Chapter 4

American Exchange Option Pricing Problems

In Chapter 3, we have proposed an FBP for pricing of the single asset American option and shown that the optimal exercise boundary of an American call is a strictly increasing function of remaining time. The results provide a useful information for finding an asymptotic formula of the optimal exercise boundary. In this chapter, we extend this method to consider the pricing of the American exchange options (AEO) that are two-variate in nature. Namely, the option value is determined by the stochastic behaviors of two underlying asset prices and the correlations between these asset prices.

An AEO is an option which give the holder a right to exchange one asset to another at any time prior the expiration date T. In Section 4.1, we derive the parabolic FBP arising from the AEO pricing problems. In Section 4.2, we show that (1) the value of the AEO and the optimal exercise ratio are both strictly increasing functions of the remaining time; and (2) the value of the alive AEO is an increasing function of S_1 and a decreasing function of S_2. We deduce an IE for the exercise ratio and provide an asymptotic solution of this IE in Section 4.3 and Section 4.4. For the infinite time horizon AEO, the exact value of the exercise ratio and the AEO are given in Section 4.5. In Section 4.6, the single asset integral recursive method [36] is extended to solve the IE numerically. We find that our asymptotic solution is very close to the numerical solution.

4.1 The Formulation of AEO

In this section, we shall derive a two variables FBP for pricing of the AEOs under the perfect market assumptions.

Let S_1 and S_2 be the price of asset 1 and asset 2. Under the risk neutral probability measure, the stochastic processes for the asset price is assumed to be

$$\frac{dS_i}{S_i} = (r - q_i)dt + \sigma_i dw_i, \ i = 1, 2,$$

where r, q_i, and σ_i are the constant risk-free interest rate, the continuous dividend rate of the asset i and the volatility the i-th asset, respectively. Here, dw_1 and dw_2 are the Wiener processes of asset 1 and asset 2 respectively, and their correlation coefficient is $\mathrm{corr}(dw_1, dw_2) = \rho dt$.

To derive the pricing operator of the exchange option, we first recall the European exchange option (EEO) pricing problems [55]. According to the definition of the exchange option, the final payoff of an EEO is given by

$$V(S_1, S_2, T) = \max(S_1 - S_2, 0), \tag{4.1}$$

where $V(S_1, S_2, t)$ denotes the value of EEO at the time $t \leq T$. As the suggestion of Margrabe [55], the value of EEO satisfies the linear homogeneous property in S_1 and

S_2, that is

$$V(\lambda S_1, \lambda S_2, t) = \lambda V(S_1, S_2, t).$$

We apply Euler theorem to the function $V(S_1, S_2, t)$ and obtain the following equation:

$$V - S_1 \frac{\partial V}{\partial S_1} - S_2 \frac{\partial V}{\partial S_2} = 0. \tag{4.2}$$

This means that a portfolio of holding $\frac{\partial V}{\partial S_i}$ units of asset i, $i = 1, 2$, becomes a replication of EEO.

By applying Itô Lemma to (4.2) and considering the instantaneous return with the dividend rate of both assets, we obtain the following pricing equation

$$V_t + \mathcal{L}_{BS} V = 0, \ t < T, \tag{4.3}$$

where the operator $\mathcal{L}_{BS}$ is defined as

$$\mathcal{L}_{BS} V = \frac{1}{2}\sigma_1^2 S_1^2 V_{S_1 S_1} + \rho \sigma_1 \sigma_2 S_1 S_2 V_{S_1 S_2} + \frac{1}{2}\sigma_2^2 S_2^2 V_{S_2 S_2} - q_1 S_1 V_{S_1} - q_2 S_2 V_{S_2}.$$

The EEO's price is then the solution of (4.3) with terminal condition (4.1).

For the AEO pricing problem, since this style of option can be exercised at any time $t < T$, the AEO pricing problem in [7] is formulated as an FBP. Let $P(S_1, S_2, t)$ be the value of an AEO, then $P(S_1, S_2, t)$ must satisfy the following inequality under the no arbitrage condition:

$$P(S_1, S_2, t) \geq \max(S_1 - S_2, 0), \ 0 \leq t \leq T.$$

Let $X_f(t)$ be the smallest value of $\frac{S_1}{S_2}$ such that

$$P(S_1, S_2, t) > \max(S_1 - S_2, 0), \ 0 \leq t \leq T.$$

Then, at any given time t, the (S_1, S_2)-plane can be separated into two distinct regions as follows:

$$\mathbf{S}(t) = \left\{ (S_1, S_2) \in \mathcal{R}^+ \times \mathcal{R}^+ | \frac{S_1}{S_2} \geq X_f(t) \right\}, \tag{4.4}$$

$$\mathbf{C}(t) = \left\{ (S_1, S_2) \in \mathcal{R}^+ \times \mathcal{R}^+ | \frac{S_1}{S_2} < X_f(t) \right\}, \tag{4.5}$$

where $\mathcal{R}^+$ denotes the set of nonnegative real numbers. The regions **S** and **C** are called the early exercise region and the holding region, and the ratio $X_f(t)$ is called the optimal exercise ratio.

According to the argument of no arbitrage, we need the following two conditions

$$P(S_1, S_2, t) = S_1 - S_2, \ \frac{\partial P}{\partial S_1}(S_1, S_2, t) = 1 \text{ and } \frac{\partial P}{\partial S_2}(S_1, S_2, t) = -1 \tag{4.6}$$

when $\frac{S_1}{S_2} = X_f(t)$. Condition (4.6) is commonly called the high contact conditions, so named because (4.6) indicates that $P(S_1, S_2, t)$, $\frac{\partial P}{\partial S_1}(S_1, S_2, t)$ and $\frac{\partial P}{\partial S_2}(S_1, S_2, t)$ are continuous across the optimal exercise boundary. Thus, the value $P(S_1, S_2, t)$ of an AEO together with the optimal exercise ratio $X(t)$ are the solution of the following FBP.

$$\mathcal{L}P = 0, \qquad (S_1, S_2) \in \mathbf{C}(t), \ 0 < t < T, \tag{4.7}$$

$$P(S_1, S_2, T) = (S_1 - S_2)^+, \qquad t = T \tag{4.8}$$

$$P(S_1, S_2, t) = S_1 - S_2, \qquad (S_1, S_2) \in \partial\mathbf{C}(t), \ 0 < t < T, \tag{4.9}$$

$$P(0, S_2, t) = 0, \qquad 0 < S_2 < \infty, \ 0 < t < T, \qquad (\mathbf{A}) \tag{4.10}$$

$$P(S_1, 0, t) = S_1, \qquad 0 < S_1 < \infty, \ 0 < t < T, \tag{4.11}$$

$$P_{S_1}(S_1, S_2, t) = 1, \qquad (S_1, S_2) \in \partial\mathbf{C}(t) \ 0 < t < T, \tag{4.12}$$

$$P_{S_2}(S_1, S_2, t) = -1, \qquad (S_1, S_2) \in \partial\mathbf{C}(t) \ 0 < t < T. \tag{4.13}$$

Here, $\partial \mathbf{C}(t)$ denotes the boundary of $\mathbf{C}(t)$.

4.2 Properties of the Free Boundary

Let $\mathbf{Q} = (0,\infty)\times(0,\infty)$ and $\Omega = \{(S_1,S_2) \in \mathbf{Q} | q_1 S_1 - q_2 S_2 > 0,\ S_1 > S_2\}$, then Ω is an open subset of $\mathbf{Q}$ such that $\mathcal{L}\psi(S_1,S_2) > 0$ for $(S_1,S_2) \in \Omega$ and $\mathcal{L}\psi(S_1,S_2) < 0$ for $(S_1,S_2) \in \mathbf{Q} - \Omega$, where $\psi(S_1,S_2) = \max\{S_1 - S_2, 0\}$. In the following theorem, we will provide the properties of problem A.

Let $\tau = T - t$ be the remaining time.

Theorem 4.1. *Let $\{P, X\}$ be the solution of problem A. Then*

(a) $\mathbf{C}(0) = \Omega$.

(b) *$P(S_1,S_2,\tau)$ is an increasing function of τ.*

(c) *$\{\mathbf{C}(\tau)\}_{\tau=0}^{T}$ is a strictly increasing sequence, that $\mathbf{C}(\tau_1) \subset \mathbf{C}(\tau_2)$ for $\tau_1 < \tau_2$.*

Proof. For (a), we first suppose that there exists an $(S_1,S_2) \in \mathbf{C}(0) - \Omega$ such that

$$\lim_{\tau\to 0} \frac{\partial P}{\partial \tau}(S_1,S_2,\tau) = \lim_{\tau\to 0} \mathcal{L}P(S_1,S_2,\tau) = \mathcal{L}\psi(x) < 0.$$

This contradicts to (4.12). Now, we suppose that $\mathbf{C}(0) \subset \Omega$. Then there exists a τ_1 such that $\mathcal{C}(\tau) \subset \Omega$ for $0 \le \tau \le \tau_1$. Let $D = \{(S_1,S_2,\tau) | (S_1,S_2) \in \mathbf{C}(\tau), 0 < \tau < \tau_1\}$ and define $v = P - \psi$. Then

$$v_\tau - \mathcal{L}_0 v = \mathcal{L}\psi > 0 \text{ for } (S_1,S_2,\tau) \in D.$$

We apply boundary point form maximum principle to this equation. Since $v(S_1, S_2, \tau) \geq 0$ for $(S_1, S_2, \tau) \in D$ and $v(S_1, S_2, \tau) = 0$ for $(S_1, S_2) \in \partial\mathbf{C}(\tau)$, $0 < \tau < \tau_1$, it shows that $v_{S_1}(S_1, S_2, \tau) < 0$. This contradicts to (4.12). So we have shown that $\mathbf{C}(0) \equiv \Omega$.

