Chapter 5

A Numerical Example

5.1 Problem Statements

Consider a network topology $G =< V, E >$ (as Figure 5.1 shows), where V ={node 1, node 2, ..., node 7} and $E = \{e_k,\ k = 1, 2, \ldots, 14\}$ denote the set of nodes and the set of links in the network respectively. Let node 1 and node 7 be the source and destination respectively. Each connection is delivered from o to d. Given the cost taking account of delay and the purchasing cost of bandwidth for each link: $\kappa_1 = \$5$, $\kappa_2 = \$6$, $\kappa_3 = \$10$, $\kappa_4 = \$5$, $\kappa_5 = \$4$, $\kappa_6 = \$11$, $\kappa_7 = \$6$, $\kappa_8 = \$8$, $\kappa_9 = \$6$, $\kappa_{10} = \$7$, $\kappa_{11} = \$12$, $\kappa_{12} = \$6$, $\kappa_{13} = \$5$, and $\kappa_{14} = \$6$. There are also given the maximal capacity of each link: $U_1 = 2,300$ kbps (i.e. kilobits/sec), $U_2 = 3,500$ kbps, $U_3 = 1,000$ kbps, $U_4 = 2,500$ kbps, $U_5 = 2,100$ kbps, $U_6 = 2,200$ kbps, $U_7 = 2,000$ kbps, $U_8 = 3,000$ kbps, $U_9 = 2,100$ kbps, $U_{10} = 2,700$ kbps, $U_{11} = 1,500$ kbps, $U_{12} = 1,800$ kbps, $U_{13} = 3,000$ kbps, and $U_{14} = 3,500$ kbps.

There are given three classes (as Table 5.1 shows), where class 1 has the highest priority and class 3 has the lowest priority. The maximal possible number of connections in each class is 10. Under the total available budget $B = \$130,000$, we want to allocate the bandwidths in order to provide each class with maximal

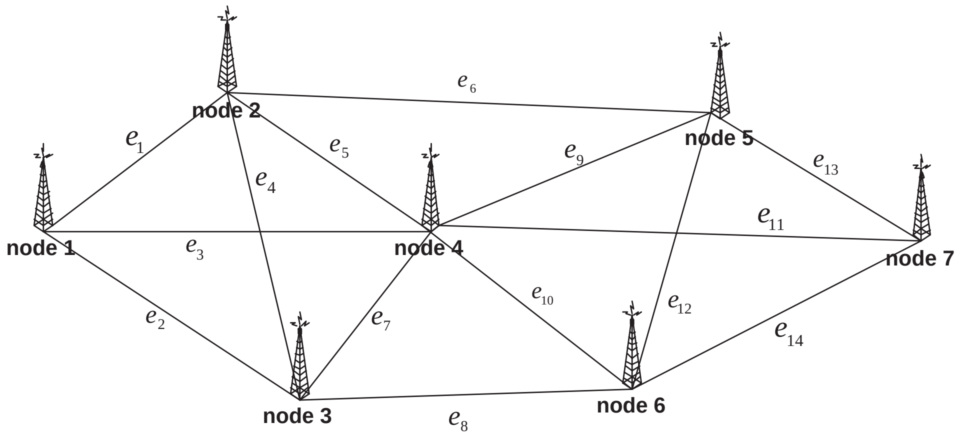

Figure 5.1: Sample Network Topology

Table 5.1: The Characteristics of Each Class

class	bandwidth requirement	aspiration level of bandwidth	reservation level of bandwidth	reserved budget	maximal number of connections
1	160 kbps	334 kbps	167 kbps	$18,000	10
2	80 kbps	166 kbps	83 kbps	$9,000	10
3	25 kbps	56 kbps	28 kbps	$3,000	10

possible quality of service (QoS) defined via (3.1).

Let x_k be the bandwidth allocated to the link $e_k \forall k = 1, 2, \ldots, 14$. We also let θ^i_j be the bandwidth allocated to the connection j of class i $\forall i = 1, 2, 3$. For each class i, we consider the objective function f_i as below:

$$f_1(\theta^1) = \log_2 \frac{\theta^1}{1670}, \tag{5.1}$$

$$f_2(\theta^2) = \log_2 \frac{\theta^2}{830}, \tag{5.2}$$

$$f_3(\theta^3) = \log_2 \frac{\theta^3}{280}, \tag{5.3}$$

where θ^i is the bandwidth allocated to class i. Suppose each objective is regarded as important as each other. Thus, each objective function has equal weight $w_1 = w_2 = w_3 = \frac{1}{3}$. Then, we can formulate the mathematical model as follows.

