0$ for $t \in (t_2, t_1)$ by (3.14) and (3.15), this implies that

$$v(y(t), t) > -m, \text{ for } t \in (t_2, t_1). \tag{3.17}$$

Since $g_{-m}(t_1) = y(t_1) = s(t_1)$ and $y(t) < g_{-m}(t) < s(t)$ for $t \in (t_2, t_1)$, there is a $\delta > 0$ such that $y'(t) > g'(t) > s'(t)$ for $t \in (t_1 - \delta, t_1)$. Since $y'(t) = m$, we have $v(s(t), t) = -s'(t) > -y'(t) = -m$ for $t \in (t_1 - \delta, t_1)$. Let $\Omega = \{(x, t) | y(t) < x < s(t),\ t_1 - \delta < t < t_1\}$. We apply extensions of maximum principle, [20], to (3.16) on Ω. Since $v(y(t), t) > -m$ for $t \in (t_2, t_1)$ by (3.17) and $v(s(t), t) > -m$ for

$t \in (t_1 - \delta, t_1)$, this implies that $v(x,t) > -m$ for $(x,t) \in \Omega$. This contradicts to that $v(g_{-m}(t), t) = -m$ for $t \in (t_1 - \delta, t_1)$. So case (1) dose not hold.

Now we consider case (2). We have know that the level curves Γ_α of a parabolic equation is continuous. We consider the line $y(t)$ for $t \in (t_2, t_1) \cup (t_3, t_0)$. In (3.17), we have $v(y(t), t) > -m$ for $t \in (t_2, t_1)$. And then, we also have $f(t_0) = 0$ and $f(t) > 0$ for $t \in (t_3, t_0)$. This implies that there is a $\delta_2 > 0$ such that $f'(t) < 0$ for $t \in (t_0 - \delta_2, t_0)$. Since $f'(t) = w_x(y(t), t)(m + v(y(t), t))$ and $f'(t) < 0$ for $t \in (t_0 - \delta_2, t_0)$, we have $v(y(t), t) > -m$ for $t \in (t_0 - \delta_2, t_0)$. Now we can select a suitable $\delta > 0$ such that $v(y(t), t) > -m$ for $t \in (t_0 - \delta, t_0) \cup (t_1 - \delta, t_1)$. Since $v(y(t_0), t_0) = -m = v(y(t_1), t_1)$ and $v(y(t), t) > -m$ for $t \in (t_0 - \delta, t_0) \cup (t_1 - \delta, t_1)$, there exists a $t' \in (t_0 - \delta, t_0)$ and a $t'' \in (t_1 - \delta, t_1)$ such that

$$v(y(t'), t') = \beta = v(y(t''), t''), \text{ for aome } \beta > -m.$$

Since the level curves of a parabolic equation is continuous, there is a level curve Γ_β connecting $(y(t'), t')$ and $(y(t''), t'')$. On the other hand, we have $(y(t''), t'') \in \Omega_1$, where $\Omega_1 = \{(x,t) \in C | g_{-m}(t) \leq x < s(t),\ t_0 < t < t_1\}$. This contradicts to that $\Gamma_{-m} \cap \Gamma_\beta \neq \emptyset$. So case (2) does not hold.

Since both case (1) and case (2) do not hold, we conclude that there is no such point t_0 with $s''(t) > 0$ for $t > 0$. Thus, $s(t)$ is a concave function. □

3.4 Application to American Call Option

In this section, we apply our results to vanilla American option. Let x denote the stock price and $K > 0$ be the strike price. The payoff function, $\psi(x)$, is given as $\max\{0, x-K\}$ for a call option and is given as $\max\{0, K-x\}$ for a put option. Then ψ is a strictly increasing function for a call option when $x \geq K$ and is a strictly decreasing function for a put option when $x \leq K$.

