2 Notations and useful equations

In this section, we shall introduce some of the notations and equations that will be used throughout the thesis.

2.1 Notations

Most of the notations used in this thesis are standard.

- $\mathbb{R}$: the real line.
- $\mathbb{C}$: the complex plane.
- (a, k): the Appell's symbol which is defined as

$$(a, k) = a(a+1)\cdots(a+k-1) \tag{2.1}$$

  where $a \in \mathbb{C}$ and k is any positive integer. In particular, we define $(a, 0) \equiv 1$.
- $\Gamma(a)$: the gamma function with parameter a.
- $B(a, b)$: the beta function with parameters a and b.
- $\delta_x(A)$: the indicator function of x in A which is defined as $\delta_x(A)$ is 1, if $x \in A$; and is 0, otherwise.
- $J_c(y)$: the Bessel function of the first kind, see Gradshteyn and Ryzhik (2000, p. 900). Gradshteyn and Ryzhik (2000, pp. 948–949).

$$J_c(y) = \frac{y^c}{2^c}\sum_{k=0}^{\infty}\frac{(-1)^k y^{2k}}{2^{2k}k!\Gamma(c+k+1)}, \ |\arg y| < \pi. \tag{2.2}$$

- $P_\nu^\mu(x)$: an associated Legendre function of the first kind. See
- ${}_2F_1(a, b; c; x)$: the ordinary or Gauss hypergeometric series, which is defined as

$${}_2F_1(a, b; c; x) = \sum_{m=0}^{\infty}\frac{(a, m)(b, m)}{m!(c, m)}x^m, \ |x| < 1.$$

- $Beta(a,b)$: the beta distribution with parameters a and b. That is, if $X \sim Beta(a,b)$, then the PDF of X is

$$f(x) = \begin{cases} \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

- $X \sim Y$: the random variables X and Y have the same distribution.

- $X_n \xrightarrow{d} X$: the sequence of random variables X_n converges in distribution to the random variable X.

- $\boldsymbol{t} \cdot \boldsymbol{s}$: the inner product of vectors $\boldsymbol{t}$ and $\boldsymbol{s}$.

- u_+: the sum of all components of $\boldsymbol{u} = (u_1, \ldots, u_L)'$, i.e., $u_+ = \sum_{j=1}^{L} u_j$.

- A': the transpose of the matrix A.

- $f'(x)$: the derivative of $f(x)$.

- A^{-1}: the inverse of a square matrix A.

2.2 Useful equations

- Gradshteyn and Ryzhik (2000, Eq. 1.320.5, p. 30)

$$\cos^{2n} x = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} 2\binom{2n}{k} \cos[2(n-k)x] + \binom{2n}{n} \right\} \tag{2.3}$$

- Gradshteyn and Ryzhik (2000, Eq. 1.511, p. 51)

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}, \ |x| < 1. \tag{2.4}$$

From Eq. (2.4), we have

$$\ln(1-x) = \sum_{k=1}^{\infty} \frac{-x^k}{k}, \ |x| < 1. \tag{2.5}$$

- Gradshteyn and Ryzhik (2000, Eq. 1.515.1, p. 52)

$$\ln(1+\sqrt{1+x^2}) = \ln 2 - \sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)!}{2^{2n}(n!)^2} x^{2n}, \ x^2 \le 1. \tag{2.6}$$

- Gradshteyn and Ryzhik (2000, Eq. 1.622.3, p. 54)

$$\arctan z = \frac{1}{2i} \ln \frac{iz - 1}{iz + 1}. \tag{2.7}$$

- Gradshteyn and Ryzhik (2000, Eq. 2.146.3, p. 74)

$$\begin{aligned} \int \frac{x^{m-1}}{1 - x^{2n}}\, dx &= \frac{1}{2n} \left\{ (-1)^{m+1} \ln(1 + x) - \ln(1 - x) \right\} \\ &\quad - \frac{1}{2n} \sum_{k=1}^{n-1} \cos \frac{km\pi}{n} \ln \left(1 - 2x \cos \frac{k\pi}{n} + x^2 \right) \\ &\quad + \frac{1}{n} \sum_{k=1}^{n-1} \sin \frac{km\pi}{n} \arctan \frac{x - \cos(k\pi/n)}{\sin(k\pi/n)} \end{aligned} \tag{2.8}$$

where $m < 2n$.

