Chapter 5

Optimization Approaches for Pricing an Option

The analytic formulas and their properties have been investigated in previous two chapters for pricing of the American options. In this chapter, we will propose an optimization approach for pricing an American option based on the binomial tree method.

The binomial tree approach is a powerful numerical method for valuation the European options. Here, we reformulate the binomial tree approach as a mixed integer nonlinear programming (MINLP) models and show that the MINLP models can be solved by their NLP relaxations in Section 5.2 and Section 5.3. The solution of the MINLP models provides an optimal exercising strategy of buyer, the complete hedging portfolio for writer and the fair price of the American option for both buyer and writer. However, the observed market price is usually not equal to the fair price. In Section 5.4, we will propose two self-finance models for writers to construct an optimal hedging portfolio. By observing the computational results, we find that the NLP relaxations reduce the computational time rapidly for computing the value of an American option and its replication portfolio. The computational results are displayed in Section 5.5.

5.1 Notations

The set of parameters and variables used in the model includes the forecasts of the stock price, the allocation of money market account and stock position, and the value of option over the investment time horizon.

Parameters

T	Investment time horizon
n	Number of time step
S_0	Initial value of the stock
K	The exercise price of the option
r	r is one plus risk-free rate
u (d)	Size of upward (downward) movement

Indices

t	Time step
$i \leq t$	State at each time step t

Variables

$x_t^i,\ y_t^i$	Allocation of money market account and stock respectively, in the portfolio, at time t and state i
z_t^i	Decision variable of the option holder at time t and state i; if holder exercise the option set $z_t^i = 1$, otherwise $z_t^i = 0$.

c_t^i	Option value at time t and state i
v_t^i	Value of the portfolio consisted of x_t^i market account and y_t^i stock

5.2 Binomial Pricing Approach

Consider the one period binomial tree model: there are two states, up and down, at time 1. The asset price and the final payoff of the European option are uS_0 and $\max\{uS_0 - K, 0\}$ for the "up" state. For the "down" state, the asset price and the final payoff are dS_0 and $\max\{dS_0 - K, 0\}$. The increment size, u and d, are selected to fit the asset's dynamics. There are several possible selection methods provided by [14] and [34] based on the assumption of the asset price dynamics.

Extending one period model to multi-period model, time interval $[0, T]$ is divided equally into n time periods. It is convenient to label the nodes in the binomial tree by (t, i) which indicates the node at time step t and state i. Hence the asset price on node (t, i) is denoted by $S_t^i = S_0 u^i d^{t-i} = S_0 u^{2i-t}$, $i = 0, 1, \dots, t$, and $t = 0, 1, \dots, n$. The last equality holds for the selection of $d = \frac{1}{u}$. By applying the no-arbitrage condition, the value of the European options at node (t, i), denoted as c_t^i, is

$$c_t^i = x_t^i + S_t^i y_t^i, \tag{5.1}$$

where (x_t^i, y_t^i) is the solution of following linear system

$$r x_t^i + S_{t+1}^{i+1} y_t^i = c_{t+1}^{i+1}, \tag{5.2}$$

$$r x_t^i + S_{t+1}^{i} y_t^i = c_{t+1}^{i}, \tag{5.3}$$

for all $i = 0, 1, \ldots, t$ and $t = 0, 1, \ldots, n-1$. At the expiration date, namely $t = n$, the value of a call option is given as $c_n^i = \max(S_n^i - K, 0)$. To avoid an arbitrage opportunity, we should make an assumption as follows [14]: $d < 1 \leq r < u$. By working backwards from node n to node 1, we can recursively solve these linear systems.

The solution of these linear systems provides a dynamic replication trading strategy for the entire binomial tree which is called hedging portfolio by the practitioners. The number of shares in the hedging portfolio is called delta. In summary, we have derived a self-financing trading strategy which costs c_0 at the initialization and without adding any sources along the way as it generates. Note that the self-financing trading strategy relates only to the interim time step $t = 1, 2, \ldots, n-1$. At $t = n$, we will collect a random amount of final payoff depending on the terminal asset price, namely $\max(S_n^i - K, 0)$ for an European call option.

