3 The c-characteristic function

We review the univariate and the multivariate c-characteristic functions in Subsections 3.1 and 3.2, respectively. We also provide important additional properties of the multivariate c-characteristic function for a spherical distribution. In Subsection 3.3, we construct inversion formulas of a univariate c-characteristic function. In Subsection 3.4, we give a relation between the univariate c-characteristic function and the Fourier transformation. In Subsection 3.5, we apply our inversion formulas to study the distributions of linear combinations of the components of a Dirichlet random vector.

3.1 The univariate c-characteristic function

First, we state the definition of a univariate c-characteristic function of a random variable.

Definition 3.1 *For any random variable X with support in a subset of $[-a, a]$, $a > 0$, we define the univariate c-characteristic function of X as $g(t; X, c) = E(1-itX)^{-c}$ where $|t| < a^{-1}$ and $c > 0$.*

Jiang (1988) showed that the univariate c-characteristic function has properties similar to those of traditional characteristic function, e.g., uniqueness and convergence theorems. In the following lemma, which follows Lemma 3.1 of Jiang (1988), there is a one-to-one correspondence between a random variable and its univariate c-characteristic function.

Lemma 3.2 *Let X_1 and X_2 be random variables with support in a subset of $[-a, a]$. For any $c > 0$, if we have $g(t; X_1, c) = g(t; X_2, c)$ for all $|t| < a^{-1}$, then $X_1 \sim X_2$.*

We state the important convergence theorem, which follows Theorem 3.5 of Jiang (1988), of a univariate c-characteristic function in the following lemma.

Lemma 3.3 *Assume that X and X_1, X_2,... are random variables with support in a subset of $[-a, a]$. Then, for any $c > 0$, $X_n \xrightarrow{d} X$ if and only if $g(t; X_n, c) \to g(t; X, c)$ for all $|t| < a^{-1}$.*

Next, we state the method, which is given by Kuo (2002) and can be used to determine any moment of a random variable through its univariate c-characteristic function.

Lemma 3.4 *For a given $c > 0$,*

$$E(X^n) = \frac{1}{i^n(c,n)} \frac{d^n}{dt^n} g(t; X, c)\bigg|_{t=0}.$$

More properties and applications of the univariate c-characteristic function can be seen in Jiang (1988) and Kuo (2002).

3.2 The multivariate c-characteristic function

First, we state the definition of the multivariate c-characteristic function of a random vector.

Definition 3.5 *If $\boldsymbol{X} = (X_1, \ldots, X_L)'$ is a random vector with support in a subset of $[-a_1, a_1] \times \cdots \times [-a_L, a_L]$, its multivariate c-characteristic function is defined as*

$$g(\boldsymbol{t}; \boldsymbol{X}, c) = E[(1 - i\boldsymbol{t} \cdot \boldsymbol{X})^{-c}], \;\; |\boldsymbol{t}| < a^{-1},$$

where $c > 0$, $a = \sqrt{\sum_{j=1}^{L} a_j^2}$, $\boldsymbol{t} = (t_1, \ldots, t_L)'$, and $|\boldsymbol{t}| = \sqrt{\sum_{j=1}^{L} t_j^2}$.

The following lemmas are provided by Jiang, Dickey, and Kuo (2004). The first two lemmas state the uniqueness and convergence properties. The third lemma states the method which can be used to generate any moment of $\boldsymbol{X}$ when $g(\boldsymbol{t}; \boldsymbol{X}, c)$ is known. The last lemma states that the multivariate c-characteristic function of $G\boldsymbol{X}$, a linear transformation of $\boldsymbol{X}$, can be obtained by $g(\boldsymbol{t}; \boldsymbol{X}, c)$.

Lemma 3.6 *Let $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ be two random vectors with support in a subset of $[-a_1, a_1] \times \cdots \times [-a_L, a_L]$ and set $a = \sqrt{\sum_{j=1}^{L} a_j^2}$. For any $c > 0$, if we have $g(\boldsymbol{t}; \boldsymbol{X}_1, c) = g(\boldsymbol{t}; \boldsymbol{X}_2, c)$ for all $|\boldsymbol{t}| < a^{-1}$, then $\boldsymbol{X}_1 \sim \boldsymbol{X}_2$.*

Lemma 3.7 *Assume that $\boldsymbol{X}$ and $\boldsymbol{X}_1$, $\boldsymbol{X}_2$,... are random vectors with support in a subset of $[-a_1, a_1] \times \cdots \times [-a_L, a_L]$ and set $a = \sqrt{\sum_{j=1}^{L} a_j^2}$. Then, for any $c > 0$, $\boldsymbol{X}_n \overset{d}{\to} \boldsymbol{X}$ if and only if $g(\boldsymbol{t}; \boldsymbol{X}_n, c) \to g(\boldsymbol{t}; \boldsymbol{X}, c)$ for all $|\boldsymbol{t}| < a^{-1}$.*