- Gradshteyn and Ryzhik (2000, Eq. 2.146.4, p. 74)

$$\begin{aligned} &\int \frac{x^{m-1}}{1 - x^{2n+1}}\, dx \\ &= \frac{-1}{2n+1} \ln(1 - x) + \frac{(-1)^{m+1}}{2n+1} \sum_{k=1}^{n} \cos \frac{m\pi(2k-1)}{2n+1} \ln \left(1 + 2x \cos \frac{(2k-1)\pi}{2n+1} + x^2 \right) \\ &\quad + \frac{2(-1)^{m+1}}{2n+1} \sum_{k=1}^{n} \sin \frac{m\pi(2k-1)}{2n+1} \arctan \frac{x + \cos((2k-1)\pi/(2n+1))}{\sin((2k-1)\pi/(2n+1))} \end{aligned} \tag{2.9}$$

where $m \leq 2n$.

- Gradshteyn and Ryzhik (2000, Eq. 2.734, p. 236)

$$\begin{aligned} &(2n+1) \int x^{2n} \ln |x^2 - a^2|\, dx \\ &\quad = x^{2n+1} \ln |x^2 - a^2| + a^{2n+1} \ln \left| \frac{x+a}{x-a} \right| - 2 \sum_{k=0}^{n} \frac{a^{2n-2k} x^{2k+1}}{2k+1} \end{aligned} \tag{2.10}$$

where $F(\cdot,\cdot)$ denotes the elliptic integral of the first kind, see Gradshteyn and Ryzhik (2000, pp. 851–852).

- Gradshteyn and Ryzhik (2000, Eq. 3.197.8, p. 315)

$$\begin{aligned} &\int_0^u x^{\nu-1} (x + \alpha)^{\lambda} (u - x)^{\mu-1}\, dx \\ &\quad = \alpha^{\lambda} u^{\mu+\nu-1} B(\mu, \nu)\, {}_2F_1 \left(-\lambda, \nu; \mu + \nu; \frac{-u}{\alpha} \right) \end{aligned} \tag{2.11}$$

where $|\arg(u/\alpha)| < \pi$, $\operatorname{Re} \mu > 0$, and $\operatorname{Re} \nu > 0$.

- Gradshteyn and Ryzhik (2000, Eq. 3.621.1, p. 389)

$$\int_0^{\pi/2} \sin^{\mu-1} x \, dx = \int_0^{\pi/2} \cos^{\mu-1} x \, dx = 2^{\mu-2} B\left(\frac{\mu}{2}, \frac{\mu}{2}\right) \tag{2.12}$$

- Gradshteyn and Ryzhik (2000, Eq. 3.621.5, p. 389)

$$\int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x \, dx = \frac{1}{2} B\left(\frac{\mu}{2}, \frac{\nu}{2}\right) \tag{2.13}$$

From Eq. (2.13), we have

$$\int_0^{\pi} \sin^{a-1} x \cos^{b-1} x \, dx = \begin{cases} \frac{B(a/2,b/2)}{2}, & \text{if } b \text{ is odd,} \\ 0, & \text{if } b \text{ is even,} \end{cases} \tag{2.14}$$

where $a, b > 0$.

- Gradshteyn and Ryzhik (2000, Eq. 3.663.1, p. 400)

$$\int_0^{u} (\cos x - \cos u)^{\nu-1/2} \cos(ax) \, dx = \sqrt{\frac{\pi}{2}} \sin^{\nu} u \left(\nu + \frac{1}{2}\right) P_{a-1/2}^{-\nu}(\cos u) \tag{2.15}$$

where $\operatorname{Re} \nu > -1/2$, $a > 0$, and $0 < u < \pi$.

- Gradshteyn and Ryzhik (2000, Eq. 4.384.1, p. 577)

$$\int_0^{1} \ln(\sin(\pi x)) \sin(2n\pi x) \, dx = 0 \tag{2.16}$$

- Gradshteyn and Ryzhik (2000, Eq. 4.384.3, p. 578)

$$\int_0^{1} \ln(\sin(\pi x)) \cos(2n\pi x) \, dx = 2 \int_0^{1/2} \ln(\sin(\pi x)) \sin(2n\pi x) \, dx \tag{2.17}$$

Notice that (2.17) is $-\ln 2$, if $n = 0$; is $-1/(2n)$, if $n > 0$.