For (b), we observe (4.9) and obtain

$$P_{S_2}(S_1, X(\tau)S_1, \tau)S_1X'(\tau) + P_\tau(S_1, X(\tau)S_1, \tau) = -S_1X'(\tau).$$

This implies that $P_\tau(S_1, X(\tau)S_1, \tau) = 0$ for $0 < S_1 < \infty$, $0 < \tau \leq T$ by (4.12). We also have $P_\tau(S_1, S_2, 0) = \mathcal{L}P(S_1, S_2, 0) = \mathcal{L}\psi(S_1, S_2) > 0$ on $\mathbf{C}(0)$ and $P_\tau(0, S_2, \tau) = \lim_{S_2\to\infty} P_\tau(S_1, S_2, \tau) = 0$. The final equation is obtained by

$$\lim_{h\to 0} \frac{P(0, S_2, \tau + h) - P(0, S_2, \tau)}{h} = 0$$
$$\lim_{h\to 0}(\lim_{S_2\to\infty} \frac{P(0, S_2, \tau + h) - P(0, S_2, \tau)}{h}) = 0.$$

since $P(0, S_2, \tau) = \lim_{S_2\to\infty} P(S_1, S_2, \tau) = 0$. Since $P_\tau - \mathcal{L}_0P = 0$ in $\mathbf{C} = \bigcup_{\tau=0}^{T} \mathbf{C}(\tau)$ and the coefficients of $\mathcal{L}_0$ do not depend on τ, we have

$$(P_\tau)_\tau - \mathcal{L}P_\tau = 0 \text{ on } \mathbf{C}.$$

We apply maximum principle to this equation. This shows that $P_\tau(S_1, S_2, \tau) \geq 0$ on $\mathbf{C}$.

For (c), let $(S_1, S_2) \in \mathbf{C}(\tau_1)$. Then

$$P(S_1, S_2, \tau_2) \geq P(S_1, S_2, \tau_1) > \psi(S_1, S_2), \text{ for } \tau_2 > \tau_1.$$

This implies that $(S_1, S_2) \in \mathbf{C}(\tau_2)$. So we obtain that $\mathbf{C}(\tau_1) \subseteq \mathbf{C}(\tau_2)$.

We now suppose that there exist τ_1 and τ_2 such that

$$\partial\mathbf{C}(\tau_1) \cap \partial\mathbf{C}(\tau_2) \neq \emptyset,$$

say $(S_1^*, S_2^*) \in \partial\mathbf{C}(\tau_1) \cap \partial\mathbf{C}(\tau_2)$. This implies that for any $\tau \in (\tau_1, \tau_2)$ we have $(S_1^*, S_2^*) \in \partial\mathbf{C}(\tau)$. Let $D = \{(S_1, S_2, \tau) | P_\tau - \mathcal{L}P = 0,\ \tau_1 < \tau < \tau_2\}$ and $w = \psi - P$. Since $D \subset \mathbf{C}$, we have $w_\tau - \mathcal{L}w = -\mathcal{L}\psi > 0$ in D by (4.7). We apply boundary point form of maximum principle to this equation. Since $w(S_1^*, S_2^*, \tau) = \psi(S_1^*, S_2^*) - P(S_1^*, S_2^*, \tau) = 0$, we have $w_{S_1} < 0$. This implies that $P_{S_1}(S_1^*, S_2^*, \tau) > \psi_{S_1}(S_1^*, S_2^*) = 1$ which contradicts to (4.12). So there exists no such point $(S_1^*, S_2^*) \in \partial\mathbf{C}(\tau_1) \cap \partial\mathbf{C}(\tau_2)$ for any τ_1 and τ_2. Thus, we have $\{\mathbf{C}(\tau)\}_{\tau=0}^T$ is a strictly increasing sequence of τ. □

Consequently, we have the following corollary.

Corollary 4.2. *Let $\{P, X\}$ be a solution of problem A. Then $X(t)$ is a strictly decreasing function with $X(T) = \max\{1, q_1/q_2\}$.*

4.3 The Integral Equation

Before discussing the solution of the AEO in the next section, we first derive an integral equation by defining new variables $y_i = \frac{-1}{\sigma_i}(q_i + \frac{1}{2}\sigma_i^2)\tau + \frac{1}{\sigma_i}\ln(S_i)$, $i = 1, 2$ and $\tau = T - t$. Let $p(y_1, y_2, \tau) = P(S_1, S_2, t)$ and $x_f(\tau) = X_f(t)$, then the original pricing problem (4.7)-(4.9) can be written in the following dimensionless form:

$$\frac{\partial p}{\partial \tau} = \mathcal{L}p, \qquad \text{in } 0 < \tau < T,\ y_1 - \frac{\sigma_2}{\sigma_1}y_2 \le x_f(\tau), \quad (4.14)$$

$$p(y_1, y_2, 0) = (e^{\sigma_1 y_1} - e^{\sigma_2 y_2})^+, \qquad \text{at } \tau = 0, \quad (4.15)$$

$$p(y_1, y_2, \tau) = e^{(q_1 + \frac{1}{2}\sigma_1^2)\tau + \sigma_1 y_1} - e^{(q_2 + \frac{1}{2}\sigma_2^2)\tau + \sigma_2 y_2}, \quad \text{at } y_1 - \frac{\sigma_2}{\sigma_1}y_2 = x_f(\tau), \quad (4.16)$$

where $\mathcal{L}$ is the operator given as

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial^2}{\partial y_1^2} + 2\rho\frac{\partial^2}{\partial y_1 \partial y_2} + \frac{\partial^2}{\partial y_2^2}\right).$$

Under this transformation, the relation between $X_f(t)$ and $x_f(\tau)$ is defined as

$$x_f(\tau) = \frac{1}{\sigma_1}\left(\ln(X_f(T-\tau)) + (q_2 + \frac{1}{2}\sigma_2^2 - q_1 - \frac{1}{2}\sigma_1^2)\tau\right). \tag{4.17}$$

By imposing (4.16) into (4.14), the problem (4.14)-(4.16) can be converted into a non-homogeneous equation as follows:

$$p_\tau - \mathcal{L}p = \begin{cases} 0, & \text{if } y_1 - \frac{\sigma_2}{\sigma_1}y_2 \le x_f(\tau), \\ q_1 e^{(q_1+\frac{1}{2}\sigma_1^2)\tau+\sigma_1 y_1} - q_2 e^{(q_2+\frac{1}{2}\sigma_2^2)\tau+\sigma_2 y_2}, & \text{if } y_1 - \frac{\sigma_2}{\sigma_1}y_2 \ge x_f(\tau). \end{cases} \tag{4.18}$$

Before solving the FBP of (4.7)-(4.9), we shall find out the representation of $x_f(t)$.

Theorem 4.3. *The optimal exercise ratio $x_f(t)$ satisfies the following integral equation:*

$$\begin{aligned} e^{(q_1+\frac{1}{2}\sigma_1^2)\tau+\sigma_1 x_f(\tau)} - e^{(q_2+\frac{1}{2}\sigma_2^2)\tau} = {} & e^{\sigma_1 x_f(\tau)+\frac{1}{2}\sigma_1^2\tau} N(\frac{\bar{a}_1}{\sigma}) - e^{\frac{1}{2}\sigma_2^2\tau} N(\frac{\bar{a}_2}{\sigma}) \\ & + e^{\sigma_1 x_f(\tau)+\frac{1}{2}\sigma_1^2\tau} \int_0^\tau q_1 e^{q_1 s} N(\frac{\bar{a}_3}{\sigma}) ds \\ & - e^{\frac{1}{2}\sigma_2^2\tau} \int_0^\tau q_2 e^{q_2 s} N(\frac{\bar{a}_4}{\sigma}) ds, \end{aligned} \tag{4.19}$$

where

$$\bar{a}_1 = \tfrac{1}{\sqrt{\tau}}(\sigma_1 x_f(\tau) + (\sigma_1^2 - \rho\sigma_1\sigma_2)\tau),$$

$$\bar{a}_2 = \tfrac{1}{\sqrt{\tau}}(\sigma_1 x_f(\tau) + (\rho\sigma_1\sigma_2 - \sigma_2^2)\tau),$$

$$\bar{a}_3 = \tfrac{1}{\sqrt{\tau-s}}(\sigma_1(x_f(\tau) - x_f(s)) + (\sigma_1^2 - \rho\sigma_1\sigma_2)(\tau - s)),$$

$$\bar{a}_4 = \tfrac{1}{\sqrt{\tau-s}}(\sigma_1(x_f(\tau) - x_f(s)) + (\rho\sigma_1\sigma_2 - \sigma_2^2)(\tau - s)),$$

and N is the cumulative distribution function of a standard normal random variable $N(0,1)$.

In order to prove the theorem, we need the following two lemma:

Lemma 4.4. *[42] The Green's function $\phi(x, y, \tau)$ for (4.18) is given by*

$$\begin{aligned}&\phi(y_1, y_2, \tau; \xi_1, \xi_2, s)\\ &= \tfrac{1}{2\pi(\tau-s)}\tfrac{1}{\sqrt{1-\rho^2}}exp(-\tfrac{(y_1-\xi_1)^2-2\rho(y_1-\xi_1)(y_2-\xi_2)+(y_2-\xi_2)^2}{2(1-\rho^2)(\tau-s)}).\end{aligned}$$

Lemma 4.5. *Let*

$$\varphi(u_1, u_2) = \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}}exp(-\frac{u_1^2 - 2\rho u_1 u_2 + u_2^2}{2(1-\rho^2)})$$

be the probability density function of the standard bivariate normal distribution with covariant correlation ρ. Then, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{a+\frac{\sigma_1}{\sigma_2}u_2}\varphi(u_1, u_2)du_1du_2 = \int_{-\infty}^{\frac{\sigma_1}{\sigma}a}\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}dv = N\left(\frac{\sigma_1}{\sigma}a\right),$$

for any real number a.

Proof.