The American option, however, gives the holder a right to exercise the option before the expiration. The value of the American option at each node (t, i) is higher than the immediate exercise price. Thus the value of American option at each interim node satisfies the following equation

$$c_t^i = \max\{v_t^i, (S_t^i - K)\} \tag{5.4}$$

at node (t, i), where v_t^i is defined as

$$v_t^i = x_t^i + S_t^i y_t^i. \tag{5.5}$$

Note that, for the case of European option, c_t^i is equal to v_t^i for all $t < n$.

5.3 MINLP Valuation Models

A rational holder does not exercise the option when the final payoff $S_n - K$ is negative. In this section, we introduce a decision variable z_n^i, which is a 0-1 variable to represent the decision of an option holder. If $z_n^i = 1$, the holder exercises the option and gets the final payoff $S_n^i - K$; otherwise, the holder disclaims the right to exercise and gets nothing. Therefore, the final payoff can be rewritten as

$$c_n^i = (S_n^i - K) z_n^i. \tag{5.6}$$

The rational option holder's task is to find c_0 which maximize the expected utility of his final payoff.

A valuation procedure for an European call option is then written as the following model:

Model E

$$\max \sum_{i=0}^{n} p_j U(c_n^i) \tag{5.7}$$

$$\begin{array}{llll}
\text{s.t. } x_t^i + S_t^i y_t^i = c_t^i, & i = 0, 1, \ldots, t, & t = 0, 1, \ldots, n-1, & (5.1) \\
r x_t^i + S_{t+1}^{i+1} y_t^i = c_{t+1}^{i+1}, & i = 0, 1, \ldots, t, & t = 0, 1, \ldots, n-1, & (5.2) \\
r x_t^i + S_{t+1}^{i} y_t^i = c_{t+1}^{i}, & i = 0, 1, \ldots, t, & t = 0, 1, \ldots, n-1, & (5.3) \\
(S_n^i - K) z_n^i = c_n^i, & i = 0, 1, \ldots, n, & & (5.6) \\
x_t^i, y_t^i \in \mathbb{R}, & i = 0, 1, \ldots, t, & t = 0, 1, \ldots, n, & (5.8) \\
z_n^i \in \{0, 1\}, & i = 0, 1, \ldots, n, & & (5.9) \\
c_n^i \geq 0, & i = 0, 1, \ldots, t, & t = 0, 1, \ldots, n. & (5.10)
\end{array}$$

where p_i, $i = 0, \ldots, n$ is the objective probability of option holder and U is option holder's utility function which is strictly increasing and concave. The solution of this model provides the hedging portfolio for each internal node and the decision of option holder on expiration.

Since z_n^i is an integer decision variable, model E is an MINLP model. Solving an MINLP program is always much harder than a similarly sized pure NLP program. The nonlinear relaxation of model E is obtained by replacing (5.9) by

$$0 \leq z_n^i \leq 1, \; i \leq n.$$

We will analytical investigate the nonlinear programming relaxation solution for model E and show that model E and its NLP relaxation have the same optimal solution.

Let $\mathbf{x}_t = (x_t^1, x_t^2, \ldots, x_t^t)'$ denote the allocation of the money market account over all the state at time t. Then $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ denote the allocation of the money market account over all the binomial tree. The same definition is also applied to $\mathbf{y}$, $\mathbf{z}$, $\mathbf{c}$, and $\mathbf{v}$.

Theorem 5.1. *Suppose that $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*, \mathbf{c}^*)$ is an optimal solution of the NLP relaxation of model E. Then $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*, \mathbf{c}^*)$ is also the solution of model E given as follows:*

(A) $z_n^{*i} = 1$ *when* $S_n^i - K > 0$ *and* $z_n^{*i} = 0$ *when* $S_n^i - K \leq 0$.