Lemma 3.8 *Let $\boldsymbol{X} = (X_1, \ldots, X_L)'$ be a random vector. Then the $\boldsymbol{b}$th moment of $\boldsymbol{X}$ can be expressed as*

$$E\left(X_1^{b_1} X_2^{b_2} \cdots X_L^{b_L}\right) = \frac{1}{(c, b_+) i^{b_+}} \left. \frac{\partial^{b_+} g(\boldsymbol{t}; \boldsymbol{X}, c)}{\partial t_1^{b_1} \partial t_2^{b_2} \cdots \partial t_L^{b_L}} \right|_{\boldsymbol{t} = \boldsymbol{0}},$$

where $\boldsymbol{b} = (b_1, \ldots, b_L)'$, $\boldsymbol{0} = (0, \ldots, 0)'$, and the elements of $\boldsymbol{b}$ are nonnegative integers.

Lemma 3.9 *Let $\boldsymbol{X}$ be an L-dimensional random vector. Then*

$$g(\boldsymbol{s}; G\boldsymbol{X}, c) = g(\boldsymbol{t}; \boldsymbol{X}, c),$$

where G is an $M \times L$ matrix of real numbers, $\boldsymbol{s} = (s_1, \ldots, s_M)'$ and $\boldsymbol{t} = G'\boldsymbol{s}$.

Next, we provide important spherical properties of the multivariate c-characteristic function. A random vector has a spherical distribution when all directions are equally probable and the distribution of magnitudes is independent of direction. That is, $\boldsymbol{X}$ has a spherical distribution if $A\boldsymbol{X} \sim \boldsymbol{X}$ for any orthogonal matrix A. Spherical distributions are often derived by projection from other spherical distributions in higher dimensional space. When $g(\boldsymbol{t}; \boldsymbol{X}, c)$ is known, the following theorem is useful in determining whether $\boldsymbol{X}$ has a spherical distribution.

Theorem 3.10 *A random vector $\boldsymbol{X}$ has a spherical distribution if and only if $g(\boldsymbol{t}; \boldsymbol{X}, c)$ is a function of $|\boldsymbol{t}|$ only.*

Proof. Suppose that $\boldsymbol{X}$ is an L-dimensional random vector. When $\boldsymbol{X}$ has a spherical distribution, its PDF $f(\boldsymbol{x})$ is a function of $|\boldsymbol{x}| = r$ only and we can write $f(\boldsymbol{x})$ as $p(r)$. The volume element is $d\boldsymbol{x} = r^{L-1}\, dr\, dS$ where S is the surface of the unit sphere in L-dimensions. Also, $\boldsymbol{x} \cdot \boldsymbol{t} = r|\boldsymbol{t}| \cos\theta$, where θ is the angle between $\boldsymbol{x}$ and $\boldsymbol{t}$. Hence, the multivariate c-characteristic function of $\boldsymbol{X}$ can be expressed as

$$g(\boldsymbol{t}; \boldsymbol{X}, c) = \int r^{L-1} p(r) \int (1 - ir|\boldsymbol{t}| \cos\theta)^{-c}\, dS\, dr.$$

The integration of the inner integral is with respect to S, whose value is independent of the direction of $\boldsymbol{t}$. Hence, the integrand of the outer integral is a function of $r|\boldsymbol{t}|$ only. Therefore, $g(\boldsymbol{t}; \boldsymbol{X}, c)$ is a function of $|\boldsymbol{t}|$ only.

Conversely, let A be any $L \times L$ orthogonal matrix. Then, by Lemma 3.9, we have $g(\boldsymbol{t}; A\boldsymbol{X}, c) = g(A'\boldsymbol{t}; \boldsymbol{X}, c)$. Since $g(\boldsymbol{t}; \boldsymbol{X}, c)$ is a function of $|\boldsymbol{t}|$, we have that $g(A'\boldsymbol{t}; \boldsymbol{X}, c)$ is a function of $|A'\boldsymbol{t}| = |\boldsymbol{t}|$. That is, $A\boldsymbol{X}$ and $\boldsymbol{X}$ have the same multivariate c-characteristic function. By Lemma 3.6, $A\boldsymbol{X} \sim \boldsymbol{X}$ for any orthogonal matrix A. Therefore, $\boldsymbol{X}$ has a spherical distribution. ■

We will call $\boldsymbol{v}$ (k-dimensional vector) a margin of $\boldsymbol{X}$ (L-dimensional vector) and the distribution of $\boldsymbol{v}$ a marginal distribution of the distribution of $\boldsymbol{X}$ if $k \leq L$ and $\boldsymbol{v}$ is a linear projection of $\boldsymbol{X}$, that is, the coordinates of $\boldsymbol{v}$ are a sublist of the coordinates of $D\boldsymbol{X}$, for some matrix D of size $L \times L$, where $DD' = I$ (identity matrix). The following corollary shows that a spherical distribution and its marginal distribution have the same c-characteristic function expression and that the c-characteristic function in lower dimension can be derived easily from that in higher dimension.