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{a+\frac{\sigma_2 u_2}{\sigma_1}}\frac{1}{2\pi\sqrt{1-\rho^2}}\varphi(u_1, u_2)du_1du_2\\ &= \int_{-\infty}^{a}\int_{-\infty}^{\infty}\frac{1}{2\pi\sqrt{1-\rho^2}}e^{-\frac{\sigma_1^2 v_1^2}{2\sigma^2}}e^{-\frac{\sigma^2}{2\sigma_1^2(1-\rho^2)}\left(v_2+\frac{\sigma_1\sigma_2-\sigma_1\rho}{\sigma^2}v_1\right)^2}dv_2dv_1\\ &= \int_{-\infty}^{a}\frac{1}{\sqrt{2\pi}}e^{-\frac{(\sigma_1 v_1)^2}{2\sigma^2}}\frac{\sigma_1}{\sigma}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{w^2}{2}}dwdv_1\\ &= \int_{-\infty}^{\frac{a\sigma_1}{\sigma}}\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}dv = N(\frac{a\sigma_1}{\sigma})\end{aligned}$$

where $v_1 = u_1 - \frac{\sigma_2}{\sigma_1}u_2$, $v_2 = u_2$, and $w = \frac{\sigma(v_2+\frac{\sigma_1\sigma_2-\sigma_1\rho}{\sigma}v_1)}{\sigma_1\sqrt{1-\rho^2}}$. □

Proof of Theorem 4.3 We apply Green's function to $p(x, y, \tau)$ as well as the fact that ϕ is in a domain bounded by the optimal exercise boundary and the line $\tau = 0$

obtaining that

$$\begin{aligned} p(y_1,y_2,\tau) = & \int_{-\infty}^{\infty}\int_{\frac{\sigma_2}{\sigma_1}\xi_1}^{\infty}(e^{\sigma_1\xi_1}-e^{\sigma_2\xi_2})\phi(y_1,y_2,\tau;\xi_1,\xi_2,0)d\xi_1 d\xi_2 \\ & +\int_0^{\tau}\int_{-\infty}^{\infty}\int_{\frac{\sigma_2}{\sigma_1}\xi_1+x_f(s)}^{\infty}(q_1e^{\sigma_1\xi_1}-q_2e^{\sigma_2\xi_2})\phi(y_1,y_2,\tau;\xi_1,\xi_2,s)d\xi_1 d\xi_2 ds \end{aligned} \tag{4.20}$$

We separate (4.20) into the following four integrals:

$$\begin{aligned} p(y_1,y_2,\tau) = & \int_{-\infty}^{\infty}\int_{\frac{\sigma_2}{\sigma_1}\xi_1}^{\infty} e^{\sigma_1\xi_1}\phi(y_1,y_2,\tau;\xi_1,\xi_2,0)d\xi_1 d\xi_2 \\ & -\int_{-\infty}^{\infty}\int_{\frac{\sigma_2}{\sigma_1}\xi_1}^{\infty} e^{\sigma_2\xi_2}\phi(y_1,y_2,\tau;\xi_1,\xi_2,0)d\xi_1 d\xi_2 \\ & +\int_0^{\tau}\int_{-\infty}^{\infty}\int_{\frac{\sigma_2}{\sigma_1}\xi_1+x_f(s)}^{\infty} q_1e^{\sigma_1\xi_1}\phi(y_1,y_2,\tau;\xi_1,\xi_2,s)d\xi_1 d\xi_2 ds \\ & -\int_0^{\tau}\int_{-\infty}^{\infty}\int_{\frac{\sigma_2}{\sigma_1}\xi_1+x_f(s)}^{\infty} q_2e^{\sigma_2\xi_2}\phi(y_1,y_2,\tau;\xi_1,\xi_2,s)d\xi_1 d\xi_2 ds \\ = & \; I^{(1)}(y_1,y_2,\tau)-I^{(2)}(y_1,y_2,\tau)+I^{(3)}(y_1,y_2,\tau)-I^{(4)}(y_1,y_2,\tau). \end{aligned} \tag{4.21}$$

In order to rewrite the integrals $I^{(1)}$-$I^{(4)}$, we let $\sqrt{\tau}u_1-\sigma_1\tau=y_1-\xi_1$ and $\sqrt{\tau}u_2-\rho\sigma_1\tau=y_2-\xi_2$ in $I^{(1)}$, $\sqrt{\tau}u_1-\rho\sigma_2\tau=y_1-\xi_1$ and $\sqrt{\tau}u_2-\sigma_2\tau=y_2-\xi_2$ in $I^{(2)}$, $\sqrt{\tau-s}u_1-\sigma_1(\tau-s)=y_1-\xi_1$ and $\sqrt{\tau-s}u_2-\rho\sigma_1(\tau-s)=y_2-\xi_2$ in $I^{(3)}$ and $\sqrt{\tau-s}u_1-\rho\sigma_2(\tau-s)=y_1-\xi_1$ and $\sqrt{\tau-s}u_2-\sigma_2(\tau-s)=y_2-\xi_2$ in $I^{(4)}$, and then the integrals $I^{(1)}$-$I^{(4)}$ can be written as the following equations:

$$\begin{aligned} I^{(1)} &= e^{(\sigma_1y_1+\frac{1}{2}\sigma_1^2\tau)}\int_{-\infty}^{\infty}\int_{-\infty}^{a_1(y_1,y_2,\tau)+bu_2}\varphi(u_1,u_2)du_1du_2, \\ I^{(2)} &= e^{(\sigma_2y_2+\frac{1}{2}\sigma_2^2\tau)}\int_{-\infty}^{\infty}\int_{-\infty}^{a_2(y_1,y_2,\tau)+bu_2}\varphi(u_1,u_2)du_1du_2, \\ I^{(3)} &= e^{(\sigma_1y_1+\frac{1}{2}\sigma_1^2\tau)}\int_0^{\tau}q_1e^{q_1s}\int_{-\infty}^{\infty}\int_{-\infty}^{a_3(y_1,y_2,\tau,x_f(\tau))+bu_2}\varphi(u_1,u_2)du_1du_2ds, \\ I^{(4)} &= e^{(\sigma_2y_2+\frac{1}{2}\sigma_2^2\tau)}\int_0^{\tau}q_2e^{q_2s}\int_{-\infty}^{\infty}\int_{-\infty}^{a_4(y_1,y_2,\tau,x_f(\tau))+bu_2}\varphi(u_1,u_2)du_1du_2ds, \end{aligned} \tag{4.22}$$

where

$$a_1(y_1,y_2,\tau)=\frac{1}{\sqrt{\tau}}(y_1-\frac{\sigma_2}{\sigma_1}y_2+(\sigma_1-\rho\sigma_2)\tau),$$
$$a_2(y_1,y_2,\tau)=\frac{1}{\sqrt{\tau}}(y_1-\frac{\sigma_2}{\sigma_1}y_2+(\rho\sigma_2-\frac{\sigma_2^2}{\sigma_1})\tau),$$
$$a_3(y_1,y_2,\tau,s,x_f(s))=\frac{1}{\sqrt{\tau-s}}(y_1-\frac{\sigma_2}{\sigma_1}y_2+(\sigma_1-\rho\sigma_2)(\tau-s)-x_f(s)),$$
$$a_4(y_1,y_2,\tau,s,x_f(s))=\frac{1}{\sqrt{\tau-s}}(y_1-\frac{\sigma_2}{\sigma_1}y_2+(\rho\sigma_2-\frac{\sigma_2^2}{\sigma_1})(\tau-s)-x_f(s)),$$
$$b=\frac{\sigma_2}{\sigma_1}.$$

Here, $\varphi(u_1,u_2)$ is a probability density function of the bivariate standard normal distribution with correlation correlation ρ. By using Lemma 4.5, we get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{a_i+bu_2}\varphi(u_1,u_2)du_1du_2=N(\frac{\sigma_1 a_i}{\sigma}),\ i=1,2,3,4.$$

Hence from (4.22), we have

$$\begin{aligned}
I^{(1)}&=e^{(\sigma_1y_1+\frac{1}{2}\sigma_1^2\tau)}N\left(\frac{\sigma_1a_1(y_1,y_2,\tau)}{\sigma}\right),\\
I^{(2)}&=e^{(\sigma_2y_2+\frac{1}{2}\sigma_2^2\tau)}N\left(\frac{\sigma_1a_2(y_1,y_2,\tau)}{\sigma}\right),\\
I^{(3)}&=e^{(\sigma_1y_1+\frac{1}{2}\sigma_1^2\tau)}\int_0^{\tau}q_1e^{q_1s}N\left(\frac{\sigma_1a_3(y_1,y_2,\tau,s,x_f(s))}{\sigma}\right)ds,\\
I^{(4)}&=e^{(\sigma_2y_2+\frac{1}{2}\sigma_2^2\tau)}\int_0^{\tau}q_2e^{q_2s}N\left(\frac{\sigma_1a_4(y_1,y_2,\tau,s,x_f(s))}{\sigma}\right)ds.
\end{aligned}\tag{4.23}$$

If the value of $\frac{S_1}{S_2}$ reaches the optimal exercise ratio at the first time, that is $\frac{S_1}{S_2}=X_f(T-\tau)$ or $\sigma_1y_1-\sigma_2y_2=\sigma_1x_f(\tau)$, it is optimal to exercise the AEO. By (4.16), (4.21) and (4.23), we get (4.19). □

After replacing $x_f(\tau)$ by $X_f(\tau)$, Theorem 4.3 can be rewritten in the following form:

Theorem 4.6. *The optimal exercise ratio $X_f(t)$ satisfies the following integral equation:*

$$\begin{aligned} X_f(T-\tau) - 1 = \ & X_f(T-\tau)e^{-q_1\tau}N(\hat{a}_1) - e^{-q_2\tau}N(\hat{a}_2) \\ & + X_f(T-\tau)e^{-q_1\tau}\int_0^\tau q_1 e^{q_1 s}N(\hat{a}_3)ds \\ & - e^{-q_2\tau}\int_0^\tau q_2 e^{q_2 s}N(\hat{a}_4)ds, \end{aligned} \tag{4.24}$$

where

$$\begin{aligned} \hat{a}_1 &= \tfrac{1}{\sigma\sqrt{\tau}}(\ln X_f(T-\tau) + \tfrac{1}{2}(\sigma^2 - 2q_1 + 2q_2)\tau), \\ \hat{a}_2 &= \tfrac{1}{\sigma\sqrt{\tau}}(\ln X_f(T-\tau) - \tfrac{1}{2}(\sigma^2 + 2q_1 - 2q_2)\tau), \\ \hat{a}_3 &= \tfrac{1}{\sigma\sqrt{\tau-s}}(\ln \tfrac{X_f(T-\tau)}{X_f(T-s)} + \tfrac{1}{2}(\sigma^2 - 2q_1 + 2q_2)(\tau - s)), \\ \hat{a}_4 &= \tfrac{1}{\sigma\sqrt{\tau-s}}(\ln \tfrac{X_f(T-\tau)}{X_f(T-s)} - \tfrac{1}{2}(\sigma^2 + 2q_1 - 2q_2)(\tau - s)), \\ \sigma^2 &= \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2. \end{aligned}$$

4.4 An Asymptotic Solution of Finite-Lived AEO

The explicit solution of (4.24) is not easy to obtain when the expiration date T is finite. In this section we will apply the properties of the complementary error function to provide an asymptotic solution for (4.24).