(B) $c_t^{*i} = \frac{1}{r}(qc_{t+1}^{*i+1} + (1-q)c_{t+1}^{*i})$ *for all* $i \leq t$ *and* $0 \leq t \leq n-1$, *where* $q = \frac{u-r}{u-d}$.

(C) $y_t^{*i} = \dfrac{c_{t+1}^{*i+1} - c_{t+1}^{*i}}{S_{t+1}^{i+1} - S_{t+1}^i}$.

(D) $y_t^{*i} = \eta y_{t+1}^{*i+1} + (1-\eta) y_{t+1}^{*i}$ *for all* $i \le t$ *and* $0 \le t \le n-1$*, where* $\eta = \frac{uq}{r}$.

Proof. Let $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*, \mathbf{c}^*)$ be the optimal solution of the NLP relaxation of model E. If there exists a node (n, k) on expiration such that $0 < z_n^k < 1$ and $S_n^k - K > 0$ then we claim that $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*, \mathbf{c}^*)$ is not the optimal solution. Namely there exists a feasible solution $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{z}}, \bar{\mathbf{c}}) \ne (\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*, \mathbf{c}^*)$ with the objective value greater than the optimal value.

Such feasible solution is constructed as follows. Let $\bar{z}_n^k = 1$ and $\bar{z}_n^i = z_n^{*i}$ if $i \ne k$. Then $\bar{c}_n^k = S_n^k - K$ and $\bar{c}_n^i = c_n^{*i}$ if $i \ne k$. Let

$$\mathcal{P} = \{\text{Any path from } (n, k) \text{ backtrack to } (0,\ 0)\}$$

and $\mathcal{C}$ be the set of (t, i) in $\mathcal{P}$ which are the node in the path from (n, k) to (0,0). If $(t, i) \in \mathcal{C}$, $(\bar{x}_t^i, \bar{y}_t^i, \bar{c}_t^i)$ are obtained iteratively by solving the system of (5.1)-(5.3) with respect to $\bar{\mathbf{z}}$. If $(t, i) \notin \mathcal{C}$, we set $(\bar{x}_t^i, \bar{y}_t^i, \bar{c}_t^i) = (x_t^{*i}, y_t^{*i}, c_t^{*i})$. Then $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{z}}, \bar{\mathbf{c}})$ is a feasible solution of the NLP relaxation of model E.

Since $\bar{c}_n^k = S_n^k - K > (S_n^k - K) z_n^{*k} = c_n^{*k}$ and $U(\cdot)$ is a strictly increasing function, we have

$$\sum_{i=0}^{n} p_i U(\bar{c}_n^i) > \sum_{i=0}^{n} p_i U(c_n^{*i}).$$

The same method can be applied to the case that the optimal solution of the NLP relaxation has a node (n, k) with $0 < z_n^{*k} < 1$ and $S_n^k - K < 0$. Thus we have shown (A). (B) and (C) can be obtained by substituting (5.2) and (5.3) into (5.1).

Finally, (D) is obtained as follows:

$$\begin{aligned} y_t^{*i} &= \frac{c_{t+1}^{*i+1} - c_{t+1}^{*i}}{(u-d)S_t^i} \\ &= \frac{qc_{t+2}^{*i+2} + (1-q)c_{t+2}^{*i+1} - qc_{t+2}^{*i+1} - (1-q)c_{t+2}^{*i}}{r(u-d)S_t^i} \\ &= \frac{uq}{r}\frac{c_{t+2}^{*i+2} - c_{t+2}^{*i+1}}{(u-d)uS_t^i} + \frac{d(1-q)}{r}\frac{c_{t+2}^{*i+1} - c_{t+2}^{*i}}{(u-d)dS_t^i} \\ &= \eta y_{t+1}^{*i+1} + (1-\eta)y_{t+1}^{*i}, \end{aligned}$$

where $\eta = \frac{uq}{r}$. □

We have shown that an MINLP European option valuation model can be solved by its NLP relaxation.

