Chapter 4

Formulation of the Bandwidth Allocation Model with Proportional Fairness

4.1 Notations

Given a network topology $G =< V, E >$, where V and E denote the set of nodes and the set of links in the network respectively. There is given a set S of m classes, i.e., $|S| = m$. We denote by S^i a set of sessions in class i. There is also given the maximal possible number K^i in each class i, that is $|S^i| = K^i$. Some notations are listed as below:

$S \triangleq$ A set of m classes.

$\mathbf{f(x)} \triangleq$ A vector-function that maps the decision space $\mathbb{R}^n$ into the criterion space $\mathbb{R}^m$.

$Q \triangleq$ A set of allocation patterns (allocation decisions), which denotes the feasible set.

$\mathbf{x} \triangleq$ The vector of decision variables in $\mathbb{R}^n$.

$\mathbf{x}^* \triangleq$ The optimal solution.

$f_i \triangleq$ The individual objective function for each class $i \in S$.

$z_i \triangleq$ The outcome of the allocation pattern $\mathbf{x}$ for service i.

$\theta^i \triangleq$ The bandwidth allocated to class i.

$x_e \triangleq$ The bandwidth allocated to link $e \in E$.

$B \triangleq$ Limited available budget.

$U_e \triangleq$ The maximal capacity of each link e.

$p_j \triangleq$ The routing path connects the original o with the destination d.

$$\chi_j(e) \triangleq \begin{cases} 1 & \text{if link } e \in p_j \\ 0 & \text{if link } e \notin p_j. \end{cases}$$

$\kappa_e \triangleq$ The cost considering bandwidth and delay for each link $e \in E$.

$C_{p_j} \triangleq$ The unit cost of bandwidth on the entire path p_j.

$a_i \triangleq$ The given aspiration level attached to the attribute z_i.

$r_i \triangleq$ The given reservation level attached to the attribute z_i.

$\mu_i(z_i) \triangleq$ The achievement function of the value z_i.

$b^i \triangleq$ The bandwidth requirement for class i.

$K^i \triangleq$ The maximal possible number in each class i.

$\pi^i \triangleq$ The reserved budget for each class i.

$\nu_i \triangleq$ The assigned weight for each objective function f_i.

4.2 Objective Functions

By using the concept of the achievement function as mentioned, we can construct the achievement functions μ_i^T and μ_i^D for the throughput T and delay D respectively for each class $i \in S$. Given the weight β_i^T and β_i^D on the throughput and delay for each class $i \in S$, and $\beta_i^T + \beta_i^D = 1$ for each $\beta_i^T, \beta_i^D \in (0,1)$. Then, for each class i, we consider the individual objective function f_i of the allocation pattern $\mathbf{x} = \{x_e \mid e \in E\}$:

$$
\begin{aligned}
f_i(\mathbf{x}) &= \beta_i^T \mu_i^T(\theta^i) + \beta_i^D \mu_i^D(\tfrac{\theta^i}{\lambda_i}) \\
&= \beta_i^T \log_{\alpha_i^T} \tfrac{\theta^i}{r_i^T} + \beta_i^D \log_{\alpha_i^D} \tfrac{\theta^i/\lambda_i}{r_i^D} \\
&= \beta_i^T \frac{\log \frac{\theta^i}{r_i^T}}{\log \alpha_i^T} + \beta_i^D \frac{\log \frac{\theta^i}{\lambda_i r_i^D}}{\log \alpha_i^D} \\
&= \log(\tfrac{\theta^i}{r_i^T})^{\frac{\beta_i^T}{\log \alpha_i^T}} + \log(\tfrac{\theta^i}{\lambda_i r_i^D})^{\frac{\beta_i^D}{\log \alpha_i^D}} \\
&= \log[(\tfrac{\theta^i}{r_i^T})^{\frac{\beta_i^T}{\log \alpha_i^T}} \cdot (\tfrac{\theta^i}{\lambda_i r_i^D})^{\frac{\beta_i^D}{\log \alpha_i^D}}] \\
&= \log[\frac{(\theta^i)^{(\frac{\beta_i^T}{\log \alpha_i^T} + \frac{\beta_i^D}{\log \alpha_i^D})}}{(r_i^T)^{(\frac{\beta_i^T}{\log \alpha_i^T})} \cdot (\lambda_i r_i^D)^{(\frac{\beta_i^D}{\log \alpha_i^D})}}] \\
&= \log[\tfrac{(\theta^i)^{G_i}}{H_i}] \\
&= G_i \log \theta^i - \log H_i \\
&\triangleq f_i(\theta^i),
\end{aligned}
$$

where λ_i represents the demand of bandwidth per unit time for class i,

$$\alpha_i^T \triangleq \frac{a_i^T}{r_i^T} \tag{4.1}$$

and

$$\alpha_i^D \triangleq \frac{a_i^D}{r_i^D} \tag{4.2}$$

are given,

$$G_i \triangleq \frac{\beta_i^T}{\log \alpha_i^T} + \frac{\beta_i^D}{\log \alpha_i^D} \tag{4.3}$$

is a constant number,

$$H_i \triangleq (r_i^T)^{(\frac{\beta_i^T}{\log \alpha_i^T})} \cdot (\lambda_i r_i^D)^{(\frac{\beta_i^D}{\log \alpha_i^D})} \tag{4.4}$$

is also a constant number, and θ^i, a function of $\mathbf{x}$, denotes the bandwidths allocated to class i. The individual objective function f_i is the function of the allocation pattern $\mathbf{x}$ and the bandwidth θ^i allocated to class i. Thus, we have the multiple objective functions $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$.