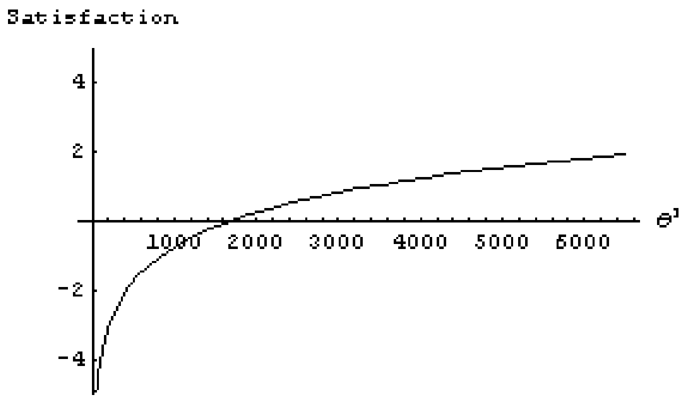

Figure 5.2: The Graph of $f_1(\theta^1)$

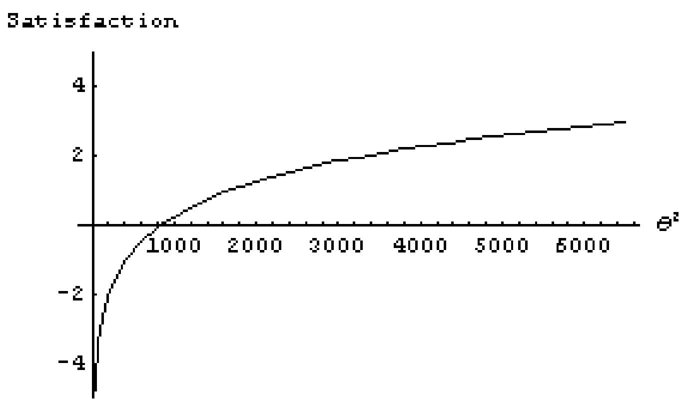

Figure 5.3: The Graph of $f_2(\theta^2)$

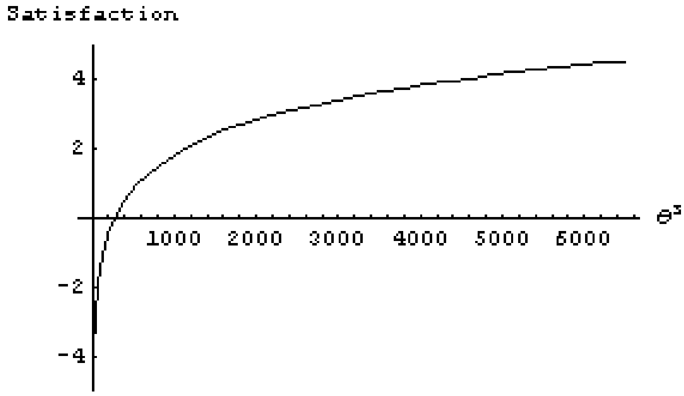

Figure 5.4: The Graph of $f_3(\theta^3)$

5.2 A Mathematical Programming

As the formulation of (MP1) in Chapter 4, we have the following mathematical model (MP4):

maximize $\frac{1}{3}\log_2\frac{\theta^1}{1670}+\frac{1}{3}\log_2\frac{\theta^2}{830}+\frac{1}{3}\log_2\frac{\theta^3}{280}$

subject to

$$5x_1+6x_2+10x_3+5x_4+4x_5+11x_6+6x_7+8x_8+6x_9$$
$$+7x_{10}+12x_{11}+6x_{12}+5x_{13}+6x_{14}=130,000$$

$$(10c^1+18,000)+(10c^2+9,000)+(10c^3+3,000)=130,000$$

$$\theta^i_j\cdot(5\chi^i_j(e_1)+6\chi^i_j(e_2)+10\chi^i_j(e_3)+5\chi^i_j(e_4)+4\chi^i_j(e_5)$$
$$+11\chi^i_j(e_6)+6\chi^i_j(e_7)+8\chi^i_j(e_8)+6\chi^i_j(e_9)+7\chi^i_j(e_{10})$$
$$+12\chi^i_j(e_{11})+6\chi^i_j(e_{12})+5\chi^i_j(e_{13})+6\chi^i_j(e_{14}))=c^i,$$
$$\forall j=1,\ldots,10,\ \forall i=1,2,3$$

$$\theta^1_1=\theta^1_2=\cdots=\theta^1_{10}\geq 160$$

$$\theta_1^2 = \theta_2^2 = \cdots = \theta_{10}^2 \geq 80$$

$$\theta_1^3 = \theta_2^3 = \cdots = \theta_{10}^3 \geq 25$$

$$\textstyle\sum_{j=1}^{10} \theta_j^i = \theta^i, \ \forall i = 1, 2, 3$$

$$\sum_{i=1}^{3} \sum_{j=1}^{10} \chi_j^i(e_k) \cdot \theta_j^i = x_k, \ \forall k = 1, \ldots, 14$$