Now, we consider the case of an American option. The current value of an American call option in the internal of binomial tree can be derived as (5.4) and (5.5). This implies that the holder exercises the call option if the exercising value is greater than the replication portfolio value; otherwise the holder keeps holding the call option. We introduce a binary variable z_t^i not only for the ending node but also for every interim node to represent the option holder whether exercise the option or not. Therefore the value of American call option can be defined as

$$c_t^i = (S_t^i - K)z_t^i + v_t^i(1 - z_t^i). \tag{5.11}$$

Then an MINLP model for valuation an American option is formulated as follows:

Model A

$$\max \sum_{t=0}^{n}\sum_{i=0}^{t}\frac{p_t^i}{r^t}U(c_t^i) \tag{5.12}$$

$$\text{s.t. } rx_t^i + S_{t+1}^{i+1}y_t^i = c_{t+1}^{i+1}, \quad i=0,1,\dots t, \quad t=0,1,\dots,n-1, \tag{5.2}$$

$$rx_t^i + S_{t+1}^{i}y_t^i = c_{t+1}^{i}, \quad i=0,1,\dots,t, \quad t=0,1,\dots,n-1, \tag{5.3}$$

$$x_t^i + S_t^i y_t^i = v_t^i, \quad i=0,1,\dots,t, \quad t=0,1,\dots,n-1, \tag{5.5}$$

$$(S_t^i - K)z_t^i + v_t^i(1-z_t^i) = c_t^i, \quad i=0,1,\dots,t, \quad t=0,1,\dots n-1, \tag{5.12}$$

$$(S_n^i - K)z_n^i = c_n^i, \quad i=0,1,\dots,n \tag{5.13}$$

$$x_t^i, y_t^i \in \mathbb{R}, \quad i=0,1,\dots,t, \quad t=0,1,\dots,n, \tag{5.8}$$

$$z_t^i \in \{0,1\}, \quad i=0,1,\dots,t, \quad t=0,1,\dots,n, \tag{5.14}$$

$$v_t^i,\ c_t^i \geq 0, \quad i=0,1,\dots,t, \quad t=0,1,\dots,n. \tag{5.15}$$

where $p_t^i,\ i=0,\dots,t,\ t=0,\cdots,n$ is the objective probability of option holder and U is option holder's utility function which is strictly increasing and concave. The American option valuation problem is now modeled as an MINLP problem. According to the special structure of model A, we show that model A and its NLP relaxation have the same optimal solution.

Suppose that $(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{c})$ satisfies (5.2)-(5.3), we have

$$\begin{bmatrix} x_t^i \\ y_t^i \end{bmatrix} = \frac{1}{r(S_{t+1}^i - S_{t+1}^{i+1})}\begin{bmatrix} S_{t+1}^i & -S_{t+1}^{i+1} \\ -r & r \end{bmatrix}\begin{bmatrix} c_{t+1}^{i+1} \\ c_{t+1}^{i} \end{bmatrix} \tag{5.16}$$

for all $i \leq t, t \leq n$.

Lemma 5.2. *Let* $(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{v}, \mathbf{c})$ *and* $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{z}}, \bar{\mathbf{v}}, \bar{\mathbf{c}})$ *be any two feasible solutions of the NLP relaxation of model A. Suppose that there is a node* (τ, k) *with* $z_\tau^k \neq \bar{z}_\tau^k$ *and*

$z_t^i = \bar{z}_t^i$ *for* $(t,i) \neq (\tau,k)$*. If* $c_\tau^k > \bar{c}_\tau^k$*, then* $v_t^i \geq \bar{v}_t^i$ *and* $c_t^i \geq \bar{c}_t^i$ *for all the predecessor node* (t,i) *of* (τ,k)*.*