Next, we want to transform the multiple-objective problems to a single objective optimization meanwhile considering the fairness for each class. We will apply an approach by analyzing aggregation of outcomes $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$. This approach is introduced by Yager [37] as the so-called Ordered Weighted Averaging Method[1].

First, we define the ordering map $\mathbf{\Phi} : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$\mathbf{\Phi}(\mathbf{f}(\mathbf{x})) = (\, \Phi_1(\mathbf{f}(\mathbf{x})), \Phi_2(\mathbf{f}(\mathbf{x})), \ldots, \Phi_m(\mathbf{f}(\mathbf{x}))\,), \tag{4.5}$$

where $\Phi_1(\mathbf{f}(\mathbf{x})) \leq \Phi_2(\mathbf{f}(\mathbf{x})) \leq \ldots \leq \Phi_m(\mathbf{f}(\mathbf{x}))$ and there exists a permutation τ of set $S = \{1, 2, \ldots, m\}$ such that $\Phi_i(\mathbf{f}(\mathbf{x})) = f_{\tau(i)}(\mathbf{x})$ for $i = 1, \ldots, m$. Then we define

[1] The problem of aggregating criteria functions to form overall decision functions is of considerable importance in many disciplines. A primary factor in the determination of the structure of such aggregation functions is the relationship between the criteria involved. At one extreme is the situation in which we desire that all the criteria be satisfied. At the other extreme is the case in which the satisfaction of any of the criteria is all we desire. Here, we adopt a family of operators that provide an aggregation which lies in between these two extremes.

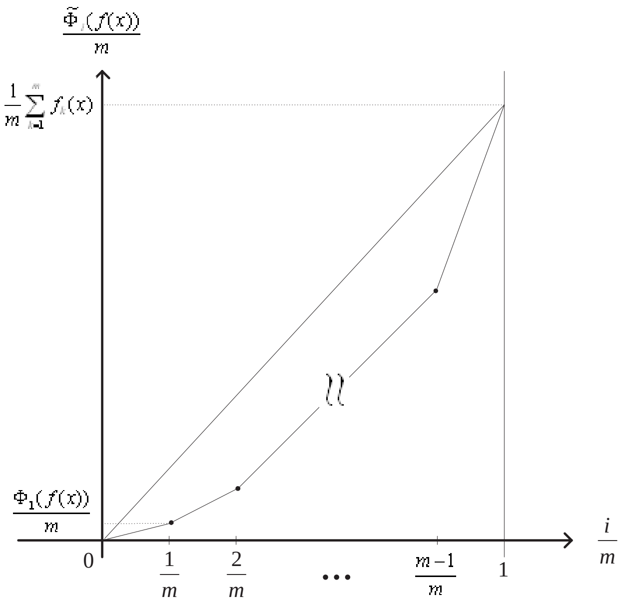

Figure 4.1: The Graph of a Absolute Lorenz Curve

the cumulative ordering map $\tilde{\mathbf{\Phi}}(\mathbf{f}(\mathbf{x})) = (\tilde{\Phi}_1(\mathbf{f}(\mathbf{x})),\ldots,\tilde{\Phi}_m(\mathbf{f}(\mathbf{x})))$ defined as

$$\tilde{\Phi}_i(\mathbf{f}(\mathbf{x})) = \sum_{k=1}^{i} \Phi_k(\mathbf{f}(\mathbf{x})), \qquad \text{for } i = 1, 2, \ldots, m. \tag{4.6}$$

The coefficients of vector $\tilde{\mathbf{\Phi}}(\mathbf{f}(\mathbf{x}))$ express, respectively: the smallest outcome, the total of the two smallest outcomes, the total of the three smallest outcomes, etc. Vector $\tilde{\mathbf{\Phi}}(\mathbf{f}(\mathbf{x}))$ can be viewed graphically with a piece wise linear curve connecting point $(0, 0)$ and points $(\frac{i}{m}, \frac{\tilde{\Phi}_i(\mathbf{f}(\mathbf{x}))}{m})$ for $i = 1, 2, \ldots, m$. Such a curve represents the absolute Lorenz curve[2] as shown in Figure 4.1.

Fair solutions to problem (3.2) can be expressed as Pareto-optimal solutions for the multiple criteria problem with objectives $\tilde{\mathbf{\Phi}}(\mathbf{f}(\mathbf{x}))$

$$\max \{(\,\tilde{\Phi}_1(\mathbf{f}(\mathbf{x})), \tilde{\Phi}_2(\mathbf{f}(\mathbf{x})), \ldots, \tilde{\Phi}_m(\mathbf{f}(\mathbf{x}))\,) : \mathbf{x} \in Q\} \tag{4.7}$$

[2] In income economics the Lorenz curve is a cumulative population versus income curve [21]. A perfectly equal distribution of income has the diagonal line as the Lorenz curve and no outcome vector can be better. The absolute Lorenz curves, we consider, are unnormalized taking into account also values of outcomes. Hence, with the relation of fair dominance an outcome vector of large unequal outcomes may be preferred to an outcome vector with small equal outcomes.

Then we have the following theorem from [25].