- Gradshteyn and Ryzhik (2000, Eq. 6.699.2, p. 723)

$$\begin{aligned} &\int_0^{\infty} x^{\lambda} J_{\nu}(ax) \cos(bx) \, dx \\ &= \frac{2^{\lambda} a^{-(1+\lambda)} \Gamma((1+\lambda+\nu)/2)}{\Gamma((\nu-\lambda+1)/2)} {}_2F_1\left(\frac{1+\lambda+\nu}{2}, \frac{1+\lambda-\nu}{2}; \frac{1}{2}; \frac{b^2}{a^2}\right), \end{aligned} \tag{2.18}$$

where $0 < b < a$ and $-\operatorname{Re} \nu < 1 + \operatorname{Re} \lambda < 3/2$.

- Gradshteyn and Ryzhik (2000, Eq. 8.380.1, p. 898)

$$B(x, y) = 2 \int_0^{1} t^{2x-1} (1 - t^2)^{y-1} \, dt, \ \operatorname{Re} x > 0, \ \operatorname{Re} y > 0. \tag{2.19}$$

- Gradshteyn and Ryzhik (2000, Eq. 8.384.4, p. 900)

$$B(x,x) = 2^{1-2x} B\left(\frac{1}{2}, x\right) \tag{2.20}$$

- Gradshteyn and Ryzhik (2000, Eq. 8.391, p. 900)

$$\int_0^x t^{p-1}(1-t)^{q-1}\,dt = \frac{x^p}{p}\,{}_2F_1(p, 1-q; p+1; x) \tag{2.21}$$

- Gradshteyn and Ryzhik (2000, Eq. 8.704, p. 949)

$$P_\nu^\mu(x) = \frac{1}{\Gamma(1-\mu)}\left(\frac{1+x}{1-x}\right)^{\mu/2} {}_2F_1\left(-\nu, \nu+1; 1-\mu; \frac{1-x}{2}\right) \tag{2.22}$$

- Prudnikov, Brychkov, and Marichev (1986, Eq. 7.3.1.27, p. 455)

$${}_2F_1(a,b;b;x) = (1-x)^{-a} \tag{2.23}$$

- Prudnikov, Brychkov, and Marichev (1986, Eq. 7.3.6.2, p. 489)

$${}_2F_1(a,b;1+a-b;-1) = 2^{-a}\sqrt{\pi}\frac{\Gamma(1+a-b)}{\Gamma((1+a)/2)\Gamma(1+a/2-b)} \tag{2.24}$$

- Prudnikov, Brychkov, and Marichev (1986, Eq. 7.4.1.367, p. 519)

$${}_3F_2\left(1,1,\frac{3}{2};2,3;z\right) = \frac{4}{z^2}\left(2\sqrt{1-z} - 2 + z - 2z\ln\frac{1+\sqrt{1-z}}{2}\right) \tag{2.25}$$

- Carlson (1977, Eq. 3.2-9, p. 54)

$$\Gamma(a+n) = \Gamma(a)(a,n), \tag{2.26}$$

- Carlson (1977, Eq. 2.2-7, p. 25)

$$(a,2n) = 2^{2n}\left(\frac{a}{2}, n\right)\left(\frac{a+1}{2}, n\right). \tag{2.27}$$

  From Eq. (2.27), we derive

$$\left(\frac{1}{2}, n\right) = \frac{(2n-1)!}{2^{2n-1}(n-1)!}. \tag{2.28}$$

- Gröbner and Hofreiter (1973, p. 105)

$$\int_0^{2\pi} (a\cos x + b\sin x)^n\,dx = \begin{cases} \frac{(1/2,n/2)2(a^2+b^2)^{n/2}\pi}{(n/2)!}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases} \tag{2.29}$$

  where $a, b \in \mathbb{R}$.

- Erdélyi, Magnus, Oberhettinger, and Tricomi (1953, p. 101, 2.8(6))

$$\left(\frac{1+\sqrt{1-z}}{2}\right)^{1-2a} = {}_2F_1\left(a-\frac{1}{2}, a; 2a; z\right). \tag{2.30}$$