Proof. Suppose both $(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{v},\mathbf{c})$ and $(\bar{\mathbf{x}},\bar{\mathbf{y}},\bar{\mathbf{z}},\bar{\mathbf{v}},\bar{\mathbf{c}})$ are feasible solutions of the NLP relaxation of model A. We have, for any interim node (t,i),

$$x_t^i = \frac{uc_{t+1}^i - dc_{t+1}^{i+1}}{r(u-d)}, \qquad \bar{x}_{ti} = \frac{u\bar{c}_{t+1}^i - d\bar{c}_{t+1}^{i+1}}{r(u-d)},$$
$$S_t^i y_t^i = \frac{c_{t+1}^{i+1} - c_{t+1}^i}{u-d}, \qquad S_t^i \bar{y}_t^i = \frac{\bar{c}_{t+1}^{i+1} - \bar{c}_{t+1}^i}{u-d}$$

by solving (5.16).

By (5.5), we calculate $v_t^i - \bar{v}_t^i = (x_t^i + S_t^i y_t^i) - (\bar{x}_t^i + S_t^i \bar{y}_t^i)$ from $\tau - 1$ to 0 iteratively. Let $(\tau-1,i)$ be a predecessor node of (τ,k). We consider the following two cases: (i) $i = k$ or (ii) $i = k-1$. For case (i), we have

$$x_{\tau-1}^k + S_{\tau-1}^k y_{\tau-1}^k - \bar{x}_{\tau-1}^k - S_{\tau-1}^k \bar{y}_{\tau-1}^k \geq (\tfrac{u}{r} - 1)(\tfrac{c_\tau^k - \bar{c}_\tau^k}{u-d}) > 0.$$

For case (ii), we have

$$x_{\tau-1}^{k-1} + S_{\tau-1}^{k-1} y_{\tau-1}^{k-1} - \bar{x}_{\tau-1}^{k-1} - S_{\tau-1}^{k-1} \bar{y}_{\tau-1}^{k-1} \geq (1 - \tfrac{d}{r})(\tfrac{c_\tau^k - \bar{c}_\tau^k}{u-d}) > 0.$$

Therefore we have $c_t^i \geq \bar{c}_t^i$ by (5.12). Replacing (τ,k) by $(\tau-1,i)$, we can show that this lemma holds for the predecessor nodes at time step $\tau - 2$. Hence, applying the same method iteratively, we prove this lemma. □

Theorem 5.3. *Suppose that* $(\mathbf{x}^*,\mathbf{y}^*,\mathbf{z}^*,\mathbf{v}^*,\mathbf{c}^*)$ *is an optimal solution of the NLP relaxation of model A. Then* $(\mathbf{x}^*,\mathbf{y}^*,\mathbf{z}^*,\mathbf{v}^*,\mathbf{c}^*)$ *is the solution of model A given as follows:*

(A) $z_t^{*i} = 1$ *when* $S_t^i - K > 0$ *and* $z_t^{*i} = 0$ *when* $S_t^i - K < 0$*.*

(B) $c_t^{*i} = \max(v_t^{*i}, S_t^i - K)$, *where* $v_t^{*i} = \frac{1}{r}(qc_{t+1}^{*i+1} + (1-q)c_{t+1}^{*i})$ *and* $q = \frac{u-r}{u-d}$.

(C) $y_t^{*i} = \dfrac{c_{t+1}^{*i+1} - c_{t+1}^{*i}}{S_{t+1}^{i+1} - S_{t+1}^{i}}$.

Proof. Let $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*, \mathbf{v}^*, \mathbf{c}^*)$ be an optimal solution of the NLP relaxation of model A. If there is (τ, k) such that $z_\tau^{*k} < 1$ when $S_\tau^k - K > v_\tau^{*k}$, we claim that there exists a feasible solution with which the object value is greater than the optimal value.

Let

$$\mathcal{P} = \{\text{Any path from } (\tau, k) \text{ backtrack to } (0,0)\}$$

and $\mathcal{C}$ be the set of (t,i) in $\mathcal{P}$ which are the node in the path from (τ, k) to (0,0). Such feasible solution is constructed as follows.