$$\chi_j^i(e_k) = 0 \text{ or } 1, \quad \forall i = 1, 2, 3, \ \forall j = 1, \ldots, 10, \text{and } k = 1, \ldots, 14$$

$$0 \leq x_k \leq U_k, \ \forall k = 1, \ldots, 14,$$

where $U_1 = 2,300$, $U_2 = 3,500$, $U_3 = 1,000$, $U_4 = 2,500$, $U_5 = 2,100$, $U_6 = 2,200$, $U_7 = 2,000$, $U_8 = 3,000$, $U_9 = 2,100$, $U_{10} = 2,700$, $U_{11} = 1,500$, $U_{12} = 1,800$, $U_{13} = 3,000$, and $U_{14} = 3,500$.

5.3 Numerical Experiments with ILOG

5.3.1 Modifications of the Objective Functions

Since the objective functions f_i in (5.1)-(5.3) are logarithmic functions which can not be solved by ILOG software. To overcome this problem, we replace f_i by piecewise linear functions $\hat{f}_i$ for each $i = 1, 2, 3$.

$$\hat{f}_1(\theta^i) = \begin{cases} 2(\theta^1 - 835) - 1 & \text{for } 0 \leq \theta^1 < 835 \\ \frac{1}{835}(\theta^1 - 835) - 1 & \text{for } 835 \leq \theta^1 < 1670 \\ \frac{0.42}{557}(\theta^1 - 1670) & \text{for } 1670 \leq \theta^1 < 2227 \\ \frac{0.32}{557}(\theta^1 - 2227) + 0.42 & \text{for } 2227 \leq \theta^1 < 2784 \\ \frac{0.26}{556}(\theta^1 - 2784) + 0.74 & \text{for } 2784 \leq \theta^i < 3340 \\ \frac{0.56}{1580}(\theta^1 - 3340) + 1 & \text{for } 3340 \leq \theta^1 < 4920 \\ \frac{0.4}{1580}(\theta^1 - 4920) + 1.56 & \text{for } 4920 \leq \theta^1 \leq 6500 \end{cases} \tag{5.4}$$

$$\hat{f}_2(\theta^2) = \begin{cases} 2(\theta^2 - 415) - 1 & \text{for } 0 \leq \theta^2 < 415 \\ \frac{1}{415}(\theta^2 - 415) - 1 & \text{for } 415 \leq \theta^2 < 830 \\ \frac{0.42}{277}(\theta^2 - 830) & \text{for } 830 \leq \theta^2 < 1107 \\ \frac{0.32}{277}(\theta^2 - 1107) + 0.42 & \text{for } 1107 \leq \theta^2 < 1384 \\ \frac{0.26}{276}(\theta^2 - 1384) + 0.74 & \text{for } 1384 \leq \theta^2 < 1660 \\ \frac{1.30}{2420}(\theta^2 - 1660) + 1 & \text{for } 1660 \leq \theta^2 < 4080 \\ \frac{0.67}{2420}(\theta^2 - 4080) + 2.30 & \text{for } 4080 \leq \theta^2 \leq 6500 \end{cases} \tag{5.5}$$

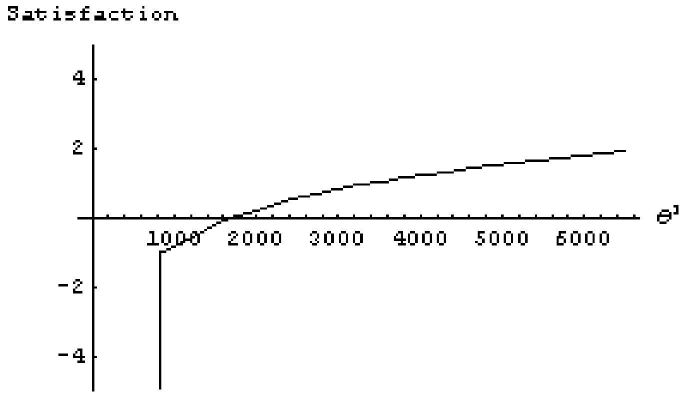

Figure 5.5: The Graph of $\hat{f}_1(\theta^1)$

$$\hat{f}_3(\theta^3) = \begin{cases} 2(\theta^3 - 140) - 1 & \text{for } 0 \leq \theta^3 < 140 \\ \frac{1}{140}(\theta^3 - 140) - 1 & \text{for } 140 \leq \theta^3 < 280 \\ \frac{0.41}{93}(\theta^3 - 280) & \text{for } 280 \leq \theta^3 < 373 \\ \frac{0.32}{93}(\theta^3 - 373) + 0.41 & \text{for } 373 \leq \theta^3 < 466 \\ \frac{0.27}{94}(\theta^3 - 466) + 0.73 & \text{for } 466 \leq \theta^3 < 560 \\ \frac{2.66}{2970}(\theta^3 - 560) + 1 & \text{for } 560 \leq \theta^3 < 3530 \\ \frac{0.88}{2970}(\theta^3 - 3530) + 3.66 & \text{for } 3530 \leq \theta^3 \leq 6500 \end{cases} \tag{5.6}$$