Theorem 4.2.1 *A feasible solution $\boldsymbol{x} \in Q$ is a fair solution of the resource allocation problem (3.2), if and only if it is a Pareto-optimal solution of the multiple criteria problem (4.7).*

In the following, we adopt an effective modeling technique for quantities $\tilde{\Phi}_i(\mathbf{f}(\mathbf{x}))$ with arbitrary i. In [25], for a given outcome vector $\mathbf{f}(\mathbf{x})$ the quantity $\tilde{\Phi}_i(\mathbf{f}(\mathbf{x}))$ may be found by solving the following linear program:

$$\begin{aligned} \tilde{\Phi}_i(\mathbf{f}(\mathbf{x})) \quad &= \quad \max\ it_i - \textstyle\sum_{k=1}^{m} d_k \\ \text{subject to} \quad & \quad t_i - f_k(\mathbf{x}) \leq d_k \\ & \quad d_k \geq 0 \\ & \quad \text{for } k = 1, \ldots, m, \end{aligned} \tag{4.8}$$

where t_i is an unrestricted variable and nonnegative variables d_k represent their downside deviations from the value of t_i for several values $f_k(\mathbf{x})$. For example, the worst outcome may be defined by the following optimization:

$$\tilde{\Phi}_1(\mathbf{f}(\mathbf{x})) = \max\ \{t_1 : t_1 \leq f_i(\mathbf{x}) \text{ for } i = 1, \ldots, m\}, \tag{4.9}$$

where t_1 is an unrestricted variable.

Formula (4.8) provides us with a computational formulation for the worst conditional mean $M_{\frac{k}{m}}(\mathbf{f}(\mathbf{x}))$ defined as the mean outcome for the k worst-off services, i.e.,

$$M_{\frac{k}{m}}(\mathbf{f}(\mathbf{x})) = \frac{1}{k}\tilde{\Phi}_k(\mathbf{f}(\mathbf{x})), \text{ for } k = 1, \ldots, m. \tag{4.10}$$

For $k = 1$, $M_{\frac{1}{m}}(\mathbf{f}(\mathbf{x})) = \tilde{\Phi}_1(\mathbf{f}(\mathbf{x})) = \Phi_1(\mathbf{f}(\mathbf{x}))$ which represents the minimum outcome. For $k = m$, $M_{\frac{m}{m}}(\mathbf{f}(\mathbf{x})) = \frac{1}{m}\tilde{\Phi}_m(\mathbf{f}(\mathbf{x})) = \frac{1}{m}\sum_{i=1}^{m}\Phi_i(\mathbf{f}(\mathbf{x})) = \frac{1}{m}\sum_{i=1}^{m} f_i(\mathbf{x})$ which represents the mean outcome.

For modeling various fair preferences one may use some combinations of the cumulative ordered outcomes $\tilde{\Phi}_i(\mathbf{f}(\mathbf{x}))$. In specific, for the weighted sum we obtain

$$\sum_{i=1}^{m} w_i \tilde{\Phi}_i(\mathbf{f}(\mathbf{x})). \tag{4.11}$$

Note that, due to the definition of map $\tilde{\Phi}_i$ with (4.6), the above function can be expressed in the form with weights $\nu_i = \sum_{j=i}^{m} w_i$ $(i = 1, \ldots, m)$ allocated to coordinates of the ordered outcome vector. When substituting w_i with ν_i, (4.11) becomes

$$\sum_{i=1}^{m} \nu_i \Phi_i(\mathbf{f}(\mathbf{x})), \tag{4.12}$$

where $\sum_{i=1}^{m} \nu_i = 1$ and $\nu_i \geq 0$, $\forall i = 1, \ldots, m$.

Applying the ordered weighted averaging method to problem (3.2), we get

$$\max \{\sum_{i=1}^{m} \nu_i \Phi_i(\mathbf{f}(\mathbf{x})) : \mathbf{x} \in Q\}. \tag{4.13}$$

If weights ν_i are strictly decreasing and positive, that is $\nu_1 > \nu_2 > \ldots > \nu_{m-1} > \nu_m > 0$, then each optimal solution of the OWA problem (4.13) is a fair solution of (3.2).

Actually, formulas (4.8) and (4.11) allow us to formulate the following mathematical programming of the original multiple criteria problem:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{m} w_i \psi_i \\ \text{subject to} & \psi_i = it_i - \sum_{k=1}^{m} d_{ki}, \ \forall i = 1, \ldots, m \\ & t_i - d_{ki} \leq f_k(\mathbf{x}), \ \forall i, \ k = 1, \ldots, m \\ & d_{ki} \geq 0, \ \forall i, \ k = 1, \ldots, m \\ & \mathbf{x} \in Q \\ & t_i \text{ unrestricted}, \forall i = 1, \ldots, m, \end{array} \tag{4.14}$$

where $w_m = \nu_m$, $w_i = \nu_i - \nu_{i+1}$ for $i = 1, \ldots, m-1$, $\nu_i \in (0, 1)$ for each i, and $\sum_{i=1}^{m} \nu_i = 1$.

4.3 Constraints

After dealing with the objective functions, we will consider the constraints. Given the total available budget B and the marginal cost κ_e of bandwidths for each link

$e \in E$, we want to allocate the bandwidths in order to provide each class with maximal possible QoS. First, let x_e and θ_j^i be the bandwidth allocated to the link e and the connection j of class i respectively. Then these decision variables must be nonnegative:

$$x_e \geq 0, \ \forall e \in E \tag{4.15}$$

$$\theta_j^i \geq 0, \ \forall j \in S^i, \ \text{for } i = 1, \dots, m. \tag{4.16}$$

Furthermore, we have the following constraints on the network:

$$\sum_{e \in E} \kappa_e x_e = B, \tag{4.17}$$

$$x_e \leq U_e, \quad \forall e \in E, \tag{4.18}$$

where U_e is the maximal capacity of each link e.