1. Let $\bar{z}_\tau^k = 1$ and $\bar{z}_t^i = z_t^{*i}$ if $(t,i) \neq (\tau, k)$.
2. If $(t,i) \notin \mathcal{C}$, set $(\bar{x}_t^i, \bar{y}_t^i, \bar{v}_t^i, \bar{c}_t^i) = (x_t^{*i}, y_t^{*i}, v_t^{*i}, c_t^{*i})$.
3. If $(t,i) \in \mathcal{C}$, $(\bar{x}_t^i, \bar{y}_t^i, \bar{v}_t^i, \bar{c}_t^i)$ are selected by solving the system of (5.2)-(5.5) with respect to $\bar{\mathbf{z}}$.

Then $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{z}}, \bar{\mathbf{v}}, \bar{\mathbf{c}})$ is a feasible solution of the NLP relaxation of model A with $\bar{c}_\tau^k = (S_\tau^k - K) > (S_\tau^k - K)z_\tau^{*k} + v_\tau^{*k}(1 - z_\tau^{*k}) = c_\tau^{*k}$.

Now, by applying lemma 5.2, we have $\bar{c}_t^i \geq c_t^{*i}$ for $(t,i) \in \mathcal{C}$ since $\bar{c}_\tau^k > c_\tau^{*k}$. This implies that

$$\sum_{t=0}^{n} \sum_{i=0}^{t} \frac{p_t^i}{r^t} U(\bar{c}_t^i) > \sum_{t=0}^{n} \sum_{i=0}^{t} \frac{p_t^i}{r^t} U(c_t^{*i}).$$

Finally, the same method can be applied to show the case that the optimal solution of the NLP relaxation has a node (τ, k) with $0 < z_\tau^{*k} < 1$ and $S_n^k - K < 0$. Thus, we have shown (A). (B) can be obtained by substituting (5.2) and (5.3) into (5.1). (C) can be obtained by solving equations (5.16). □

By solving model A, we obtain the following information: (1) optimal exercising strategy for buyer, (2) the complete hedge strategy for writer, and (3) the cost for constructing the complete hedge portfolio. Under the no-arbitrage condition, the cost must be the "fair" price of the American option for both the writers and the buyers.

5.4 Writer's Problems

However, the observed market price is always not equal to the fair price. Knowing the optimal exercising strategy of buyer and the observed market price, we shall propose a self-finance model minimizing the expected loss when the market price is less than the fair price.

The optimal exercising strategy obtained from model A and the observed market price are given as z_t^i, $i = 0, 1, \ldots, t$, $t = 0, 1, \ldots, n$ and M_0, respectively.

If the American option is not exercised, the hedging portfolio satisfies the self-finance trading strategy, which is formulated as (5.2)-(5.5), and

$$v_t^i(1 - z_t^i) = c_t^i(1 - z_t^i), i = 0, \ldots, t,\ t = 0, 1, \ldots, n-1, \qquad (5.17)$$

in the internal of binomial tree. To analyze (5.17), we find that $v_t^i = c_t^i$ if $z_t^i = 0$ and $1 - z_t^i = 0$ if $z_t^i = 1$. This implies that the self-finance trading strategy only holds for hedging the alive American option.

To measure the difference between the exercise payoff and the value of hedging portfolio, we add four deviation variables D_t^i, d_t^i, E_t^i, e_t^i, $i = 0, 1, \ldots, t$, $t = 0, 1, \ldots, n$ and have

$$(S_{t+1}^{i+1} - K)z_{t+1}^{i+1} - c_{t+1}^{i+1} = D_{t+1}^{i+1} - d_{t+1}^{i+1}, \ i = 0, 1, \ldots, t-1, \ t = 1, 2, \ldots, n-1. \quad (5.18)$$

$$(S_{t+1}^{i} - K)z_{t+1}^{i} - c_{t+1}^{i} = E_{t+1}^{i} - e_{t+1}^{i}, \ i = 0, 1, \ldots, t, \ t = 1, 2, \ldots, n-1. \quad (5.19)$$