The break points for $\hat{f}_1(\theta^1)$ are 0, 835, 1670, 2227, 2784, 3340, 4920, and 6500, we proceed as follows:

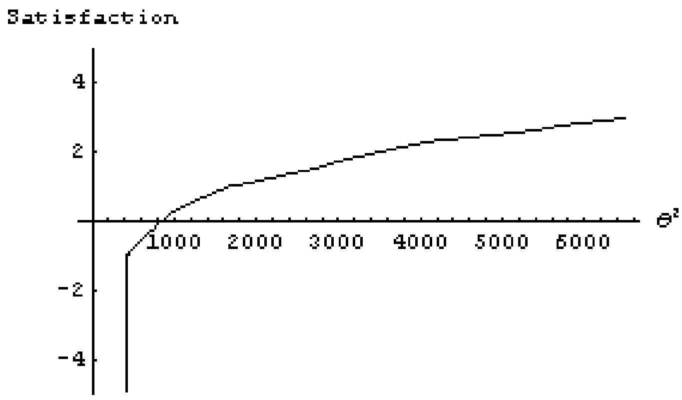

Figure 5.6: The Graph of $\hat{f}_2(\theta^2)$

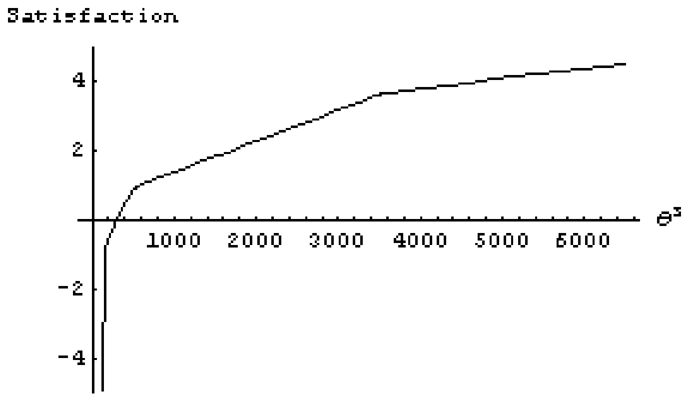

Figure 5.7: The Graph of $\hat{f}_3(\theta^3)$

Step 1 Replace $\hat{f}_1(\theta^1)$ by

$$\begin{aligned}\hat{f}_1(\theta^1) &= z_1^1\hat{f}_1(0) + z_2^1\hat{f}_1(835) + z_3^1\hat{f}_1(1670) + z_4^1\hat{f}_1(2227)\\ &\quad +z_5^1\hat{f}_1(2784) + z_6^1\hat{f}_1(3340) + z_7^1\hat{f}_1(4920) + z_8^1\hat{f}_1(6500).\\ &= -1671z_1^1 - 1z_2^1 + 0.42z_4^1 + 0.74z_5^1 + z_6^1 + 1.56z_7^1 + 1.96z_8^1.\end{aligned} \tag{5.7}$$

Step 2 Add the following constraints:

$$\begin{aligned}\theta^1 &= 0z_1^1 + 835z_2^1 + 1670z_3^1 + 2227z_4^1\\ &\quad +2784z_5^1 + 3340z_6^1 + 4920z_7^1 + 6500z_8^1\end{aligned} \tag{5.8}$$

$$z_1^1 \leq y_1^1 \tag{5.9}$$

$$z_k^1 \leq y_{k-1}^1 + y_k^1,\ \forall k = 2, \dots, 7 \tag{5.10}$$

$$z_8^1 \leq y_7^1 \tag{5.11}$$

$$\sum_{k=1}^{8} z_k^1 = 1 \tag{5.12}$$

$$\sum_{k=1}^{7} y_k^1 = 1 \tag{5.13}$$

$$y_k^1 = 0 \text{ or } 1, \forall k = 1, 2, \dots, 7 \tag{5.14}$$

$$z_k^1 \geq 0, \forall k = 1, 2, \dots, 8. \tag{5.15}$$

The break points for $\hat{f}_2(\theta^2)$ are 0, 415, 830, 1107, 1384, 1660, 4080, and 6500, we proceed as follows:

Step 3 Replace $\hat{f}_2(\theta^2)$ by

$$\begin{aligned}\hat{f}_2(\theta^2) &= z_1^2\hat{f}_2(0) + z_2^2\hat{f}_2(415) + z_3^2\hat{f}_2(830) + z_4^2\hat{f}_2(1107)\\ &\quad +z_5^2\hat{f}_2(1384) + z_6^2\hat{f}_2(1660) + z_7^2\hat{f}_2(4080) + z_8^2\hat{f}_2(6500).\\ &= -831z_1^2 - 1z_2^2 + 0.42z_4^2 + 0.74z_5^2 + z_6^2 + 2.30z_7^2 + 2.97z_8^2.\end{aligned} \tag{5.16}$$

Step 4 Add the following constraints:

$$\begin{aligned}\theta^2 &= 0z_1^2 + 415z_2^2 + 830z_3^2 + 1107z_4^2\\ &\quad +1384z_5^2 + 1660z_6^2 + 4080z_7^2 + 6500z_8^2\end{aligned} \tag{5.17}$$

$$z_1^2 \leq y_1^2 \tag{5.18}$$

$$z_k^2 \leq y_{k-1}^2 + y_k^2, \ \forall k = 2, \dots, 7 \tag{5.19}$$

$$z_8^2 \leq y_7^2 \tag{5.20}$$

$$\sum_{k=1}^{8} z_k^2 = 1 \tag{5.21}$$

$$\sum_{k=1}^{7} y_k^2 = 1 \tag{5.22}$$

$$y_k^2 = 0 \text{ or } 1, \forall k = 1, 2, \dots, 7 \tag{5.23}$$

$$z_k^2 \geq 0, \forall k = 1, 2, \dots, 8. \tag{5.24}$$

The break points for $\hat{f}_3(\theta^3)$ are 0, 140, 280, 373, 466, 560, 3530, and 6500, we proceed as follows:

Step 5 Replace $\hat{f}_3(\theta^3)$ by

$$\begin{aligned} \hat{f}_3(\theta^3) &= z_1^3 \hat{f}_3(0) + z_2^3 \hat{f}_3(140) + z_3^3 \hat{f}_3(280) + z_4^3 \hat{f}_3(373) \\ &\quad + z_5^3 \hat{f}_3(466) + z_6^3 \hat{f}_3(560) + z_7^3 \hat{f}_3(3530) + z_8^3 \hat{f}_3(6500). \\ &= -281 z_1^3 - 1 z_2^3 + 0.41 z_4^3 + 0.73 z_5^3 + z_6^3 + 3.66 z_7^3 + 4.54 z_8^3. \end{aligned} \tag{5.25}$$

Step 6 Add the following constraints:

$$\begin{aligned} \theta^3 &= 0 z_1^3 + 140 z_2^3 + 280 z_3^3 + 373 z_4^3 \\ &\quad + 466 z_5^3 + 560 z_6^3 + 3530 z_7^3 + 6500 z_8^3 \end{aligned} \tag{5.26}$$

$$z_1^3 \leq y_1^3 \tag{5.27}$$

$$z_k^3 \leq y_{k-1}^3 + y_k^3, \ \forall k = 2, \dots, 7 \tag{5.28}$$

$$z_8^3 \leq y_7^3 \tag{5.29}$$

$$\sum_{k=1}^{8} z_k^3 = 1 \tag{5.30}$$

$$\sum_{k=1}^{7} y_k^3 = 1 \tag{5.31}$$

$$y_k^3 = 0 \text{ or } 1, \forall k = 1, 2, \ldots, 7 \tag{5.32}$$

$$z_k^3 \geq 0, \forall k = 1, 2, \ldots, 8. \tag{5.33}$$

Next, combining (5.7), (5.16), and (5.25), we can replace the objective function

$$\frac{1}{3}\log_2\frac{\theta^1}{1670} + \frac{1}{3}\log_2\frac{\theta^2}{830} + \frac{1}{3}\log_2\frac{\theta^3}{280}$$

by

$$\begin{aligned}
& \tfrac{1}{3}\hat{f}_1(\theta^1) + \tfrac{1}{3}\hat{f}_2(\theta^2) + \tfrac{1}{3}\hat{f}_3(\theta^3) \\
= \; & \tfrac{1}{3}(-1671z_1^1 - 1z_2^1 + 0.42z_4^1 + 0.74z_5^1 + z_6^1 + 1.56z_7^1 + 1.96z_8^1) \\
& +\tfrac{1}{3}(-831z_1^2 - 1z_2^2 + 0.42z_4^2 + 0.74z_5^2 + z_6^2 + 2.30z_7^2 + 2.97z_8^2) \\
& +\tfrac{1}{3}(-281z_1^3 - 1z_2^3 + 0.41z_4^3 + 0.73z_5^3 + z_6^3 + 3.66z_7^3 + 4.54z_8^3)
\end{aligned}$$