In each class i, every connection use the same bandwidth and has the same bandwidth requirement, so we have:

$$\theta_1^i = \theta_2^i = \cdots = \theta_{K^i}^i \tag{4.19}$$

$$\theta_j^i \geq b^i, \ \text{for } j = 1, \dots, K^i \ , \tag{4.20}$$

where b^i is the bandwidth requirement for class i. Next, for each session j of class i, we denote the routing path connecting the source o and destination d by p_j. We define

$$\chi_j^i(e) = \begin{cases} 1 & \text{if link } e \in p_j \\ 0 & \text{if link } e \notin p_j. \end{cases} \tag{4.21}$$

Thus, we have the constraint:

$$\sum_i \sum_j \chi_j^i(e) \theta_j^i = x_e, \quad \forall e \in E. \tag{4.22}$$

Moreover, for each class i,

$$\theta_j^i \cdot \sum_e \kappa_e \chi_j^i(e) = c^i, \tag{4.23}$$

where c^i is a budget allocated to each connection of class i. Then,

$$\sum_i (K^i \cdot c^i + \pi^i) = B, \tag{4.24}$$

where π^i is the reserved budget for each class i.

4.4 A Mathematical Model

As mentioned in Sections 4.2 and 4.3, when using the achievement function and applying Ordered Weighted Averaging method, we will get the following mathematical model (MP1):

$$\text{Maximize} \quad \sum_{i=1}^{m} w_i \psi_i$$

$$\text{Subject to} \quad \sum_{e \in E} \kappa_e x_e = B$$

$$\sum_i \sum_j \chi_j^i(e) \theta_j^i = x_e, \ \forall e \in E$$

$$\sum_i (K^i \cdot c^i + \pi^i) = B$$

$$\theta_j^i \cdot \sum_e \kappa_e \chi_j^i(e) = c^i, \ \forall j \in S^i, \ \text{for } i = 1, \ldots, m$$

$$x_e \leq U_e, \ \forall e \in E$$

$$\psi_i = it_i - \sum_{k=1}^{m} d_{ki}, \ \forall i = 1, \ldots, m$$

$$t_i - d_{ki} \leq f_k(\mathbf{x}), \ \forall i, \ k = 1, \ldots, m$$

$$d_{ki} \geq 0, \ \forall i, \ k = 1, \ldots, m$$

$$\theta_1^i = \theta_2^i = \cdots = \theta_{K^i}^i, \ \forall i = 1, \ldots, m$$

$$\theta_j^i \geq b^i, \ \forall j \in S^i, \ \text{for } i = 1, \ldots, m$$

$$x_e \geq 0, \ \forall e \in E$$

$$\theta_j^i \geq 0, \ \forall j \in S^i, \ \text{for } i = 1, \ldots, m$$

$$\chi_j^i(e) = 0 \text{ or } 1, \ \forall e \in E,$$

$$t_i \text{ unrestricted}, \forall i = 1, \ldots, m,$$

where $w_m = \nu_m$, $w_i = \nu_i - \nu_{i+1}$ for $i = 1, \ldots, m-1$, $\nu_i \in (0,1)$ is given for each i, and $\sum_{i=1}^{m} \nu_i = 1$. The individual function ψ_i is the first i sum of the ordered multiple objective functions $\mathbf{\Phi}(\mathbf{f}(\mathbf{x}))$ in the allocation pattern $\mathbf{x} = \{x_e \mid e \in E\}$ and the bandwidth θ^i allocated to class i. Here, we let K^i in (4.19) be a fixed number for the discussion under deterministic assumption of feasibility of (MP1). In general, K^i may be random which is beyond scope of the thesis.

4.5 Modifications of Nonlinear Parts

4.5.1 Objectives

Since

$$\psi_i = it_i - \sum_{k=1}^{m} d_{ki}, \ \forall i = 1, \ldots, m \tag{4.25}$$

in (MP1), we rewrite the objective function in the model (MP1) as follows:

$$\begin{aligned}\sum_{i=1}^{m} w_i\psi_i &= \sum_{i=1}^{m} w_i(it_i - \sum_{k=1}^{m} d_{ki}) \\ &= \sum_{i=1}^{m} iw_it_i - \sum_{i=1}^{m} w_i \sum_{k=1}^{m} d_{ki} \\ &= \sum_{i=1}^{m} iw_it_i - \sum_{i=1}^{m}\sum_{k=1}^{m} w_i d_{ki}.\end{aligned} \tag{4.26}$$

4.5.2 Constraints

Since the achievement function μ_i in (3.1) is not a linear function. To overcome this problem, we resort to a piecewise linear function[3] to approximate the achievement function. The piecewise linearity of our tricks has the advantage that using a linear programming (LP) solver, e.g. ILOG.

We define

$$\hat{\mu}_i(\theta^i) = \begin{cases} m_1 \cdot (\theta^i - b_{i,1}) + \mu_i(b_{i,1}) & \text{for } 0 \le \theta^i < b_{i,1} \\ m_2 \cdot (\theta^i - b_{i,1}) + \mu_i(b_{i,1}) & \text{for } b_{i,1} \le \theta^i < r_i \\ m_3 \cdot (\theta^i - r_i) & \text{for } r_i \le \theta^i < b_{i,2} \\ m_4 \cdot (\theta^i - b_{i,2}) + \mu_i(b_{i,2}) & \text{for } b_{i,2} \le \theta^i < b_{i,3} \\ m_5 \cdot (\theta^i - b_{i,3}) + \mu_i(b_{i,3}) & \text{for } b_{i,3} \le \theta^i < a_i \\ m_6 \cdot (\theta^i - a_i) + 1 & \text{for } a_i \le \theta^i < b_{i,4} \\ m_7 \cdot (\theta^i - b_{i,4}) + \mu_i(b_{i,4}) & \text{for } b_{i,4} \le \theta^i \le M_i \end{cases} \tag{4.27}$$

[3]A **piecewise linear function** consists of several straight line segments.