Note that $D_t^i > 0$ and $E_t^i > 0$ are the loss of writers at (t, i). Though the option should be traded with its market price M_0, the writer could make a loan of money $b > 0$ for constructing a minimum loss portfolio. Therefore, the initial value of the hedging portfolio can be represented as

$$c_0 = M_0 + b.$$

The goal of this model is to minimize the expected value of $U(b + \sum_{t=0}^{T} \sum_{i=0}^{t} \frac{p_t^i}{r^t} D_t^i + E_t^i)$, where $U(x)$ is the utility function of writers.

Now we obtain the following minimum loss hedging model.

Minimum loss model

$$\min U(b + \sum_{t=0}^{T} \sum_{i=0}^{t} \frac{p_t^i}{r^t} D_t^i + E_t^i)$$

$$\text{s.t. } x_t^i + S_t^i y_t^i = v_t^i, \qquad i = 0, 1, \ldots, t, \qquad t = 0, 1, \ldots, n-1,$$

$$r x_t^i + S_{t+1}^{i+1} y_t^i = c_{t+1}^{i+1}, \qquad i = 0, 1, \ldots, t, \qquad t = 0, 1, \ldots, n-1,$$

$$r x_t^i + S_{t+1}^{i} y_t^i = c_{t+1}^{i}, \qquad i = 0, 1, \ldots, t, \qquad t = 0, 1, \ldots, n-1,$$

$$v_t^i(1-z_t^i) = c_t^i(1-z_t^i), \qquad i=0,1,\ldots,t, \qquad t=0,1,\ldots,n-1,$$

$$(S_{t+1}^{i+1}-K)z_{t+1}^{i+1} - c_{t+1}^{i+1} = D_{t+1}^{i+1} - d_{t+1}^{i+1}, \qquad i=0,1,\ldots,t, \qquad t=1,2,\ldots,n-1.$$

$$(S_{t+1}^{i}-K)z_{t+1}^{i} - c_{t+1}^{i} = E_{t+1}^{i} - e_{t+1}^{i}, \qquad i=0,1,\ldots,t, \qquad t=1,2,\ldots,n-1.$$

$$c_0 \leq M_0 + b,$$

$$x_t^i,\ y_t^i \in \mathbb{R}, \qquad i=0,1,\ldots,t, \qquad t=0,1,\ldots,n,$$

$$b, D_t^i,\ d_t^i,\ E_t^i,\ e_t^i,\ c_n^i \geq 0, \qquad i=0,1,\ldots,t, \qquad t=0,1,\ldots,n.$$

Here, p_t^i, $i=0,1,\ldots,t$, $t=0,1,\ldots,N$ is the objective probability of the writer.

5.5 Numerical Results

In this section, we shall compare the computational time of Model A with its relaxation. Here,all models are coded by GAMS. Note that Model A, which is an MINLP problem, is solved by BARON solver, and its NLP relaxation is solved by MINOS solver. We will find that the relaxation model reduces the computational time rapidly. In the following results, the objective probability at state (t,i) and the utility function of the writer is given as $p_t^i = \frac{C_i^t}{2^t}$ and $U(x)=x$, respectively.

Example 5.4. *We assume that the asset dynamics satisfies the geometric Brownain motion and price a one-year maturity, at-the-money American call option with the current price at 100. The risk-free interest rate and the volatility of the asset are assumed to be 6% and 16%, respectively. That is,* $K=100$, $T=1$, $S=100$, $r=0.06$. *The increment size is given as* $u=\exp(\sigma\sqrt{(T/(N-1))})$ *and* $d=1/u$, *where* N *is the number of steps.*

time steps	fair price	Model A (Sec.)	relaxation (Sec.)
20	9.780	45.47	0.219
40	9.745	3568	1.105
60	9.734	NA	5.563
80	9.728	NA	17.917
100	9.725	NA	44.188
150	9.720	NA	246.922