5.3.2 Modifications of the Constraints

We proceed to consider the following constraints in (MP4):

$$\begin{aligned}
& \theta_j^i \cdot (5\chi_j^i(e_1) + 6\chi_j^i(e_2) + 10\chi_j^i(e_3) + 5\chi_j^i(e_4) + 4\chi_j^i(e_5) \\
& +11\chi_j^i(e_6) + 6\chi_j^i(e_7) + 8\chi_j^i(e_8) + 6\chi_j^i(e_9) + 7\chi_j^i(e_{10}) \\
& +12\chi_j^i(e_{11}) + 6\chi_j^i(e_{12}) + 5\chi_j^i(e_{13}) + 6\chi_j^i(e_{14})) = c^i, \\
& \forall j = 1, \ldots, 10, \;\; \forall i = 1, 2, 3
\end{aligned} \tag{5.34}$$

and

$$\sum_{i=1}^{3}\sum_{j=1}^{10} \chi_j^i(e_k) \cdot \theta_j^i = x_k, \;\; \forall k = 1, \ldots, 14. \tag{5.35}$$

Since 0-1 variables $\chi_j^i(e_k)$ multiplied by decision variables θ_j^i are nonlinear, we replace $\chi_j^i(e_k)\theta_j^i$ by nonnegative variables $A_j^i(e_k)$. Then (5.34) and (5.35) become

$$\begin{aligned}
& 5A_j^i(e_1) + 6A_j^i(e_2) + 10A_j^i(e_3) + 5A_j^i(e_4) + 4A_j^i(e_5) \\
& +11A_j^i(e_6) + 6A_j^i(e_7) + 8A_j^i(e_8) + 6A_j^i(e_9) + 7A_j^i(e_{10}) \\
& +12A_j^i(e_{11}) + 6A_j^i(e_{12}) + 5A_j^i(e_{13}) + 6A_j^i(e_{14}) = c^i, \\
& \forall j = 1, \ldots, 10, \;\; \forall i = 1, 2, 3
\end{aligned} \tag{5.36}$$

and

$$\sum_{i=1}^{3}\sum_{j=1}^{10} A_j^i(e_k) = x_k,\ \forall k = 1,\dots,14. \tag{5.37}$$

Simultaneously,

$$\theta_1^1 = \theta_2^1 = \cdots = \theta_{10}^1 \geq 160, \tag{5.38}$$

$$\theta_1^2 = \theta_2^2 = \cdots = \theta_{10}^2 \geq 80, \tag{5.39}$$

and

$$\theta_1^3 = \theta_2^3 = \cdots = \theta_{10}^3 \geq 25 \tag{5.40}$$

can be rewritten respectively as

$$A_j^1(e_k) \geq 160\chi_j^1(e_k),\ \forall k = 1,\dots,14,\ \forall j = 1,\dots,10, \tag{5.41}$$

$$A_j^2(e_k) \geq 80\chi_j^2(e_k),\ \forall k = 1,\dots,14,\ \forall j = 1,\dots,10, \tag{5.42}$$

and

$$A_j^3(e_k) \geq 25\chi_j^3(e_k),\ \forall k = 1,\dots,14,\ \forall j = 1,\dots,10. \tag{5.43}$$

Then we have the constraints of the form

$$-A_j^1(e_k) + 160 \leq 0 \tag{5.44}$$

$$-A_j^1(e_k) \leq 0, \tag{5.45}$$

$$-A_j^2(e_k) + 80 \leq 0 \tag{5.46}$$

$$-A_j^2(e_k) \leq 0, \tag{5.47}$$

and

$$-A_j^3(e_k) + 25 \leq 0 \tag{5.48}$$

$$-A_j^3(e_k) \leq 0. \tag{5.49}$$

Adding the two constraints (5.50) and (5.51) to the model will ensure that at least one of (5.44) and (5.45) is satisfied:

$$-A_j^1(e_k) + 160 \leq M \cdot \chi_j^1(e_k) \tag{5.50}$$

$$-A_j^1(e_k) \leq M \cdot (1 - \chi_j^1(e_k)). \tag{5.51}$$

Similarly, adding the two constraints (5.52) and (5.53) to the model will ensure that at least one of (5.46) and (5.47) is satisfied:

$$-A_j^2(e_k) + 80 \leq M \cdot \chi_j^2(e_k) \tag{5.52}$$

$$-A_j^2(e_k) \leq M \cdot (1 - \chi_j^2(e_k)). \tag{5.53}$$

Next, adding the two constraints (5.54) and (5.55) to the model will ensure that at least one of (5.48) and (5.49) is satisfied:

$$-A_j^3(e_k) + 25 \leq M \cdot \chi_j^3(e_k) \tag{5.54}$$

$$-A_j^3(e_k) \leq M \cdot (1 - \chi_j^3(e_k)). \tag{5.55}$$

In (5.50)-(5.55), $\chi_j^i(e_k)$ is a 0-1 variable for each i, j and M is a number chosen large enough to ensure that

$$-A_j^1(e_k) + 160 \leq M,$$

$$-A_j^1(e_k) \leq M,$$

$$-A_j^2(e_k) + 80 \leq M,$$

$$-A_j^2(e_k) \leq M,$$

$$-A_j^3(e_k) + 25 \leq M,$$

and

$$-A_j^3(e_k) \leq M$$

are satisfied.

5.3.3 A Mixed-Integer Programming Model

From the above discussion, we present a Mixed-Integer programming model (MP5):

$$\begin{aligned}
\text{maximize} \quad & \tfrac{1}{3}(-1671z_1^1 - 1z_2^1 + 0.42z_4^1 + 0.74z_5^1 + z_6^1 + 1.56z_7^1 + 1.96z_8^1) \\
& + \tfrac{1}{3}(-831z_1^2 - 1z_2^2 + 0.42z_4^2 + 0.74z_5^2 + z_6^2 + 2.30z_7^2 + 2.97z_8^2) \\
& + \tfrac{1}{3}(-281z_1^3 - 1z_2^3 + 0.41z_4^3 + 0.73z_5^3 + z_6^3 + 3.66z_7^3 + 4.54z_8^3)
\end{aligned}$$

$$\begin{aligned}
\text{subject to} \quad & 5x_1 + 6x_2 + 10x_3 + 5x_4 + 4x_5 + 11x_6 + 6x_7 + 8x_8 + 6x_9 \\
& +7x_{10} + 12x_{11} + 6x_{12} + 5x_{13} + 6x_{14} = 130,000
\end{aligned}$$

$$(10c^1 + 18,000) + (10c^2 + 9,000) + (10c^3 + 3,000) = 130,000$$

$$\begin{aligned}
& 5A_j^i(e_1) + 6A_j^i(e_2) + 10A_j^i(e_3) + 5A_j^i(e_4) + 4A_j^i(e_5) \\
& +11A_j^i(e_6) + 6A_j^i(e_7) + 8A_j^i(e_8) + 6A_j^i(e_9) + 7A_j^i(e_{10}) \\
& +12A_j^i(e_{11}) + 6A_j^i(e_{12}) + 5A_j^i(e_{13}) + 6A_j^i(e_{14}) = c^i, \\
& \forall j = 1, \ldots, 10,\ \forall i = 1, 2, 3
\end{aligned}$$

$$\begin{aligned}
& -A_j^1(e_k) + 160 - M\chi_j^1(e_k) \leq 0,\ \forall j = 1, \ldots, 10,\ \forall k = 1, \ldots, 14 \\
& -A_j^1(e_k) - M(1 - \chi_j^1(e_k)) \leq 0,\ \forall j = 1, \ldots, 10,\ \forall k = 1, \ldots, 14
\end{aligned}$$

$$\begin{aligned}
& -A_j^2(e_k) + 80 - M\chi_j^2(e_k) \leq 0,\ \forall j = 1, \ldots, 10,\ \forall k = 1, \ldots, 14 \\
& -A_j^2(e_k) - M(1 - \chi_j^2(e_k)) \leq 0,\ \forall j = 1, \ldots, 10,\ \forall k = 1, \ldots, 14
\end{aligned}$$

$$\begin{aligned}
& -A_j^3(e_k) + 25 - M\chi_j^3(e_k) \leq 0,\ \forall j = 1, \ldots, 10,\ \forall k = 1, \ldots, 14 \\
& -A_j^3(e_k) - M(1 - \chi_j^3(e_k)) \leq 0,\ \forall j = 1, \ldots, 10,\ \forall k = 1, \ldots, 14
\end{aligned}$$

$$\sum_{j=1}^{10} \theta_j^i = \theta^i,\ \forall i = 1, 2, 3$$

$$\sum_{i=1}^{3} \sum_{j=1}^{10} A_j^i(e_k) = x_k,\ \forall k = 1, \ldots, 14$$

$$\chi_j^i(e_k) = 0 \text{ or } 1, \ \forall i = 1,2,3, \ j = 1,\ldots,10, \text{ and } k = 1,\ldots,14$$