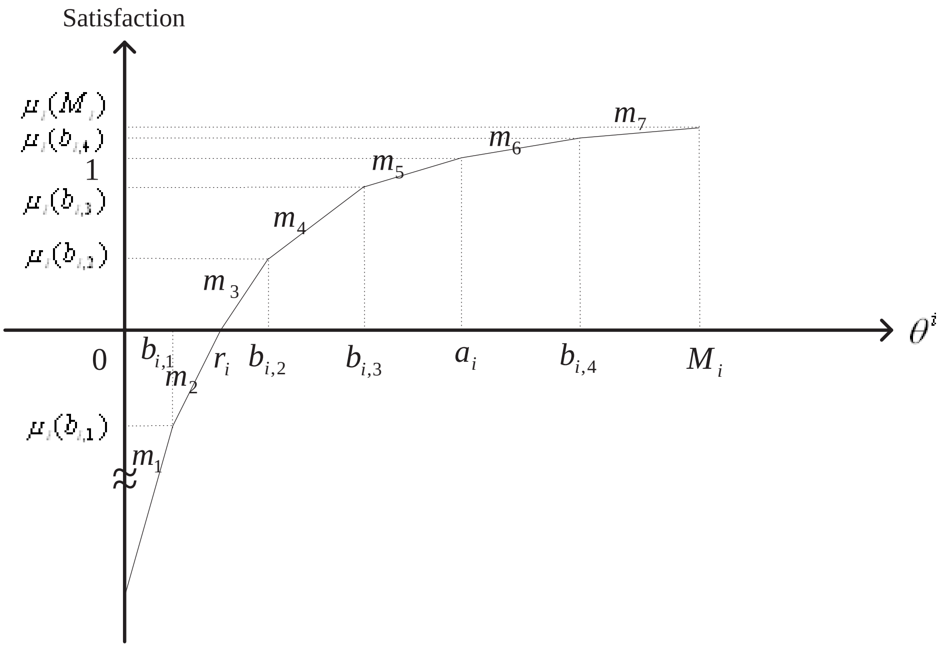

Figure 4.2: The Graph of $\hat{\mu}_i(\theta^i)$

as a continuous function with $\hat{\mu}_i(0) = -m_1 b_{i,1} + \mu_i(b_{i,1})$ and derivative

$$\hat{\mu}_i'(\theta^i) = \begin{cases} m_1 & \text{for } 0 \leq \theta^i < b_{i,1} \\ m_2 & \text{for } b_{i,1} \leq \theta^i < r_i \\ m_3 & \text{for } r_i \leq \theta^i < b_{i,2} \\ m_4 & \text{for } b_{i,2} \leq \theta^i < b_{i,3} \\ m_5 & \text{for } b_{i,3} \leq \theta^i < a_i \\ m_6 & \text{for } a_i \leq \theta^i < b_{i,4} \\ m_7 & \text{for } b_{i,4} \leq \theta^i \leq M_i, \end{cases} \tag{4.28}$$

where M_i is a sufficiently large number, m_i represents the slope on i-th line segment for $i = 1, 2, \ldots, 7$, and $m_k > m_{k+1}$ for $k = 1, 2, \ldots, 6$.

Since a piecewise linear function is not a linear function, one might think that linear programming could not be used to solve optimization problems involving these functions. By using 0-1 variables, however, piecewise linear functions can be represented in linear form. The piecewise linear function $\hat{\mu}_i$ that we constructed above has break points[4] $0,\ b_{i,1},\ r_i,\ b_{i,2},\ b_{i,3},\ a_i,\ b_{i,4},\ M_i$. We use the following method to express the above piecewise linear function via linear constraints and 0-1 variables:

Step 1. Wherever $\hat{\mu}_i(\theta^i)$ occurs in the optimization problem, replace $\hat{\mu}_i(\theta^i)$ by

$$z_1^i\hat{\mu}_i(0) + z_2^i\hat{\mu}_i(b_{i,1}) + z_3^i\hat{\mu}_i(r_i) + z_4^i\hat{\mu}_i(b_{i,2})$$
$$+z_5^i\hat{\mu}_i(b_{i,3}) + z_6^i\hat{\mu}_i(a_i) + z_7^i\hat{\mu}_i(b_{i,4}) + z_8^i\hat{\mu}_i(M_i). \tag{4.29}$$

Step 2. Add the following constraints to the problem:

$$\theta^i = 0z_1^i + z_2^i b_{i,1} + z_3^i r_i + z_4^i b_{i,2}$$
$$+z_5^i b_{i,3} + z_6^i a_i + z_7^i b_{i,4} + z_8^i M_i \tag{4.30}$$
$$z_1^i \leq y_1^i \tag{4.31}$$
$$z_k^i \leq y_{k-1}^i + y_k^i,\ \forall k = 2, \dots, 7 \tag{4.32}$$
$$z_8^i \leq y_7^i \tag{4.33}$$
$$\sum_{k=1}^{8} z_k^i = 1 \tag{4.34}$$
$$\sum_{k=1}^{7} y_k^i = 1 \tag{4.35}$$
$$y_k^i = 0 \text{ or } 1, \forall k = 1, 2, \dots, 7 \tag{4.36}$$
$$z_k^i \geq 0, \forall k = 1, 2, \dots, 8. \tag{4.37}$$

Next, we deal with the constraints involving the logarithmic functions for each i:

$$t_h - d_{ih} \leq f_i(\theta^i),\ \forall h = 1, \dots, m,$$

[4]The points where the slope of the piecewise linear function changes (or the range of definition of the function ends) are called the **break points** of the function.