When the evolution of the stock price satisfies the geometric Brownian motion, Black and Scholes [6] showed that the option's fair price satisfies the fundamental equation

$$\mathcal{L}V = 0, \tag{3.18}$$

with the initial condition

$$V(x,0) = \psi(x)$$

in the domain $\{(x,\tau)|0 \leq \tau \leq T, 0 \leq x < \infty\}$, $T < \infty$. In the BS framework, the operator is given as $\mathcal{L} = \mathcal{L}_0 - \frac{\partial}{\partial \tau}$ and

$$\mathcal{L}_0 = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + (r-q)x\frac{\partial}{\partial x} - r,$$

where q, σ^2 and r denote respectively the constant dividend yield, the volatility of the asset, and the risk-free interest rate in the market, respectively.

Merton [59] derived that the fair price of the American option satisfies (3.18) in the continuation region; moreover the value v together with $s(\tau)$ is a solution of the following FBP

$$\begin{aligned}
&\mathcal{L}v = 0 && \text{on } \mathcal{C}, \\
&v(x,0) = \psi(x) && \text{for } x \geq 0, \\
&v(0,\tau) = \psi(0) && \text{for } 0 \leq \tau < T, \\
&v(\tau, s(\tau)) = \psi(s(\tau)) && \text{for } 0 \leq \tau < T, \\
&v_x(\tau, s(\tau)) = \psi'(s(\tau)) && \text{for } 0 \leq \tau < T.
\end{aligned} \tag{3.19}$$

So far, many authors, such as [11], [12] and [17], write (3.19) in a non-dimensional form by letting

$$
\begin{aligned}
&k = 2r/\sigma^2,\ h = 2d/\sigma^2,\ x = Ke^y, \\
&v(x,t) = Kp(y,t), s(t) = Ke^{z(\tau)}.
\end{aligned}
\tag{3.20}
$$

They showed that $z(t)$ is a monotonically decreasing convex function. Hence, $s(t) = e^{z(\tau)}$ for an American put is a monotonically increasing convex function of t since e^x is a strictly increasing convex function. Now we consider the American call by using the same method. Although we can show that $z(\tau)$ is a concave function for the American call, we do not know the concavity of $s(t) = e^{z(\tau)}$. Therefore, their methods do not work for showing the concavity of $s(t) = e^{z(\tau)}$ for the American call. By applying the results in the previous two sections, we will provide a rigorous proof for the concavity of $s(\tau)$ for an American call.

Let $C_K = \{(x,\tau)|K < x < s(\tau),\ 0 < t \leq T\}$, $T < \infty$. We apply $\mathcal{L}_0$ to ψ and obtain $\mathcal{L}_0\psi = rK - qx$ in C_K. This implies that

$$
\mathcal{L}_0\psi(x) \begin{cases} > 0, & x < \frac{r}{q}K \\ < 0, & x > \frac{r}{q}K \end{cases}
$$

in C_K for the case of $r > q$. On the other hand, we have $\mathcal{L}_0\psi(x) > 0$ in C_K for the case $r < q$. We also have that $\psi(x)(= x - K)$ is a smooth positive increasing function in C_K. Let

$$
M_0 = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + (r - q + \sigma^2)x\frac{\partial}{\partial x} - q,
$$

and $M = M_0 - \frac{\partial}{\partial \tau}$. Then we have $M_0\psi'(x) = -q < 0$ for $x > K$.

In order to apply Theorem 3.8, we need the following lemma.

Lemma 3.9. *Let (s, v) be a solution of (3.19) and $w = v - \psi$ be respectively defined on C_K if $r < q$ and on $C_{\frac{r}{q}K}$ if $r > q$. Then*

(a) $s(\tau)$ is a strictly increasing function.

(b) $s(\tau)$ is bounded above.

(c) $w_\tau(s(\tau), \tau) = 0$ and $w_x(s(\tau), \tau) = 0$.

(d) $w_x < 0$ in C_K if $r < q$ or in $C_{\frac{r}{q}K}$ if $r > q$.