$$A_j^i(e_k) \geq 0, \ \forall i = 1,2,3, \ j = 1,\ldots,10, \text{ and } k = 1,\ldots,14$$

$$0 \leq x_k \leq U_k, \ \forall k = 1,\ldots,14$$

$$\theta^1 - 835z_2^1 - 1670z_3^1 - 2227z_4^1 - 2784z_5^1 - 3340z_6^1 - 4920z_7^1 - 6500z_8^1 = 0$$

$$\theta^2 - 415z_2^2 - 830z_3^2 - 1107z_4^2 - 1384z_5^2 - 1660z_6^2 - 4080z_7^2 - 6500z_8^2 = 0$$

$$\theta^3 - 140z_2^3 - 280z_3^3 - 373z_4^3 - 466z_5^3 - 560z_6^3 - 3530z_7^3 - 6500z_8^3 = 0$$

$$z_1^i - y_1^i \leq 0, \ \forall i = 1,2,3$$

$$z_k^i - y_{k-1}^i - y_k^i \leq 0, \ \forall k = 2,\ldots,7, \ \forall i = 1,2,3$$

$$z_8^i - y_7^i \leq 0, \ \forall i = 1,2,3$$

$$\sum_{k=1}^{8} z_k^i = 1, \ \forall i = 1,2,3$$

$$\sum_{k=1}^{7} y_k^i = 1, \ \forall i = 1,2,3$$

$$y_k^i = 0 \text{ or } 1, \ \forall k = 1,2,\ldots,7, \ \forall i = 1,2,3$$

$$z_k^i \geq 0, \ \forall k = 1,2,\ldots,8, \ \forall i = 1,2,3,$$

where $U_1 = 2,300$, $U_2 = 3,500$, $U_3 = 1,000$, $U_4 = 2,500$, $U_5 = 2,100$, $U_6 = 2,200$, $U_7 = 2,000$, $U_8 = 3,000$, $U_9 = 2,100$, $U_{10} = 2,700$, $U_{11} = 1,500$, $U_{12} = 1,800$, $U_{13} = 3,000$, and $U_{14} = 3,500$.

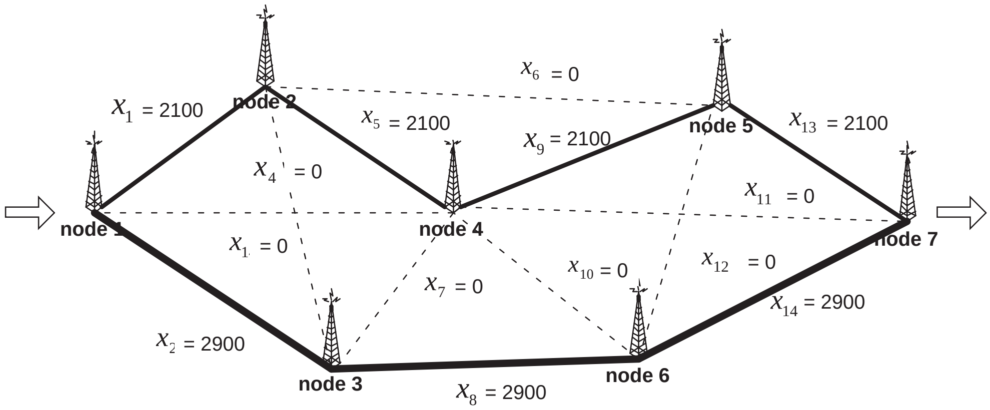

Figure 5.8: The Allocation of Bandwidths in Sample Network Topology

Using this model, we can find a pareto optimal allocation of bandwidth on the network (as Figure 5.8) under a budget $B = \$130,000$. The Pareto optimal solution is: $x_3 = x_4 = x_6 = x_7 = x_{10} = x_{11} = x_{12} = 0$ kbps, $x_1 = x_5 = x_9 = x_{13} = 2,100$ kbps, $x_2 = x_8 = x_{14} = 2,900$ kbps, $\theta_j^1 = 300$ kbps, $\theta_j^2 = 150$ kbps, $\theta_j^3 = 50$ kbps for all j. We find the bandwidth allocated to class 1 is $\theta^1 = 3,000$ kbps, the bandwidth allocated to class 2 is $\theta^2 = 1,500$ kbps, and the bandwidth allocated to class 3 is $\theta^3 = 500$ kbps. This allocation can provide proportional fairness to every class, and the satisfaction of each class equals 0.848. The optimal paths (in Figure 5.8) are 1-2-4-5-7 and 1-3-6-7, and the cost per unit bandwidth of the optimal path is \$20. We also find the bottleneck links are e_5 and e_9.