where $f_i(\theta^i) = G_i \log \theta^i - \log H_i$. We approximate $f_i(\mathbf{x})$ by

$$\hat{f}_i(\theta^i) = \begin{cases} G_i m_{i,1}\theta^i + G_i m_{i,1}(-m_{i,1}b_{i,1} + \mu_i(b_{i,1})) - \log H_i & \text{for } 0 \le \theta^i < b_{i,1} \\ G_i m_{i,2}\theta^i + G_i m_{i,2}(-m_{i,2}b_{i,1} + \mu_i(b_{i,1})) - \log H_i & \text{for } b_{i,1} \le \theta^i < 1 \\ G_i m_{i,3}\theta^i + G_i m_{i,3}(-m_{i,3}) - \log H_i & \text{for } 1 \le \theta^i < b_{i,2} \\ G_i m_{i,4}\theta^i + G_i m_{i,4}(-m_{i,4}b_{i,2} + \mu_i(b_{i,2})) - \log H_i & \text{for } b_{i,2} \le \theta^i < b_{i,3} \\ G_i m_{i,5}\theta^i + G_i m_{i,5}(-m_{i,5}b_{i,3} + \mu_i(b_{i,3})) - \log H_i & \text{for } b_{i,3} \le \theta^i < 10 \\ G_i m_{i,6}\theta^i + G_i m_{i,6}(-10m_{i,6} + 1) - \log H_i & \text{for } 10 \le \theta^i < b_{i,4} \\ G_i m_{i,7}\theta^i + G_i m_{i,7}(-m_{i,7}b_{i,4} + \mu_i(b_{i,4})) - \log H_i & \text{for } b_{i,4} \le \theta^i \le M_i, \end{cases} \tag{4.38}$$

where M_i is a sufficiently large number. Employing the method mentioned above, $f_i(\theta^i)$ is replaced by

$$z_1^i \hat{f}_i(0) + z_2^i \hat{f}_i(b_{i,1}) + z_3^i \hat{f}_i(1) + z_4^i \hat{f}_i(b_{i,2})$$

$$+z_5^i \hat{f}_i(b_{i,3}) + z_6^i \hat{f}_i(10) + z_7^i \hat{f}_i(b_{i,4}) + z_8^i \hat{f}_i(M_i). \tag{4.39}$$

Then add the following constraints to the model:

$$\theta^i = 0z_1^i + z_2^i b_{i,1} + z_3^i + z_4^i b_{i,2}$$

$$+z_5^i b_{i,3} + 10z_6^i + z_7^i b_{i,4} + z_8^i M_i \tag{4.40}$$

$$z_1^i \le y_1^i \tag{4.41}$$

$$z_k^i \le y_{k-1}^i + y_k^i,\ \forall k = 2, \ldots, 7 \tag{4.42}$$

$$z_8^i \le y_7^i \tag{4.43}$$

$$\sum_{k=1}^{8} z_k^i = 1 \tag{4.44}$$

$$\sum_{k=1}^{7} y_k^i = 1 \tag{4.45}$$

$$y_k^i = 0 \text{ or } 1, \forall k = 1, 2, \ldots, 7 \tag{4.46}$$

$$z_k^i \ge 0, \forall k = 1, 2, \ldots, 8. \tag{4.47}$$

We proceed to consider the following constraints in the mathematical programming model (MP1):

$$\sum_i \sum_j \chi_j^i(e)\theta_j^i = x_e, \ \forall e \in E \tag{4.48}$$

and

$$\theta_j^i \cdot \sum_e \kappa_e \chi_j^i(e) = c^i, \ \forall j \in S^i, \text{ for } i = 1, \dots, m. \tag{4.49}$$

Since 0-1 variables $\chi_j^i(e)$ multiplied by decision variables θ_j^i are nonlinear, we replace $\chi_j^i(e)\theta_j^i$ by nonnegative variables $A_j^i(e)$. Then (4.48) and (4.49) become

$$\sum_i \sum_j A_j^i(e) = x_e, \ \forall e \in E \tag{4.50}$$

and

$$\sum_e \kappa_e A_j^i(e) = c^i, \ \forall j \in S^i, \text{ for } i = 1, \dots, m. \tag{4.51}$$

Simultaneously,

$$\theta_j^i \geq b^i, \ \forall j \in S^i, \text{ for } i = 1, \dots, m \tag{4.52}$$

can be rewritten as

$$A_j^i(e) \geq b^i \chi_j^i(e), \ \forall e \in E, \ \forall j \in S^i, \text{ for } i = 1, \dots, m. \tag{4.53}$$

Then we get the two constraints of the form

$$-A_j^i(e) + b^i \leq 0 \tag{4.54}$$

$$-A_j^i(e) \leq 0. \tag{4.55}$$

We want to ensure that at least one of (4.54) and (4.55) is satisfied. Adding the two constraints (4.56) and (4.57) to the formulation will ensure that at least one of (4.54) and (4.55) is satisfied:

$$-A_j^i(e) + b^i \leq M \cdot \chi_j^i(e) \tag{4.56}$$

$$-A_j^i(e) \leq M \cdot (1 - \chi_j^i(e)), \tag{4.57}$$

where $\chi_j^i(e)$ is a $0-1$ variable, and M is a number chosen large enough to ensure that $-A_j^i(e) + b^i \leq M$ and $-A_j^i(e) \leq M$ are satisfied. Next, let us show that the following result holds.

Theorem 4.5.1 *The inclusion of constraints (4.56) and (4.57) is equivalent to at least one of (4.54) and (4.55) being satisfied. Either* $\chi_j^i(e) = 0$ *or* $\chi_j^i(e) = 1$.