Proof. In this lemma, we show the case of $r > q$ and the same method can be applied to the case of $r < q$. To show this lemma, we apply Lemma 3.2, Theorem 3.6 to (3.19). Since $\frac{1}{2}\sigma^2 x^2 > 0$ if $x > K$, $-r < 0$ and $\limsup_{x\to\infty} \mathcal{L}_0\psi(x) = \limsup_{\xi\to\infty}(rK - qx) < 0$, this implies that $s(t)$ is a bounded strictly increasing function by (a) in Theorem 3.6 and (d) in Lemma 3.2. Since $u_x(s(\tau), \tau) = \psi'(s(\tau))$, we have $w_x(s(\tau), \tau) = 0$ and $w_\tau(s(\tau), \tau) = 0$ by Lemma 3.7. Since $-r < 0$, $\psi'(x) = 1 > 0$ if $x > K$ and $M_0\psi'(x) = -q < 0$, we have $v_x < \psi'(x)$ in $C_{\frac{r}{q}K}$. Hence $w_x(x, \tau) = v_x(x, \tau) - \psi'(x) < 0$ in $C_{\frac{r}{q}K}$. □

By using Lemma 3.9 directly, we obtain the following theorem.

Theorem 3.10. *Let $\{s, v\}$ be a solution of (3.19). Then $s(\tau)$ is a strictly increasing concave function for $\tau > 0$.*

3.5 An Asymptotic Solution for the Early Exercise Boundary

Following the spirit of [12], the concavity of $s(\tau)$ provides useful information to obtain its asymptotic formula as time near to expiration. In order to obtain the asymptotic formula, we rewrite (3.19) in the following non-dimensional form by letting (3.20):

$$
\begin{aligned}
&Lp = 0, && \text{on } \mathcal{C}, \\
&p(y,0) = \max\{e^y - 1, 0\}, && \text{for } y \geq 0, \\
&p(\tau, s(\tau)) = e^{z(\tau)} - 1, && \text{for } 0 \leq \tau < T, \\
&p_y(\tau, s(\tau)) = e^{z(\tau)}, && \text{for } 0 \leq \tau < T,
\end{aligned}
$$

where $L = L_0 - \frac{\partial}{\partial \tau}$ and

$$L_0 = \frac{\partial}{\partial y^2} + (k-d-1)\frac{\partial}{\partial y} - k.$$

Here, we define $k = \frac{2r}{\sigma^2}$ and $d = \frac{2q}{\sigma^2}$.

The fundamental solution $\Gamma(y,\tau)$ of this operator is

$$\Gamma(y,\tau) = \frac{1}{2\sqrt{\pi\tau}} \exp[-\frac{y^2}{4\tau} - (\frac{k-d-1}{2})y - \frac{(k-d-1)^2+4k}{4}\tau]$$

and the solution of this problem is

$$p(y,\tau) = \int_0^\infty (e^\zeta - 1)\Gamma(y-\zeta,\tau)d\zeta + \int_0^\tau \int_{z(\tau-\lambda)}^\infty (de^\zeta - k)\Gamma(y-\zeta,\lambda)d\zeta d\lambda. \quad (3.21)$$

Applying the integration by parts and $L\Gamma = 0$ to the first integral of (3.21), we obtain that (see [70])

$$p(y,\tau) = \max\{e^y - 1, 0\} + \int_0^\tau \Gamma(y,\tau) + \int_0^{z(\tau-\lambda)} (k - de^\zeta)\Gamma(y-\zeta,\lambda)d\zeta d\lambda.$$

Since $p_\tau(z(\tau),\tau)=0$, we have

$$\lim_{y\to z(\tau)} \left\{\Gamma(y,\tau)+\int_0^{z(0)}(k-de^{\zeta})\Gamma(y-\zeta,\tau)d\zeta + \int_0^{\tau}(k-de^{\eta})\Gamma(y-z(\eta),\tau-\eta)z'(\eta)d\eta\right\}=0, \tag{3.22}$$

where $\eta = t-\lambda$.