Table 5.1: American call

The computational results of Example 5.4 is listed in Table 5.1. In Table 5.1, we display the American put's fair price and The execution times for solving model A and its relaxation model from column 2 to column 4. We find that the execution time for model A is 45.47 seconds and 3568.66 seconds for the 20 and 40 time steps, respectively. However, the execution time for the relaxation model are 0.219 seconds and 1.105 seconds for the 20 and 40 time steps, respectively. Moreover, when the time steps are greater than 60, the execution time of the MINLP model raises to several hours. Compared the execution time with both two models, we have that the relaxation model reduces the computation time rapidly.

Example 5.5. *For the same situation, we price a one-year maturity, at-the-money American put option with the current price at 100.*

The computational results of Example 5.5 is listed in Table 5.2. Table 5.2 displays the American put's fair price and the execution times for solving model A and its relaxation model from column 2 to column 4. We find that the model A and its relaxation have the same optimal solution and the relaxation model reduces the computation time. However, we find that when the time steps are greater than 40, the execution time of the MINLP model raises to several hours. For $N = 20$, the optimal exercising strategy for buyers, which will be used in Example 5.7, is displayed in Table 5.3.

time steps	fair price	Model A	relaxation (Sec.)
20	4.636	3624	1.319
40	4.593	NA	2.484
60	4.557	NA	10.750
80	4.547	NA	37.375
100	4.542	NA	148.094
150	4.534	NA	1174.680

Table 5.2: American put

If we assume that the asset process does not satisfy the geometric Brownian motion, we give an example that assume the asset process satisfies pure Poisson process. Model A can also provide the optimal exercising strategy and the complete hedging portfolio.

Example 5.6. *We consider the same put of the previous example but assume that the asset price satisfies the pure poisson process.*

Since the pure poisson process does not drop down, the rational buyer does not

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
																				0
																			0	0
																		0	0	0
																	0	0	0	0
																0	0	0	0	0
															0	0	0	0	0	0
														0	0	0	0	0	0	0
													0	0	0	0	0	0	0	0
z												0	0	0	0	0	0	0	0	0
											0	0	0	0	0	0	0	0	0	0
										0	0	0	0	0	0	0	0	0	0	1
									0	0	0	0	0	0	0	0	0	0	1	1
								0	0	0	0	0	0	0	0	0	0	1	1	1
							0	0	0	0	0	0	0	0	0	1	1	1	1	1
						0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
					0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
				0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1
			0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
		0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 5.3: Optimal decision variable of buyer

time steps	fair price	Model A	relaxation (Sec.)
20	5.541	32.220	0.313
40	5.682	NA	2.734
60	5.635	NA	15.539
80	5.603	NA	81.859
100	5.541	NA	129.734

Table 5.4: American put under pure poission process

exercise the American option prior the expiration date. Therefore, we find that the decision variables $z_t^i = 0$, for all i and t. In Table 5.4, we display the American put's fair price and the execution times for solving model A and its relaxation model from column 2 to column 4. Compared column 3 and column 4, we also find that the relaxation model reduces the computation time rapidly.

Example 5.7. *For the writer's problem, we consider the same put option in Example 5.5. In this case, we assume that the market price is 4.6.*

The writer can construct a perfect hedging portfolio, which is a feasible solution of minimum loss model, with initial value 4.636. The loss of constructing a perfect hedging portfolio is 0.036. However, the optimal solution of the minimum loss model can provide a portfolio, that has no loss for all states except 0.735 at state $(5, 1)$, with market price 4.6. This implies that discounted expected loss 0.0175, which is less than 0.036. Therefore, the minimum loss portfolio has a nice performance of constructing a hedging portfolio.

Note that constructing a perfect hedging portfolio is a feasible solution of minimum loss model, so the objective value of the minimum loss model is always less than or equal to the difference between the fair price or the market price.