Proof. If $\chi_j^i(e) = 0$, then (4.56) and (4.57) becomes $-A_j^i(e) + b^i \leq 0$ and $-A_j^i(e) \leq M$. Thus, if $\chi_j^i(e) = 0$, then (4.54) must be satisfied. Similarly, if $\chi_j^i(e) = 1$, then (4.56) and (4.57) becomes $-A_j^i(e) + b^i \leq M$ and $-A_j^i(e) \leq 0$. Thus, if $\chi_j^i(e) = 1$, then (4.55) must be satisfied. Therefore, whether $\chi_j^i(e) = 0$ or $\chi_j^i(e) = 1$, (4.56) and (4.57) ensure that at least one of (4.54) and (4.55) is satisfied. □

4.5.3 A Mixed-Integer Programming Model

Making use of techniques mentioned above, we rewrite (MP1) as the following model (MP2).

Maximize

$$\sum_{i=1}^{m} i w_i t_i - \sum_{i=1}^{m} \sum_{k=1}^{m} w_i d_{ki} \tag{4.58}$$

subject to

$$\sum_{e \in E} \kappa_e x_e = B \tag{4.59}$$

$$\sum_i \sum_j A_j^i(e) = x_e, \ \forall e \in E \tag{4.60}$$

$$\sum_i (K^i \cdot c^i + \pi^i) = B \tag{4.61}$$

$$\sum_e \kappa_e A_j^i(e) = c^i, \ \forall j \in S^i, \ \text{for } i = 1, \ldots, m \tag{4.62}$$

$$x_e \leq U_e, \ \forall e \in E \tag{4.63}$$

$$\begin{aligned} t_i - d_{ki} - z_1^i \hat{f}_i(0) - z_2^i \hat{f}_i(b_{i,1}) - z_3^i \hat{f}_i(1) - z_4^i \hat{f}_i(b_{i,2}) - z_5^i \hat{f}_i(b_{i,3}) \\ -z_6^i \hat{f}_i(10) - z_7^i \hat{f}_i(b_{i,4}) - z_8^i \hat{f}_i(M_i) \leq 0, \ \forall i, \ k = 1, \ldots, m \end{aligned} \tag{4.64}$$

$$d_{ki} \geq 0, \ \forall i, \ k = 1, \ldots, m \tag{4.65}$$

$$-A_j^i(e) + b^i \leq M \cdot \chi_j^i(e), \ \forall e \in E, \ \forall j \in S^i, \ \text{for } i = 1, \ldots, m \tag{4.66}$$

$$-A_j^i(e) \leq M \cdot (1 - \chi_j^i(e)), \ \forall e \in E, \ \forall j \in S^i, \ \text{for } i = 1, \ldots, m \tag{4.67}$$

$$\theta^i = 0z_1^i + z_2^i b_{i,1} + z_3^i + z_4^i b_{i,2} + z_5^i b_{i,3} + 10z_6^i + z_7^i b_{i,4} + z_8^i M_i, \ \text{for } i = 1, \ldots, m \tag{4.68}$$

$$z_1^i \leq y_1^i, \ \text{for } i = 1, \ldots, m \tag{4.69}$$

$$z_k^i \leq y_{k-1}^i + y_k^i, \ \forall k = 2, \ldots, 7, \ i = 1, \ldots, m \tag{4.70}$$

$$z_8^i \leq y_7^i, \ \text{for } i = 1, \ldots, m \tag{4.71}$$

$$\sum_{k=1}^{8} z_k^i = 1, \ \text{for } i = 1, \ldots, m \tag{4.72}$$

$$\sum_{k=1}^{7} y_k^i = 1, \ \text{for } i = 1, \ldots, m \tag{4.73}$$

$$y_k^i = 0 \text{ or } 1, \forall k = 1, 2, \ldots, 7, \ i = 1, \ldots, m \tag{4.74}$$

$$z_k^i \geq 0, \forall k = 1, 2, \ldots, 8, \ i = 1, \ldots, m \tag{4.75}$$

$$x_e \geq 0, \ \forall e \in E \tag{4.76}$$

$$\chi_j^i(e) = 0 \text{ or } 1, \ \forall e \in E, \ \forall j \in S^i, \ \text{for } i = 1, \ldots, m \tag{4.77}$$

$$A_j^i(e) \geq 0, \ \forall e \in E, \ \forall j \in S^i, \ \text{for } i = 1, \ldots, m \tag{4.78}$$

$$t_i \text{ unrestricted}, \ \forall i = 1, \ldots, m, \tag{4.79}$$

where $w_m = \nu_m$, $w_i = \nu_i - \nu_{i+1}$ for $i = 1, \ldots, m-1$, $\nu_i \in (0, 1)$ is given for each i, and $\sum_{i=1}^{m} \nu_i = 1$.

4.6 Pareto Optimal Solutions

Using the mathematical programming model, given a limited available budget B, we can get the optimal solutions x_e^* and θ^{i*} which represent the optimal bandwidth allocation for each link e and for each class i. Bandwidth are allocated along less expensive paths that connect the origin o and the destination d. Since each edge e on the network has cost κ_e, we can find the minimal cost path by the optimization procedure. The set of all possible optimal paths from the origin to the destination denotes R^*, that is, $R^* = \{p_j | p_j$ is the optimal path from o to $d\}$. After the optimization of the bandwidth allocation model, we can obtain the optimal path p_j from o to d $j = 1, \ldots, \sharp(R^*)$. The optimal path p_j may not be unique. The set R^* includes the inexpensive routes from the origin to the destination on the network.