Let erfc(x) denotes the complementary error function, i.e.

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt.$$

The relation between error function and normal distribution function is

$$N(x) = 1 - \frac{1}{2}\text{erfc}(\frac{x}{\sqrt{2}}). \tag{4.25}$$

By using the Taylor expansion and integration by parts, the complementary error function is asymptotic to

$$\mathrm{erfc}(x) = \frac{e^{-x^2}}{\sqrt{\pi}x}(1 - \frac{1}{2x^2} + - \cdots) \sim \frac{e^{-x^2}}{\sqrt{\pi}x}, \text{ as } x \to \infty. \tag{4.26}$$

In this section, we will replace $N(x)$ in terms of $\mathrm{erfc}(\frac{x}{\sqrt{2}})$ and provide an asymptotic solution for the optimal exercise ratio $X(T-\tau)$ as the remaining time near to zero.

Before deriving the asymptotic expression of $X(T-\tau)$, we introduce the following lemma which has been provided in [18].

Lemma 4.7. *Let $B(z,\tau)$ be a monotone decreasing function of z on $[0,1]$. Suppose that there is a $z_0 \in [0,1]$ such that $B(z_0,\tau) = 0$ for all τ and that $B^2(z,\tau) \to \infty$ for all $z \neq z_0$ as $\tau \to 0$. Then, as τ is near to 0, we have following two asymptotic formulas:*

$$\int_0^1 A(z)e^{-B^2(z)}dz \sim A(z_0)\frac{\sqrt{\pi}}{|B_z(z_0)|}, \tag{4.27}$$

$$\frac{1}{\sqrt{\pi}}\int_0^1 B^{-1}e^{-B^2}dz \sim -\frac{B_{zz}(z_0)}{2|B_z(z_0)|^3}. \tag{4.28}$$

Proof. Since $B^2(z,\tau) \to \infty$ for all $z \neq z_0$ as $\tau \to 0$ and $B^2(z_0,\tau) = 0$ for all τ then $e^{-B^2(z,\tau)} \to 0$ for all $z \neq z_0$ as $\tau \to 0$ and $e^{-B^2(z_0,\tau)} = 1$ for all τ. This implies that the neighborhood of z_0 provides the main contribution to the value of the Laplace integral as τ is near to 0. Thus, we expand $B^2(z,\tau)$ at $z = z_0$ by using Taylor expansion and obtain that

$$\begin{aligned} B^2(z,\tau) &= B^2(z_0,\tau) + 2B(z_0,\tau)B_z(z_0,\tau)(z-z_0) + B_z^2(z_0,\tau)(z-z_0)^2 + \cdots \\ &\sim B_z^2(z_0,\tau)(z-z_0)^2 \end{aligned}$$

since $B(z_0,\tau) = 0$. And then, we use this expansion formula in the exponent. As τ

is near to 0, the Laplace integral will approximate to the following Gaussian integral

$$\int_0^1 A(z)e^{-B^2(z,\tau)} \sim A(z_0)\int_0^1 e^{-B_z^2(z_0,\tau)(z-z_0)^2}dz.$$

Now, we rewrite the above Gaussion integral by its asymptotic formula and obtain that

$$A(z_0)\int_0^1 e^{-B_z^2(z_0,\tau)(z-z_0)^2}dz \sim A(z_0)\frac{\sqrt{\pi}}{|B_z(z_0,\tau)|}.$$

Since $B(z_0) = 0$ then $B^{-1} \to \infty$ as $z \to z_0$. The above result can not be applied when $A(z) = B^{-1}(z,\tau)$. Now, we rewrite $B^{-1}(z,\tau)$ as follows:

$$\begin{aligned} B^{-1} &= B^{-1} - [B_z(z_0,\tau)(z-z_0)]^{-1} + [B_z(z_0,\tau)(z-z_0)]^{-1} \\ &= \frac{B_z(z_0)(z-z_0) - B(z)}{B(z)B_z(z_0)(z-z_0)} + [B_z(z_0)(z-z_0)]^{-1} \\ &\sim -\frac{B_{zz}(z_0,\tau)}{2B_z^2(z_0)} + [B_z(z_0)(z-z_0)]^{-1} \end{aligned}$$

Here, the final term of above equation is obtain by applying Tayor expansion to $B(z)$ at $z = z_0$. Now, we use this asymptotic formula to substitute B^{-1} and obtain the following formula

$$\begin{aligned} \frac{1}{\sqrt{\pi}}\int_0^1 B^{-1}e^{-B^2}dz &\sim \frac{1}{\sqrt{\pi}}\int_0^1 \left(-\frac{B_{zz}(z_0,\tau)}{2B_z^2(z_0)} + [B_z(z_0)(z-z_0)]^{-1}\right)e^{-B_z^2(z_0)(z-z_0)^2}dz \\ &\sim -\frac{B_{zz}(z_0)}{2|B_z(z_0)|^3}. \end{aligned}$$

□

We now begin to derive the asymptotic expression of $X(T-\tau)$ as τ is near to 0.

Theorem 4.8. *The asymptotic solution of (4.24), when τ is close to zero, is as follows:*

1. *For* $q_1 > q_2$,

$$X(T-\tau) \sim (1 + \frac{\sigma^2\tau(\frac{q_1}{q_1-q_2}) + d}{1+\sigma^2\tau(\frac{q_1}{q_1-q_2})^2})e^{(q_1-q_2)\tau}, \tag{4.29}$$

where $d = \sqrt{\sigma^4\tau^2(\frac{q_1}{q_1-q_2})^2 - 2\sigma^2\tau\log[\frac{(q_1-q_2)\pi\sqrt{2\tau}}{\sigma^2}](1+\sigma^2\tau(\frac{q_1}{q_1-q_2})^2)}$.

2. *For* $q_1 = q_2$,

$$X(T-\tau) \sim e^{-\left[-2\sigma^2\tau\log\left(\sqrt{2}\pi\tau\sigma^{-2}q_1\right)\right]^{1/2}}. \tag{4.30}$$

Proof. By defining $Y(T-\tau) = X(T-\tau)e^{(q_2-q_1)\tau}$, (4.24) can be converted as follows:

$$\begin{aligned} Y(T-\tau)e^{q_1\tau} - e^{q_2\tau} = {} & Y(T-\tau)N(\frac{\tilde{a}_1(\tau)}{\sigma}) - N(\frac{\tilde{a}_2(\tau)}{\sigma}) \\ & +Y(T-\tau)\int_0^\tau q_1 e^{q_1 s}N(\frac{\tilde{a}_3(\tau,s)}{\sigma})ds \\ & -\int_0^\tau q_2 e^{q_2 s}N(\frac{\tilde{a}_4(\tau,s)}{\sigma})ds, \end{aligned} \tag{4.31}$$

where

$$\begin{aligned} \tilde{a}_1(\tau) &= \tfrac{\log Y(T-\tau)}{\sqrt{\tau}} + \tfrac{1}{2}\sigma^2\sqrt{\tau} = \tilde{a}_2(\tau) + \sigma^2\sqrt{\tau}, \\ \tilde{a}_3(\tau,s) &= \tfrac{\log\frac{Y(T-\tau)}{Y(T-s)}}{\sqrt{\tau-s}} + \tfrac{1}{2}\sigma^2\sqrt{\tau-s} = \tilde{a}_4(\tau,s) + \sigma^2\sqrt{\tau-s}. \end{aligned}$$

By applying (4.25), we express (4.31) in terms of the complementary error function as follows:

$$\begin{aligned} & Y(T-\tau)\text{erfc}(\tfrac{1}{\sqrt{2}}(\tfrac{\log Y(T-\tau)}{\sigma\sqrt{\tau}} + \tfrac{1}{2}\sigma\sqrt{\tau})) - \text{erfc}(\tfrac{1}{\sqrt{2}}(\tfrac{\log Y(T-\tau)}{\sigma\sqrt{\tau}} - \tfrac{1}{2}\sigma\sqrt{\tau})) \\ & = \lim_{y\to Y(T-\tau)}\{y\int_0^\tau q_1 e^{q_1 s}\text{erfc}(\frac{1}{\sqrt{2}}(\frac{\log\frac{y}{Y(T-s)}}{\sigma\sqrt{\tau-s}} + \frac{1}{2}\sigma\sqrt{\tau-s}))ds \\ & \quad -\int_0^\tau q_2 e^{q_2 s}\text{erfc}(\frac{1}{\sqrt{2}}(\frac{\log\frac{y}{Y(T-s)}}{\sigma\sqrt{\tau-s}} - \frac{1}{2}\sigma\sqrt{\tau-s}))ds\}. \end{aligned} \tag{4.32}$$