Let $B(\tau,\eta;y) = \frac{y-z(\eta)}{2\sqrt{\tau-\eta}}$ and $\theta(\tau,\eta;y) = \frac{y-z(\eta)}{2(\tau-\eta)z'(\eta)}$. In abbreviation, we write $B = B(\tau,\eta;z(\tau))$ and $\theta = \theta(\tau,\eta;z(\tau))$ when $y = z(\tau)$. Since $s(\tau) = Ke^{z(\tau)}$ is a strictly increasing concave function, we have $z''(\tau) < 0$ and $z'(\tau)$ is a positive decreasing function. This implies that $\theta = \frac{z'(\xi)}{2z'(\eta)} \in (0,1/2)$ where $\xi \in (\eta,\tau)$ by using the mean value theorem. Hence $\mu(\tau)$ in (3.23) is bounded. Then we obtain the following lemma by using the concavity of $z(\tau)$.

Lemma 3.11. *Let*

$$\mu(\tau) = \frac{1}{\sqrt{\pi}}\int_0^{\alpha(\tau)} e^{-B^2}\frac{e^{\delta(z(\eta))}}{1-\theta}dB, \tag{3.23}$$

where $\alpha(\tau) = \frac{z(\tau)}{2\sqrt{\tau}}$ *and* $\delta(z(\eta)) = (\frac{k-d-1}{2}z(\eta)+(\frac{(k-d-1)^2}{4})\eta)$. *Then* $\mu(\tau)\to 1$ *as* $\tau\to 0$.

Since the argument for deriving this lemma has been provided by [12], we omit the proof of this lemma.

Now, Lemma 3.11 can be used to obtain the following asymptotic formula of $z(t)$. Evans *et al.* [18] also obtain the similar formula by dealing with the different equality of $z(\tau)$.

Theorem 3.12. *Near to expiry, the optimal exercise boundary of an American call option on an asset with dividends satisfies*

$$s(\tau) = Ke^{z(\tau)} \sim K + Kz(\tau) \sim K + 2K\tau^{1/2}\left[\log\frac{\sigma^2}{4(r-q)\sqrt{\pi\tau}}\right]^{1/2}, \qquad \text{for } q>r,$$

$$s(\tau) = Ke^{z(\tau)} \sim K + Kz(\tau) \sim K + 2K\tau^{1/2}\left[\log\frac{\sigma^2}{8\sqrt{\pi r\tau}}\right]^{1/2}, \qquad \text{for } q=r.$$

Proof. Applying (b) in Lemma 3.2, we have $s(0) = K$ when $q \geq r$. This implies that $z(0) = 0$ when $q \geq r$. We first consider the case of $q > r \geq 0$. Since $z(0) = 0$, (3.22) can be written as

$$\Gamma(z(\tau), \tau) + \int_0^\tau (k - de^{z(\eta)})\Gamma(z(\tau) - z(\eta), \tau - \eta)z'(\eta)d\eta = 0.$$

Dividing by

$$\exp[-(\frac{k-d-1}{2})z(\tau) - \frac{(k-d-1)^2 + 4k}{4}\tau],$$

yields

$$\frac{1}{2\sqrt{\pi\tau}}e^{-\alpha^2(\tau)} + k\int_0^\tau \frac{z'(\eta)}{2\sqrt{\pi(\tau-\eta)}}e^{a(\eta)}d\eta - d\int_0^\tau \frac{z'(\tau)}{2\sqrt{\pi(\tau-\eta)}}e^{b(\eta)}d\eta = 0,$$

where

$$a(\eta) = -B^2(\eta) + \tfrac{k-d-1}{2}z(\eta) - \tfrac{(k-d-1)^2+4k}{4}\eta,$$
$$b(\eta) = -B^2(\eta) + \tfrac{k-d+1}{2}z(\eta) - \tfrac{(k-d-1)^2+4k}{4}\eta.$$