If the optimal bandwidths allocated to each class i are $\theta^{i*} \geq 0$, then θ^{i*} is the unique solution and it can provide the proportional fairness to every class. If, for each class i, the bandwidth allocated to each optimal path p_j is $\theta^i_{p_j} \geq 0$, then

$$\sum_j \theta^i_{p_j} = \theta^{i*} \tag{4.80}$$

and

$$0 \leq \sum_i \theta^i_{p_j} \leq \min_{e \in p_j} U_e \tag{4.81}$$

Next, we discuss the sensitivity in the maximal number K^i of each class i. When $K^i \to K^i + \epsilon$, the constraint

$$\sum_i (K^i \cdot c^i + \pi^i) = B \tag{4.82}$$

becomes

$$\sum_i ((K^i + \epsilon) \cdot c^i + \pi^i) = B \tag{4.83}$$

which equivalent to

$$\sum_i (K^i \cdot c^i + \pi^i) = B - \epsilon c^i. \tag{4.84}$$

If the shadow price of (4.82) is ς, then the impact on the objective function value is $\epsilon c^i \varsigma$ as $K^i \to K^i + \epsilon$.

We find a Pareto optimal allocation $\mathbf{x}^*$ of bandwidths on the network under a limited available budget B. This allocation $\mathbf{x}^*$ can provide the so-called proportional fairness to every class i, that is, this allocation can provide the similar satisfaction to each user in all classes. We also can find the bandwidths θ^{i*} allocated to each class i, because $\theta^{i*} = \sum_{j=1}^{K^i} \theta^{i*}_j$. Moreover, we also attain the maximal rate (capacity)

R_e^i, which the link e can offer for a new connection of class i. Next, using the output of this model, we will try to provide a routing scheme with End-to-End QoS guarantees.

4.7 A Routing Scheme with End-to-End QoS Guarantees

4.7.1 QoS Guarantees in Deterministic Case

From the optimization of the model mentioned above, we have a network $G = (V, E')$, where V is the set of nodes and E' is the set of links given from the pareto optimal allocation $\mathbf{x}^*$, namely, $\mathbf{x}^* = \{x_e^* \mid e \in E'\}$. For each connection j in class i, we also obtain $\theta_j^{i*} = r^i$. Each link $e \in E'$ is characterized by the following values:

i. A maximal bandwidth (rate) U_e^i, which the link e can offer to a new connection of class i. When a new connection j of class i with a rate $\theta_j^i < U_e^i$ is established through link e, the value of U_e^i becomes $U_e^i - \theta_j^i$.

ii. A constant delay d_e, related to the link's speed, propagation delay and maximal transfer unit.

A connection j of class i in the network is characterized [26] by the following values:

i. A source node o and a destination node d.

ii. A mean packet size c^i of a connection in each class i.

iii. A bandwidth (rate) $\theta_j^{i*} = r^i$.

iv. A maximal end-to-end delay constraint D^i.

In each class i, a connection j should be routed through some path p_j between the corresponding source and destination nodes. We shall denote by $n(p_j)$ the number of links of a path p_j. We assume that the scheduling policy in the network belongs to the rate-based class. When a connection j of class i is routed over a path p_j with a rate r^i, the following end-to-end delay $D(p_j)$ applies:

$$D(p_j) = \frac{n(p_j) \cdot c^i}{r^i} + \sum_{e \in p_j} d_e, \quad \forall j \in R^*. \tag{4.85}$$

A path p_j between o and d is feasible for connection j of class i if $D(p_j) \leq D^i$. A connection j of class i is feasible if it has a feasible path.

4.7.2 Optimization of the Path Selection

We begin with the basic problem of identifying feasible paths, where multiple connections can be established. Given a network $G = (V, E')$, with a maximal rate U_e^i and a delay d_e for each link $e \in E'$. For each connection of class i, we find a feasible path p_j such that:

$$\frac{n(p_j) \cdot c^i}{r^i} + \sum_{e \in p_j} d_e \leq D^i, \quad \forall j \in R^*, \quad \text{for } i = 1, \ldots, m \tag{4.86}$$

$$N(e) \cdot r^i \leq R_e^i, \quad \forall e \in E', \ \text{for } i = 1, \ldots, m, \tag{4.87}$$

where $N(e)$ means the number $\sharp\{p_j \in R^* | e \in p_j\}$. The ability to identify a feasible path for a connection does not yield yet a satisfactory QoS routing solution. In order to supervise multiple connections across the network, the routing algorithm must consider the efficient use of the consumed bandwidth. There does not seem to be a precise definition for the optimality of a path in this context, yet it is clear that an efficient scheme should aim at balancing the loads across the network in [26]. In the following, we present one scheme for achieving such goals.

Here, we consider one scheme that aims at balancing the loads across the network. The way that the path rate affects the available rate of its links depends on the current rate value at each link [7]. Therefore, a better measure for balancing the loads over the network [26] may be one that aims at seeking a path for which the residual maximal rate (i.e., after establishing the new connection) of its bottleneck link is maximal. More precisely, in each class i, given a connection j and a feasible path p_j between its source and destination nodes, denote by A_{p_j} the (largest possible value of the) residual maximal rate of its bottleneck link, i.e.,

$$A_{p_j} = \min_{e \in p_j} (R_e^i - r^i). \tag{4.88}$$

The problem, then, is to find a feasible path p_j^* that maximizes A_{p_j}. This routing scheme distributes the connection among the paths so as to avoid overloaded links.

For each class i, we give the following mathematical model (MP3) of the routing with End-to-End QoS guarantees.

$$\begin{aligned} \text{maximize} \quad & A_{p_j} \\ \text{subject to} \quad & \tfrac{n(p_j)\cdot c^i}{r^i} + \sum_{e \in p_j} d_e \le D^i,\ \forall j \in R^* \\ & N(e)\cdot r^i \le R_e^i,\ \forall e \in E',\ \forall j \in S^i \\ & A_{p_j} \le R_e^i - r^i,\ \ \forall e \in p_j,\ \forall j \in R^*. \end{aligned}$$

The optimization goals of this scheme is to enhance the performance of IP traffic while utilizing the bandwidth on All-IP networks economically. This QoS routing is to make more efficient use of bandwidth on the network.