As τ near to zero, since $X(T-\tau) > X(T) \geq 1$ and $Y(T-\tau) = X(T-\tau)e^{(q_2-q_1)\tau}$ then $\frac{1}{\sqrt{2}}(\frac{\log Y(T-\tau)}{\sigma\sqrt{\tau}} \pm \frac{1}{2}\sigma\sqrt{\tau}) = \frac{1}{\sqrt{2}}(\frac{\log X(T-\tau)}{\sigma\sqrt{\tau}} \pm \frac{1}{2}(\sigma + \frac{q_2-q_1}{\sigma})\sqrt{\tau})$ tends to infinity Thus, component of LHS of (4.32), $\text{erfc}(\frac{1}{\sqrt{2}}(\frac{\log Y(T-\tau)}{\sigma\sqrt{\tau}} \pm \frac{1}{2}\sigma\sqrt{\tau}))$, has the following asymptotic form:

$$\begin{aligned}&\text{erfc}(\frac{1}{\sqrt{2}}(\frac{\log Y(T-\tau)}{\sigma\sqrt{\tau}} \pm \frac{1}{2}\sigma\sqrt{\tau}))\\ &\sim \frac{1}{\sqrt{\pi}}\frac{1}{\frac{\log Y(T-\tau)}{\sigma\sqrt{\tau}} \pm \frac{1}{2}\sqrt{\sigma^2\tau}}e^{-\frac{1}{2}(\frac{\log Y(T-\tau)}{\sigma\sqrt{\tau}}\pm\frac{1}{2}\sigma\sqrt{\tau})^2},\end{aligned} \tag{4.33}$$

as $\tau \to 0$. By applying the integral mean value theorem, the integrand of RHS of (4.32) can be rewritten as

$$\begin{aligned}&\text{erfc}(\frac{1}{\sqrt{2}}(\frac{\log\frac{y}{Y(T-s)}}{\sigma\sqrt{\tau-s}} \pm \frac{1}{2}\sigma\sqrt{\tau-s}))\\ &= \text{erfc}(\frac{1}{\sqrt{2}}(\frac{\log\frac{y}{Y(T-s)}}{\sigma\sqrt{\tau-s}})) \mp \frac{2}{\sqrt{\pi}}\int_{\frac{1}{\sqrt{2}}(\frac{\log\frac{y}{Y(T-s)}}{\sigma\sqrt{\tau-s}})}^{\frac{1}{\sqrt{2}}(\frac{\log\frac{y}{Y(T-s)}}{\sigma\sqrt{\tau-s}}\pm\frac{1}{2}\sigma\sqrt{\tau-s})} e^{-\eta^2}d\eta\\ &= \text{erfc}(\frac{1}{\sqrt{2}}(\frac{\log\frac{y}{Y(T-s)}}{\sigma\sqrt{\tau-s}})) \mp \frac{1}{\sqrt{\pi}}\sigma\sqrt{\tau-s}e^{-c^2},\end{aligned}$$

where c lies between $\frac{1}{\sqrt{2}}(\frac{\log\frac{y}{Y(T-s)}}{\sigma\sqrt{\tau-s}})$ and $\frac{1}{\sqrt{2}}(\frac{\log\frac{y}{Y(T-s)}}{\sigma\sqrt{\tau-s}} \pm \frac{1}{2}\sigma\sqrt{\tau-s})$. By setting $s = \tau z$, and considering τ near to zero, we have $c \sim \frac{1}{\sqrt{2}}(\frac{\log\frac{y}{Y(T-s)}}{\sigma\sqrt{\tau-s}})$ and $e^{q_i\tau} \sim 1$, $i = 1, 2$. Then the RHS of (4.32) has the following asymptotic form

$$\begin{aligned}&\lim_{y\to Y(T-\tau)}\{(q_1y-q_2)\tau\int_0^1 \text{erfc}(\frac{1}{\sqrt{2}}\frac{\log\frac{y}{Y(T-\tau z)}}{\sqrt{\sigma^2\tau(1-z)}})dz\\ &-(q_1y-q_2)\frac{\sigma^2\tau^{\frac{3}{2}}}{\sqrt{\pi}}\int_0^1\sqrt{1-z}e^{\frac{-\log^2\frac{y}{Y(T-\tau z)}}{2\sigma^2\tau(1-z)}}dz\}.\end{aligned}$$

Therefore, we derive the following asymptotic equation of (4.32):

$$\begin{aligned}&\sqrt{\frac{2}{\pi}}\frac{\sigma^2\tau^{\frac{3}{2}}}{\log^2 Y(T-\tau)}e^{-\frac{1}{2}(\frac{\log Y(T-\tau)}{\sqrt{\sigma^2\tau}}-\frac{1}{2}\sqrt{\sigma^2\tau})^2}\\ &\sim \lim_{y\to Y(T-\tau)}\{(q_1y-q_2)\tau\int_0^1 \text{erfc}(\frac{1}{\sqrt{2}}\frac{\log\frac{y}{Y(T-\tau z)}}{\sqrt{\sigma^2\tau(1-z)}})dz\\ &-(q_1y-q_2)\sigma^2\tau^{\frac{3}{2}}\int_0^1\sqrt{\frac{1-z}{\pi}}e^{\frac{-\log^2\frac{y}{Y(T-\tau z)}}{2\sigma^2\tau(1-z)}}dz\}.\end{aligned} \tag{4.34}$$

Note that

$$Y(T) = \max(1, \frac{q_2}{q_1}).$$

Now we consider the case of $q_1 \geq q_2$ and let

$$\alpha(\tau) = \frac{-\log Y(T-\tau)}{\sqrt{\tau}}, \text{ for } q_1 \geq q_2,$$

and then (4.34) can be converted as follows:

$$\begin{aligned}
&\frac{\sigma^2\tau^{3/2}}{\tau\alpha^2(\tau)} e^{-\frac{\tau\alpha^2(\tau)}{2\sigma^2\tau}} \\
&\sim \lim_{y\to Y(T-\tau)} \sqrt{\frac{\pi}{2}} \left(q_1 e^{-\sqrt{\tau}\alpha(\tau)} - q_2\right) \tau \int_0^1 \operatorname{erfc}\left(B(z,\alpha(z\tau),y)\right) dz \\
&\quad - \left(q_1 e^{-\sqrt{\tau}\alpha(\tau)} - q_2\right) \frac{\sigma^2\tau^{3/2}}{\sqrt{2}} \int_0^1 \sqrt{1-z}\, e^{-B^2(z,\alpha(z\tau),y)} dz,
\end{aligned} \tag{4.35}$$

where

$$B(z,\alpha(\tau z),y) = \frac{\sqrt{z}\alpha(\tau z) - \frac{\log \frac{q_2}{q_1 y}}{\sqrt{\tau}}}{\sigma\sqrt{2(1-z)}}.$$

By applying the definition of $\alpha(\tau)$ and take the limit under the integral, we have

$$B(z,\alpha(\tau z),\alpha(\tau)) = \frac{\sqrt{z}\alpha(\tau z) - \alpha(\tau)}{\sigma\sqrt{2(1-z)}}.$$

For convenient we denote $B(z,\tau,y)$ as $B(z)$. Since $Y(T-\tau z)$ is a monotone increasing function of z, then there is an unique number z_0 such that

$$Y(T-\tau z_0) = y$$

and $Y(T-\tau z_0) < y$ for $z < z_0$ and $Y(T-\tau z) > y$ for $z > z_0$. This implies that $B(z,\tau,y) \to \infty$ for all z in $[0,z_0)$ and $B(z,\tau,y) \to -\infty$ for all z in $(z_0,1]$, as $\tau \to 0$. Thus, we replace $\operatorname{erfc}(B(z))$ by using (4.26) for $B(z,\tau) \to \pm\infty$ when τ is small. Then the first integral of (4.35) can be rewritten as the following asymptotic formula :

$$\begin{aligned}
\int_0^1 \operatorname{erfc}(B(z))dz &\sim \frac{1}{\sqrt{\pi}} \int_0^{z_0(x)} B^{-1}(z) e^{-B^2(z)} dz + \int_{z_0(x)}^1 (2 + B^{-1}(z)\frac{e^{-B^2(z)}}{\sqrt{\pi}})dz \\
&= 2[1-z_0] + \frac{1}{\sqrt{\pi}} \int_0^1 B^{-1}(z) e^{-B^2(z)} dz.
\end{aligned}$$

In order to find out the asymptotic solution of (4.35), we consider that y approaches to $Y(T-\tau)$ and sets $z_0 = 1$. The remainder is to evaluate the following two integrals

$$\frac{1}{\sqrt{\pi}}\int_0^1 B^{-1}(z)e^{-B^2(z)}dz \text{ and } \int_0^1 \sqrt{1-z}e^{-B^2(z)}dz.$$

Since $B(z_0,\tau) = 0$ for all τ and, for $z \neq z_0$, $B^2(z,\tau) \to \infty$ as $\tau \to 0$ then

$$\frac{1}{\sqrt{\pi}}\int_0^1 B^{-1}(z)e^{-B^2(z)}dz \sim -\frac{B_{zz}(z_0)}{2|B_z(z_0)|^3}, \tag{4.36}$$

$$\int_0^1 \sqrt{1-z}e^{-B^2(z)}dz \sim \sqrt{1-z_0}\frac{\sqrt{\pi}}{|B_z(z_0)|}, \tag{4.37}$$

by using lemma 4.7.

The limit of the first integral in (4.35) is asymptotic to the RHS of (4.36). We see that this asymptotic expression is

$$\int_0^1 \text{erfc}B(z)dz \sim \frac{2\sqrt{\pi}}{\alpha^2(\tau)} \tag{4.38}$$

as $z_0 \to 1$. By applying (4.37), the second integral of (4.35) tends to zero as $z_0 \to 1$. So we obtain the following equation:

$$e^{-\frac{\alpha^2(\tau)}{2\sigma^2}} \sim \frac{\sqrt{2\tau}\pi}{\sigma^2}\left(q_1 e^{-\sqrt{\tau}\alpha(\tau)} - q_2\right). \tag{4.39}$$

Since $Y(T-\tau) = e^{-\sqrt{\tau}\alpha(\tau)}$, (4.39) can be rewritten as

$$e^{-\frac{\log^2 Y(T-\tau)}{2\sigma^2\tau}} \sim \frac{\sqrt{2\tau}\pi}{\sigma^2}(q_1 Y(T-\tau) - q_2). \tag{4.40}$$

Let $Y(T-\tau) = 1 + y(\tau)$, then (4.40) can be rewritten as

$$e^{-\frac{\log^2(1+y(\tau))}{2\sigma^2\tau}} \sim \frac{\sqrt{2\tau}\pi}{\sigma^2}(q_1 y(\tau) + q_1 - q_2).$$

And then, we have

$$-\frac{\log^2(1+y(\tau))}{2\sigma^2\tau} \sim \log[\frac{(q_1-q_2)\pi\sqrt{2\tau}}{\sigma^2}] + \log(\frac{q_1}{q_1-q_2}y(\tau)+1).$$