Expanding $e^{b(\eta)}$ at $a(\eta)$ by using the Taylor expansion, we have

$$\frac{1}{2\sqrt{\pi\tau}}e^{-\alpha^2(\tau)} + (k-d)\int_0^\tau \frac{z'(\eta)}{2\sqrt{\pi(\tau-\eta)}}e^{a(\eta)}d\eta \sim 0,$$

as τ near to 0. Calculating $\frac{dB}{d\eta}$ directly, we have

$$\frac{1}{1-\theta}dB = \frac{-z'(\eta)}{2\sqrt{\tau-\eta}}d\eta.$$

Therefore, we have

$$\int_0^\tau \frac{z'(\eta)}{2\sqrt{\pi(\tau-\eta)}}e^{a(\eta)}d\eta = \int_0^{\alpha(\tau)} e^{-B^2}\frac{e^{\delta(z(\eta))}}{1-\theta}dB \sim 1,$$

as $\tau \to 0$, by using Lemma 3.11. Consequently, we have

$$\frac{e^{-\alpha^2(\tau)}}{\sqrt{\pi}} \sim 2(k-d)\sqrt{\tau}.$$

This implies that

$$s(\tau) = Ke^{z(\tau)} \sim K + Kz(\tau) \sim K + 2K\tau^{1/2}\left[\log \frac{\sigma^2}{4(r-q)\sqrt{\pi\tau}}\right]^{1/2}.$$

since $k = \frac{2r}{\sigma^2}$ and $d = \frac{2q}{\sigma^2}$.

On the other hand, we consider the case of $r = q > 0$ and we have $z(0) = 0$ in this case. (3.22) can be rewritten as

$$\lim_{y \to z(\tau)} \{\Gamma(y,\tau) + k\int_0^\tau (1 - e^{z(\eta)})\Gamma(y - z(\eta), \tau - \eta)z'(\eta)d\eta\} = 0.$$

This implies that

$$\lim_{y \to z(\tau)} \left\{ \frac{1}{2\sqrt{\pi\tau}} e^{-\alpha^2(\tau)} + k\int_0^\tau \frac{z'(\eta)}{2\sqrt{\pi(\tau-\eta)}}(e^{a(\eta)} - e^{b(\eta)})d\eta \right\} = 0,$$

where

$$a(\eta) = -B^2(\tau,\eta;y) - \frac{1}{2}z(\eta) - \frac{1+4k}{4}\eta$$

and

$$b(\eta) = -B^2(\tau,\eta;y) + \frac{1}{2}z(\eta) - \frac{1+4k}{4}\eta.$$

By using the Taylor expansion to $e^{b(\eta)}$ at $a(\eta)$, we have the following approximation formula

$$\lim_{y \to z(\tau)} \left\{ \frac{1}{2\sqrt{\pi\tau}} e^{-\alpha^2(\tau)} - k\int_0^\tau \frac{z'(\eta)z(\eta)}{\sqrt{\pi(\tau-\eta)}} e^{-B^2(\eta;y)} e^{\delta(\eta)} d\eta \right\} = 0, \text{ as } \tau \to 0,$$

where $\delta(\eta) = -(\frac{1}{2}z(\eta) + \frac{1+4k}{4}\eta)$. Rewriting $-z'(\eta)z(\eta) = z'(\eta)(y - z(\eta)) - yz'(\eta)$, we have

$$\begin{aligned} \lim_{y \to z(\tau)} \quad & \left\{ \frac{1}{2\sqrt{\pi\tau}} e^{-\alpha^2(\tau)} - k\int_0^\tau \frac{yz'(\eta)}{\sqrt{\pi(\tau-\eta)}} e^{-B^2(\eta;y)} e^{\delta(\eta)} d\eta \right. \\ & \left. + k\int_0^\tau \frac{z'(\eta)(y - z(\eta))}{\sqrt{\pi(\tau-\eta)}} e^{-B^2(\eta;y)} e^{\delta(\eta)} d\eta \right\} = 0. \end{aligned} \tag{3.24}$$

Taking limit at $z(\tau)$ and applying Lemma 3.11, the first integral of above equation approximates to $2z(\tau)$ as $\tau \to 0$.