Multiplying above equation by $2\sigma^2\tau$ and expanding $\log^2(1+y(\tau))$ and $\log(\frac{q_1}{q_1-q_2}y(\tau)+1)$ at 1, we obtain that

$$-y^2(\tau) \sim 2\sigma^2\tau\log[\frac{(q_1-q_2)\pi\sqrt{2\tau}}{\sigma^2}] + 2\sigma^2\tau(\frac{q_1}{q_1-q_2}y(\tau) + \frac{q_1^2}{2(q_1-q_2)^2}y^2(\tau)).$$

This implies that

$$[1+\sigma^2\tau\frac{q_1^2}{(q_1-q_2)^2}]y^2(\tau) + 2\sigma^2\tau(\frac{q_1}{q_1-q_2})y(\tau) + 2\sigma^2\tau\log[\frac{(q_1-q_2)\pi\sqrt{2\tau}}{\sigma^2}] = 0,$$

and that the solution of this quadratic equation is

$$y(\tau) = \frac{\sigma^2\tau(\frac{q_1}{q_1-q_2})+d}{1+\sigma^2\tau(\frac{q_1}{q_1-q_2})^2},$$

where

$$d = \sqrt{\sigma^4\tau^2(\frac{q_1}{q_1-q_2})^2 - 2\sigma^2\tau\log[\frac{(q_1-q_2)\pi\sqrt{2\tau}}{\sigma^2}](1+\sigma^2\tau(\frac{q_1}{q_1-q_2})^2)}.$$

Here, we select positive term to make sure $y(\tau) \geq 0$. So, we have

$$Y(T-\tau) = 1 + \frac{\sigma^2\tau(\frac{q_1}{q_1-q_2})+d}{1+\sigma^2\tau(\frac{q_1}{q_1-q_2})^2},$$

and

$$X(T-\tau) = (1+\frac{\sigma^2\tau(\frac{q_1}{q_1-q_2})+d}{1+\sigma^2\tau(\frac{q_1}{q_1-q_2})^2})e^{(q_1-q_2)\tau}.$$

However, the above approximation can not be applied to the case $q_1 = q_2$. We use first order approximation to $q_1e^{-\sqrt{\tau}\alpha(\tau)} - q_2$ and obtain

$$q_1e^{-\sqrt{\tau}\alpha(\tau)} - q_1 \sim -q_1\sqrt{\tau}\alpha(\tau), \text{ as } \tau \to 0.$$

Now, (4.39) can be rewritten as follows:

$$e^{-\frac{\alpha^2(\tau)}{2\sigma^2}} \sim -\sqrt{2}\pi\tau\sigma^{-2}q_1\alpha(\tau).$$

Beginning the iteration scheme from the initial value $\alpha_0 = 0$, we obtain that

$$\alpha(\tau) \sim \left[-2\sigma^2 \log\left(\sqrt{2}\pi\tau\sigma^{-2}q_1\right)\right]^{1/2}.$$

Then

$$Y(T-\tau) \sim e^{-\left[-2\sigma^2\tau \log\left(\sqrt{2}\pi\tau\sigma^{-2}q_1\right)\right]^{1/2}},$$

and

$$X(T-\tau) \sim e^{-\left[-2\sigma^2\tau \log\left(\sqrt{2}\pi\tau\sigma^{-2}q_1\right)\right]^{1/2}}.$$

□

Finally, we convert (4.21) in terms of S_1, S_2 and $X_f(T-\tau)$ as follows:

$$\begin{aligned} P(S_1,S_2,\tau) = {} & S_1e^{-q_1\tau}N(\frac{a_1}{\sigma}) - S_2e^{-q_2\tau}N(\frac{a_2}{\sigma}) \\ & +q_1S_1e^{-q_1\tau}\int_0^\tau e^{q_1s}N(\frac{a_3}{\sigma})ds - q_2S_2e^{-q_2\tau}\int_0^\tau N(\frac{a_4}{\sigma})ds, \end{aligned} \tag{4.41}$$

where

$$\begin{aligned} a_1 &= \tfrac{1}{\sqrt{\tau}}\left(\ln(\tfrac{S_1}{S_2}) + \delta_1\tau\right), \\ a_2 &= \tfrac{1}{\sqrt{\tau}}\left(\ln(\tfrac{S_1}{S_2}) - \delta_2\tau\right), \\ a_3 &= \tfrac{1}{\sqrt{\tau-s}}\left(\ln(\tfrac{S_1}{S_2X_f(\tau-s)}) + \delta_1(\tau-s)\right), \\ a_4 &= \tfrac{1}{\sqrt{\tau-s}}\left(\ln(\tfrac{S_1}{S_2X_f(\tau-s)}) - \delta_2(\tau-s)\right). \end{aligned}$$

4.5 The Exact Solution of the Perpetual AEO

A perpetual option is an option which does not have an expiry date but rather an infinite time horizon. Namely, the expiry date T for a perpetual option is equal to infinity at the initial. The optimal exercise ratio for the perpetual AEO in (4.24) is a time-invariant constant [36], denoted as $X_f(\infty)$. Since τ is any point in $(0, T)$ and $0 < s < \tau$, $X_f(T-\tau)$ and $X_f(T-s)$ in (4.24) are both replaced by $X_f(\infty)$. Therefore, the optimal exercise ratio $X_f(\infty)$ satisfies the following equation

$$\begin{aligned} X_f(\infty) - 1 = \; & X_f(\infty)e^{-q_1\tau}N(\hat{a}_1) - e^{-q_2\tau}N(\hat{a}_2) \\ & + X_f(\infty)e^{-q_1\tau}\int_0^\tau q_1 e^{q_1 s}N(\hat{a}_3)ds - e^{-q_2\tau}\int_0^\tau q_2 e^{q_2 s}N(\hat{a}_4)ds, \end{aligned} \tag{4.42}$$

where

$$\begin{aligned} \hat{a}_1 &= \frac{\ln X_f(\infty) + \delta_1\tau}{\sigma\sqrt{\tau}}, \; \hat{a}_2 = \frac{\ln X_f(\infty) - \delta_2\tau}{\sigma\sqrt{\tau}}, \\ \hat{a}_3 &= \frac{\delta_1}{\sigma}\sqrt{\tau - s}, \; \hat{a}_4 = \frac{-\delta_2}{\sigma}\sqrt{\tau - s}, \\ \delta_1 &= \frac{1}{2}(\sigma^2 - 2q_1 + 2q_2), \; \delta_2 = \frac{1}{2}(\sigma^2 + 2q_1 - 2q_2). \end{aligned}$$

Theorem 4.9. *The value of the optimal exercise ratio $X_f(\infty)$ is*

$$X_f(\infty) = \frac{\left(1 + \sqrt{\frac{\delta_2^2}{\delta_2^2 + 2q_2\sigma^2}}\right)}{\left(1 - \sqrt{\frac{\delta_1^2}{\delta_1^2 + 2q_1\sigma^2}}\right)}. \tag{4.43}$$

Proof. Let $u = \tau - s$, then $\hat{a}_3 = \frac{\delta_1}{\sigma}\sqrt{u}$. Using integration by parts to the third term on the RHS of (4.42), we obtain

$$\begin{aligned} & \int_0^\tau q_1 e^{q_1(\tau-u)}N\left(\frac{\delta_1}{\sigma}\sqrt{u}\right)du \\ & = -N(\frac{\delta_1}{\sigma}\sqrt{\tau}) + \frac{1}{2}e^{q_1\tau} + \frac{1}{2}e^{q_1\tau}\frac{\delta_1}{\sigma}\int_0^\tau \frac{1}{\sqrt{2\pi u}}e^{-(q_1+(\frac{\delta_1}{\sqrt{2}\sigma})^2)u}du. \end{aligned} \tag{4.44}$$

Applying the same argument to the forth term in (4.42), we also obtain

$$\begin{aligned} & \int_0^\tau q_2 e^{q_2(\tau-u)}N\left(-\frac{\delta_2}{\sigma}\sqrt{u}\right)du \\ & = -N(-\frac{\delta_2}{\sigma}\sqrt{\tau}) + \frac{1}{2}e^{q_2\tau} - \frac{1}{2}e^{q_2\tau}\frac{\delta_2}{\sigma}\int_0^\tau \frac{1}{\sqrt{2\pi u}}e^{-(q_2+(\frac{\delta_2}{\sqrt{2}\sigma})^2)u}du. \end{aligned} \tag{4.45}$$

Substituting (4.44)-(4.45) into (4.42), and letting the remaining time τ tend to infinity, since the terms $e^{-q_1\tau}N\left(\frac{\delta_1\sqrt{\tau}}{\sigma}\right)$ and $e^{-q_2\tau}N\left(\frac{-\delta_2\sqrt{\tau}}{\sigma}\right)$ both tend to zero, then we get the following equation

$$\begin{aligned}&\left(\frac{1}{2}-\frac{\delta_1}{2\sigma}\int_0^\infty \frac{1}{\sqrt{2\pi u}}e^{-(\frac{\delta_1^2}{2\sigma^2}+q_1)u}du\right)X_f(\infty)\\ &=\left(\frac{1}{2}+\frac{\delta_2}{2\sigma}\int_0^\infty \frac{1}{\sqrt{2\pi u}}e^{-(\frac{\delta_2^2}{2\sigma^2}+q_2)u}du\right).\end{aligned} \tag{4.46}$$

By the well-known result

$$\frac{1}{\sqrt{2\pi}}\int_0^\tau \frac{1}{\sqrt{u}}e^{-cu}du \to \sqrt{\frac{1}{2c}},\ \text{as } \tau\to\infty$$

and (4.46), we get (4.43). □

Note that $X_f(\infty)$ in the above theorem is a constant when the parameters σ_1, σ_2, q_1, q_2 and ρ are given. Substituting $X_f(\infty)$ into (4.21), the early exercise premium for the perpetual AEO can be obtained as follows:

Theorem 4.10. *The early exercise premium of the perpetual AEO is*

$$\left(\frac{S_1}{2}\left(\frac{\delta_1}{D_1}-1\right)e^{-\left(\frac{D_1+\delta_1}{\sigma^2}\right)}+\frac{S_2}{2}\left(\frac{\delta_2}{D_2}+1\right)e^{-\left(\frac{D_2-\delta_2}{\sigma^2}\right)}\right)\left(\frac{S_1}{S_2X_f(\infty)}\right),$$

where $D_i=\sqrt{\delta_i^2+2q_i\sigma^2}$, $i=1,2$.

In order to prove this theorem, we need to use the moment generating function of Inverse Gaussian distribution. This function is given as follows, see Berg [5]:

Lemma 4.11. *Let* X *be an Inverse Gaussian random variable with mean* μ *and variance* μ^2/ν *and its probability density funciton is written as*

$$h(x|\mu,\nu)=\left(\frac{\mu\nu}{2\pi x^3}\right)^{1/2}e^{-\frac{\nu(x-\mu)^2}{2\mu x}}$$

then the moment generating function of Inverse Gaussian random variable is

$$\mathrm{E}\left[e^{tX}\right] = \int_0^\infty e^{tx} h(x|\mu,\nu)dx = e^{\nu-\eta(t)},$$

where $\eta(t) = \sqrt{\nu^2 - 2t\mu\nu}$.

Proof of Theorem 4.10. We first express (4.21) in terms of S_1, S_2 and $X_f(T-\tau)$ as follows:

$$\begin{aligned} P(S_1,S_2,\tau) = \; & S_1 e^{-q_1\tau} N\left(\frac{\ln(\frac{S_1}{S_2}) + \delta_1\tau}{\sigma\sqrt{\tau}}\right) - S_2 e^{-q_2\tau} N\left(\frac{\ln(\frac{S_1}{S_2}) - \delta_2\tau}{\sigma\sqrt{\tau}}\right) \\ & + q_1 S_1 e^{-q_1\tau}\int_0^\tau e^{q_1 s} N\left(\frac{a_3}{\sigma}\right)ds - q_2 S_2 e^{-q_2\tau}\int_0^\tau e^{q_2 s} N\left(\frac{a_4}{\sigma}\right)ds, \end{aligned} \tag{4.47}$$

where

$$\begin{aligned} a_3 &= \tfrac{1}{\sqrt{\tau-s}}\left(\ln\left(\tfrac{S_1}{S_2X_f(\tau-s)}\right) + \delta_1(\tau-s)\right), \\ a_4 &= \tfrac{1}{\sqrt{\tau-s}}\left(\ln\left(\tfrac{S_1}{S_2X_f(\tau-s)}\right) - \delta_2(\tau-s)\right). \end{aligned}$$

Replacing $X_f(\tau-s)$ by $X_f(\infty)$ and letting $u = \tau - s$, the early exercise premium of the perpetual AEO is reduced to

$$P(S_1,S_2) = S_1\int_0^\infty q_1 e^{-q_1 u} N\left(\frac{a_3}{\sigma}\right)du - S_2\int_0^\infty q_2 e^{-q_2 u} N\left(\frac{a_4}{\sigma}\right)du, \tag{4.48}$$

where

$$\begin{aligned} a_3(S_1,S_2) &= \frac{1}{\sqrt{u}}\left(\ln\left(\frac{S_1}{S_2X_f(\infty)}\right) + \delta_1 u\right), \\ a_4(S_1,S_2) &= \frac{1}{\sqrt{u}}\left(\ln\left(\frac{S_1}{S_2X_f(\infty)}\right) - \delta_2 u\right), \end{aligned}$$

when the remaining time τ tends to infinity. Here, the first two terms of RHS in (4.47) are both converge to zero since $N(x)$ is bounded, and $e^{-q_1\tau}$, $e^{-q_2\tau}$ both converge to zero.

Applying integration by parts to the first integral of (4.48), we obtain that

$$\begin{aligned} & q_1 \int_0^\infty e^{-q_1 u} N\left(\frac{\delta_1 u + \delta_1}{\sigma\sqrt{u}}\right) du \\ = & \frac{-A}{2\sigma}\left[\int_0^\infty e^{-q_1 u}\left(\frac{1}{2\pi u^3}\right)^{1/2} e^{-\frac{(\delta_1 u + A)^2}{2\sigma^2 u}} du\right] \\ & + \frac{\delta_1}{2\sigma}\left[\int_0^\infty e^{-q_1 u}\left(\frac{1}{2\pi u}\right)^{1/2} e^{-\frac{(\delta_1 u + A)^2}{2\sigma^2 u}} du\right], \end{aligned} \tag{4.49}$$

where $A = \ln\left(\frac{S_1}{S_2 X_f(\infty)}\right)$.

Let $\mu_1 = \frac{-A}{\delta_1}$ and $\nu_1 = \frac{-A\delta_1}{\sigma^2}$, then $\mu_1\nu_1 = \frac{A^2}{\sigma^2}$. Using Lemma 4.11 to the first term of RHS in (4.49), we obtain that

$$\int_0^\infty e^{-q_1 u}\left(\frac{\mu_1\nu_1}{2\pi u^3}\right)^{1/2} e^{-\nu_1\frac{(u-\mu_1)^2}{2\mu_1 u}} du = e^{\nu_1 - \sqrt{\nu_1^2 + 2q_1\nu_1\mu_1}} = e^{\frac{\delta_1 + D_1}{\sigma^2}A}, \tag{4.50}$$

where $D_1 = \sqrt{\delta_1^2 + 2q_1\sigma^2}$.

By expanding $(\delta_1 u + A)^2$ and letting $u = v^2$, we have

$$\frac{\delta_1}{2\sigma}\int_0^\infty e^{-q_1 u}\left(\frac{1}{2\pi u}\right)^{1/2} e^{-\frac{(\delta_1 u + A)^2}{2\sigma^2 u}} du = \frac{\delta_1 e^{-\frac{\delta_1 A}{\sigma^2}}}{\sigma}\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\left(\alpha_1 v^2 + \beta v^{-2}\right)} dv$$

in (4.49), where $\alpha_1 = \frac{\delta_1^2}{2\sigma^2} + q_1$ and $\beta = \frac{A^2}{2\sigma^2}$. Note that the following identity can be obtained from the integral table:

$$\int_0^\infty e^{-(\alpha_1 v^2 + \beta v^{-2})} dv = \frac{1}{2}\sqrt{\frac{\pi}{\alpha_1}} e^{-2\sqrt{\alpha_1\beta}}$$

for any positive real number α_1, β. Hence, we obtain

$$\frac{\delta_1}{2\sigma}\int_0^\infty e^{-q_1 u}\left(\frac{1}{2\pi u}\right)^{1/2} e^{-\frac{(\delta_1 u + A)^2}{2\sigma^2 u}} du = \frac{\delta_1}{2D_1} e^{-\frac{(D_1 + \delta_1)A}{\sigma^2}}. \tag{4.51}$$

in (4.49). By (4.49), (4.50) and (4.51), we get

$$\int_0^\infty q_1 e^{-q_1 u} N\left(\frac{a_3}{\sigma}\right) du = \frac{1}{2}\left(\frac{\delta_1}{D_1} - 1\right) e^{-\left(\frac{D_1 + \delta_1}{\sigma^2}\right)}\left(\frac{S_1}{S_2 X_f(\infty)}\right).$$

in (4.48).

The same process can also be applied to the second integral of (4.48) and we have derived an explict pricing formula for the early exercise premium of AEO. □

To demonstrate the pricing formula in Theorem 4.2, let us consider the following example.

Example 4.12. *The current price and the constant volatility of asset 1 are given as 40 and* 40%. *For asset 2, the current price and the constant volatility are given as 35 and* 40%. *The correlation coefficient between two assets is given as* $\rho = 10\%$. *Consider the following three cases:* (q_1, q_2)=*(0.1, 0.01), (0.05, 0.01) and (0.02, 0.01). From case 1 to case 3, the optimal exercise ratio are 2.1644, 3.5219, and 10.0365, respectively, and the early exercise premium are 12.5432, 9.2213, and 3.4067, respectively.*

4.6 Integral Recursive Methods

Many numerical methods have been discussed for the FBP derived from American style options (Kim [36], Ju and Zhong [35]). We use the integral recursive (IR) method, which was proposed by Kim [36], to calculate a numerical solution of (4.19).

In our numerical procedure, all integrals in (4.19) are approximated by the trapezoid rule. Let $\{\tau_i\}_{i=0}^{n}$ be a partition of $[0, \tau]$, $\Delta\tau \equiv \tau_{i+1} - \tau_i = \tau/n$ and x_i denote the numerical solution of $x(\tau_i)$, $i = 1, 2, \cdots, n$. We have known that $x_0 = x(\tau_0) = \max(0, \sigma_1 \log(\frac{q_2}{q_1}))$. As $i = 1$, x_1 is approximated by solving the following nonlinear algebra equation

$$
\begin{aligned}
& e^{(q_1+\frac{1}{2}\sigma_1^2)\tau_1+\sigma_1 x_1} - e^{(q_2+\frac{1}{2}\sigma_2^2)\tau_1} \\
& = e^{\sigma_1 x_1+\frac{1}{2}\sigma_1^2\tau_1} N(\frac{\overline{a}_1(x_1,\tau_1)}{\sigma}) - e^{\frac{1}{2}\sigma_2^2\tau_1} N(\frac{\overline{a}_2(x_1,\tau_1)}{\sigma}) \\
& \quad + e^{\sigma_1 x_1+\frac{1}{2}\sigma_1^2\tau_1} \frac{\Delta\tau}{2}(q_1 N(\frac{\overline{a}_3(x_0,\tau_0,x_0)}{\sigma}) + q_1 e^{q_1\tau_1} N(\frac{\overline{a}_3(x_1,\tau_1,x_0)}{\sigma})) \\
& \quad - e^{\frac{1}{2}\sigma_2^2\tau_1} \frac{\Delta\tau}{2}(q_2 N(\frac{\overline{a}_4(x_0,\tau_0,x_0)}{\sigma}) + q_2 e^{q_2\tau_1} N(\frac{\overline{a}_4(x_1,\tau_1,x_0)}{\sigma})),
\end{aligned}
\tag{4.52}
$$

where $\overline{a}_1(x,\tau) = \frac{1}{\sqrt{\tau}}(x+(\sigma_1^2-\rho\sigma_1\sigma_2)\tau)$, $\overline{a}_2(x,\tau) = \frac{1}{\sqrt{\tau}}(x+(\rho\sigma_1\sigma_2-\sigma_2^2)\tau)$, $\overline{a}_3(x,\tau,y) = \frac{1}{\sqrt{\tau}}(x+(\sigma_1^2-\rho\sigma_1\sigma_2)\tau-y)$, and $\overline{a}_4(x,\tau,y) = \frac{1}{\sqrt{\tau}}(x+(\rho\sigma_1\sigma_2-\sigma_2^2)\tau-y)$. Here, x_1, which is the only unknown number in (4.52), can be obtained by using the root-finding method.