Now we claim that the second integral of (3.24) approximates to 0 as $t \to 0$. Since $z(\eta)$ is a strictly increasing function, we set $\xi = z(\eta)$. Then $\eta = z^{-1}(\xi)$. The second integral of (3.24) can be written as

$$\int_0^{z(\tau)} B(\tau, z^{-1}(\xi); y) e^{-B^2(\tau, z^{-1}(\xi); y)} e^{\delta(z^{-1}(\xi))} d\xi,$$

where $B(\tau, z^{-1}(\xi); y) = \frac{y-\xi}{2\sqrt{\pi\tau(1-z^{-1}(\xi)/\tau)}}$. Since $z(\eta)$ is a strictly increasing function from 0 to $z(\tau)$, for any y there exists a unique η_0, which depends on y, such that $y = z(\eta_0) = \xi_0$. This implies that $B(\tau, z^{-1}(\xi_0); y) = 0$, $B(\tau, z^{-1}(\xi); y) > 0$ for any $\xi \in (0, \xi_0)$ and $B(\tau, z^{-1}(\xi); y) < 0$ for any $\xi \in (\xi_0, s(\tau))$.

Since $z(\eta)$ is a strictly increasing function from 0 to $z(\tau)$, we have $\eta = z^{-1}(\xi)$ lies between 0 and τ. This implies that $\frac{z^{-1}(\xi)}{\tau}$ lies between 0 and 1. As a result, we have

$$B^2(\tau, \xi; y) \to \infty$$

as $t \to 0$ for $\xi \neq \xi_0$ and

$$B^2(\tau, \xi_0; y) = 0$$

for all τ.

So we have that $\exp(-B^2(\tau, \xi; y) + \delta(z^{-1}(\xi))) \to 0$ as $\tau \to 0$ for $\xi \neq \xi_0$ and $\exp(-B^2(\tau, \xi_0; y) + \delta(z^{-1}(\xi_0))) = \exp(\delta(z^{-1}(\xi_0))) \to 1$ as $\tau \to 0$. Hence the main contribution of the second integral in (3.24) comes form the neighbor of ξ_0 as t is close to 0. When ξ nears to ξ_0, the expansion of B at ξ_0 is

$$\begin{aligned} B(\tau, z^{-1}(\xi); y) &= B(\tau, z^{-1}(\xi_0); y) + B_\xi(\tau, z^{-1}(\xi_0); y)(\xi - \xi_0) + \cdots \\ &= B_\xi(\tau, z^{-1}(\xi_0); y)(\xi - \xi_0) + \cdots . \end{aligned}$$

Since $B(\tau, z^{-1}(\xi_0); y) = 0$, we have

$$\int_0^\tau \frac{z'(\eta)(y - z(\eta))}{2\sqrt{\pi(\tau - \eta)}e^{-B^2(\eta;y)}e^{\delta(\eta)}} d\eta$$
$$\sim B(\tau, z^{-1}(\xi_0); y) \int_0^{z(\tau)} (\xi - \xi_0) e^{-B^2(\tau, z^{-1}(\xi); y)} dz \to 0$$

as $\tau \to 0$.

Hence, we have

$$\frac{e^{-\alpha^2(\tau)}}{2k\sqrt{\pi\tau}} - 2z(\tau) \sim 0.$$

Since $z(\tau) = 2\sqrt{\tau}\alpha(\tau)$, we have

$$e^{-\alpha^2(\tau)} \sim 4\sqrt{\pi}k\tau\alpha(\tau).$$

Therefore, the solution of (3.24) for α is

$$\alpha(\tau) \sim \sqrt{\log \frac{1}{4\sqrt{\pi}k\tau}}.$$

This implies that

$$s(\tau) = Ke^{z(\tau)} \sim K + Kz(\tau) \sim K + 2K\tau^{1/2}\left[\log \frac{\sigma^2}{8\sqrt{\pi}r\tau}\right]^{1/2}.$$

□