Recursively, the general algebra equation for x_i, $i = 2, 3, \ldots, n$ can be reduced by

$$
\begin{aligned}
& e^{(q_1+\frac{1}{2}\sigma_1^2)\tau_i+\sigma_1 x_i} - e^{(q_2+\frac{1}{2}\sigma_2^2)\tau_i} \\
& = e^{\sigma_1 x_i+\frac{1}{2}\sigma_1^2\tau_i} N(\frac{\overline{a}_1(x_i,\tau_i)}{\sigma}) - e^{\frac{1}{2}\sigma_2^2\tau_i} N(\frac{\overline{a}_2(x_i,\tau_i)}{\sigma}) \\
& \quad + e^{\sigma_1 x_i+\frac{1}{2}\sigma_1^2\tau_i} \frac{\Delta\tau}{2}\{N(\frac{\overline{a}_3(x_0,\tau_0,x_0)}{\sigma}) \quad + 2\sum_{j=1}^{i-1} q_1 e^{q_1\tau_j} N(\frac{\overline{a}_3(x_j,\tau_j,x_{j-1})}{\sigma}) \\
& \quad + q_1 e^{q_1\tau_i} N(\frac{\overline{a}_3(x_i,\tau_i,x_{i-1})}{\sigma})\} \quad - e^{\frac{1}{2}\sigma_2^2\tau_i} \frac{\Delta\tau}{2}\{N(\frac{\overline{a}_4(x_0,\tau_0,x_0)}{\sigma}) \\
& \quad + 2\sum_{j=1}^{i-1} q_2 e^{q_2\tau_j} N(\frac{\overline{a}_4(x_j,\tau_j,x_{j-1})}{\sigma}) \quad + q_2 e^{q_2\tau_i} N(\frac{\overline{a}_4(x_i,\tau_i,x_{i-1})}{\sigma})\},
\end{aligned}
$$

where x_i, $i = 1, 2, \cdots, n$ are solved sequentially. As n large enough, the optimal exercise ratio $x(\tau)$ can be approximated to sufficient accuracy as desired.

Now we substitute $\{x_i\}_{i=0}^n$ into (4.47). The value of AEO can be approximated by following equation.

$$
\begin{aligned}
& p_n(y_1, y_2, \tau_n) \\
= \quad & \bar{p}(y_1, y_2, \tau_n) + e^{\sigma_1 y_1 + \frac{1}{2}\sigma_1^2 \tau_n} \frac{\Delta\tau}{2}(N(\frac{a_3(y_1, y_2, \tau_0, x_0)}{\sigma}) \\
& + 2\sum_{j=1}^{n-1}(q_1 e^{q_1 \tau_j} N(\frac{a_3(y_1, y_2, \tau_j, x_{j-1})}{\sigma})) + q_1 e^{q_1 \tau_n} N(\frac{a_3(y_1, y_2, \tau_n, x_n)}{\sigma})) \\
& - e^{\sigma_1 y_1 + \frac{1}{2}\sigma_2^2 \tau_n} \frac{\Delta\tau}{2}(N(\frac{a_4(y_1, y_2, \tau_0, x_0)}{\sigma}) + 2\sum_{j=1}^{n-1}(q_2 e^{q_2 \tau_j} N(\frac{a_4(y_1, y_2, \tau_j, x_{j-1})}{\sigma})) \\
& + q_2 e^{q_2 \tau_n} N(\frac{a_4(y_1, y_2, \tau_n, x_n)}{\sigma})),
\end{aligned}
$$

where $\bar{p}(y_1, y_2, \tau) = e^{(\sigma_1 y_1 + \frac{1}{2}\sigma_1^2 \tau)} N(\frac{\sigma_1 a_1(y_1, y_2, \tau)}{\sigma}) - e^{(\sigma_2 y_2 + \frac{1}{2}\sigma_2^2 \tau)} N(\frac{\sigma_1 a_2(y_1, y_2, \tau)}{\sigma})$. And then, we convert $(p_n(y_1, y_2, \tau_n), x(\tau_n))$ back to $(P_n(S_1, S_2, \tau_n), X(\tau_n))$ and obtain the price of AEO numerically. Obviously, the limit of P_n tends to $P(S_1, S_2, \tau)$ as n tends to infinity. Here, we do not approximate (4.24) directly because of that the integral region in (4.24) is described by the nonlinear function $\log S_1$.

4.7 Numerical Results

In this section, the numerical solution obtained from IR method is compared to our asymptotic formula. Figure 4.1 displays the graph of the case of that $\sigma_1 = \sigma_2 = 0.5$, $q_1 = 0.02$, $q_2 = 0.01$ and $\rho = 0.5$. Figure 4.2 displays the graph of the case of that $\sigma_1 = \sigma_2 = 0.5$, $q_1 = q_2 = 0.01$ and $\rho = 0.5$. The solid curve is numerically computed by IR method and the dash curve is computed by asymptotic formulas (4.43). These figures show that the results from our asymptotic formula and IR method are very close as time near to expiration date.

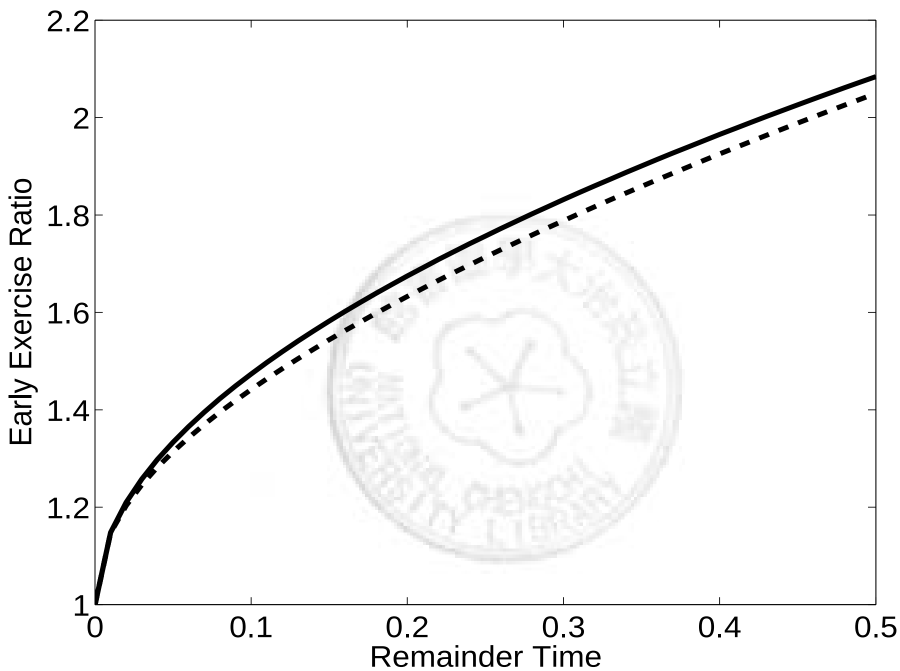

Figure 4.1: The optimal exercise ratio $X(\tau)$ as a function of $\tau = T - t$ for $q_1 = 0.02$, $q_2 = 0.01$ with given by (4.29)(dash curve) and recursive integration method(solid curve)

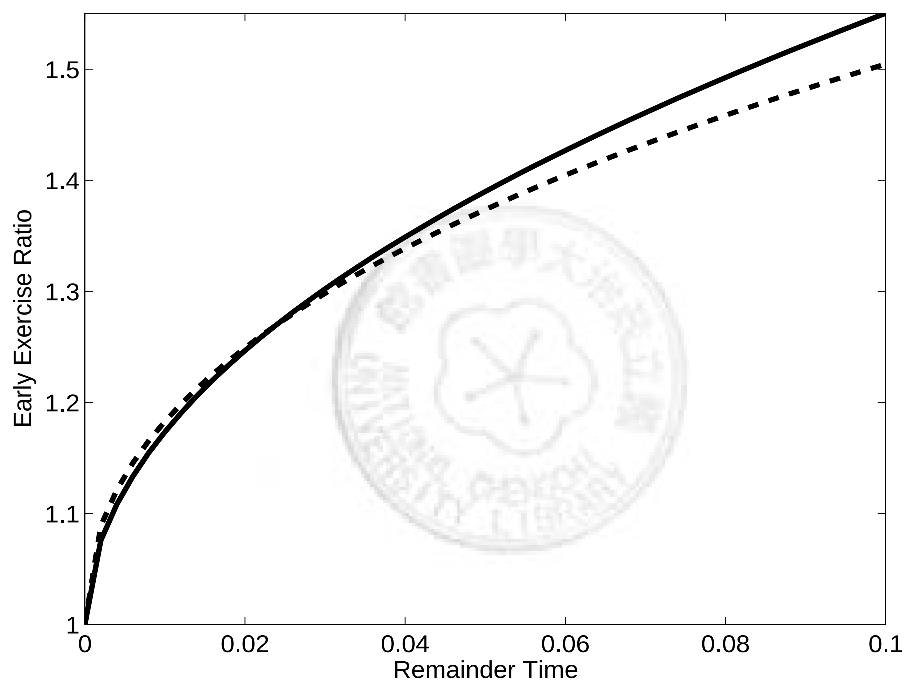

Figure 4.2: The optimal exercise ratio $X(\tau)$ as a function of $\tau = T - t$ for $q_1 = q_2 = 0.01$ with given by (4.30)(dash curve) and recursive integration method(solid curve)