Corollary 3.11 *Let $\boldsymbol{X}$ be a random vector having a spherical distribution and $\boldsymbol{Y}$ be any margin of $\boldsymbol{X}$. If $g(\boldsymbol{t}; \boldsymbol{X}, c) = h(|\boldsymbol{t}|)$ for some function h, then $g(\boldsymbol{s}; \boldsymbol{Y}, c) = h(|\boldsymbol{s}|)$.*

Proof. There exists a matrix G such that $\boldsymbol{Y} = G\boldsymbol{X}$ and $GG' = I$ (identity matrix). By Lemma 3.9, we have

$$g(\boldsymbol{s}; \boldsymbol{Y}, c) = g(\boldsymbol{s}; G\boldsymbol{X}, c) = g(G'\boldsymbol{s}; \boldsymbol{X}, c) = h(|G'\boldsymbol{s}|) = h(|\boldsymbol{s}|). \ ■$$

With Theorem 3.10 and Corollary 3.11, we see that any marginal of a spherical distribution is also a spherical distribution. More properties and applications of the multivariate c-characteristic function can be seen in Kuo (2002) and Jiang, Dickey, and Kuo (2004). Further discussion can be seen in Dickey, Jiang, and Kuo (2008).

3.3 Inversion formulas of a univariate c-characteristic function

Knowing the univariate c-characteristic function of a random variable X, it is natural to ask its PDF or CDF expression. In this subsection, we construct inversion formulas of a univariate c-characteristic function to address this problem.

Definition 3.12 *The generalized Stieltjes transform of index $c > 0$ of a measure $d\alpha(x)$ is defined as*

$$F_c(z) = \int_0^\infty \frac{d\alpha(x)}{(z+x)^c}. \tag{3.1}$$

The generalized Stieltjes transform of index $c > 0$ of a function $\phi(x) \in L(0,\infty)$ is defined as

$$F_c(z) = \int_0^\infty \frac{\phi(x)}{(z+x)^c}\, dx. \tag{3.2}$$

Useful references on generalized Stieltjes transform include Zayed (1996, pp. 175–189) and Widder (1946, pp. 325–391).

Let $f(x)$ and $F(x)$ be the PDF and the CDF of a random variable X, respectively. According to Definition 3.1, the univariate c-characteristic function of X can be expressed as

$$g(t; X, c) = \int_{-a}^{a} \frac{f(x)}{(1-itx)^c}\, dx = \int_{-a}^{a} \frac{dF(x)}{(1-itx)^c}.$$

Replacing t by i/z, we then have

$$z^{-c} g(i/z; X, c) = \int_{-a}^{a} \frac{f(x)}{(z+x)^c}\, dx \tag{3.3}$$

$$= \int_{-a}^{a} \frac{dF(x)}{(z+x)^c}. \tag{3.4}$$

Notice that the right hand sides of Eq. (3.3) and Eq. (3.4) are the generalized Stieltjes transforms of $f(x)$ and $dF(x)$, respectively. That is, there is a relation between the generalized Stieltjes transform and the univariate c-characteristic function.

In the next lemma, we state an inversion formula, which is given by Sumner (1949, Theorem 4a), of a generalized Stieltjes transform.

Lemma 3.13 *If $\phi(x) \in L[0,b]$ for all positive b and Eq. (3.2) converges, then*

$$\lim_{\epsilon \to 0^+} \frac{-1}{2\pi i} \int_{C_{\epsilon,x}} (z+x)^{c-1} F_c'(z)\, dz = \frac{\phi(x^+) + \phi(x^-)}{2}$$

for any positive x at which both $\phi(x^+)$ and $\phi(x^-)$ exist, where $C_{\epsilon,x}$ is the contour which starts at the point $-x - i\epsilon$, proceeds along the straight line $Im\, z = -\epsilon$ to the point $-i\epsilon$, then along the semi-circle $|z| = \epsilon$, $Re\, z \geq 0$, to the point $i\epsilon$, and finally along the line $Im\, z = \epsilon$ to the point $-x + i\epsilon$, see Figure 1.

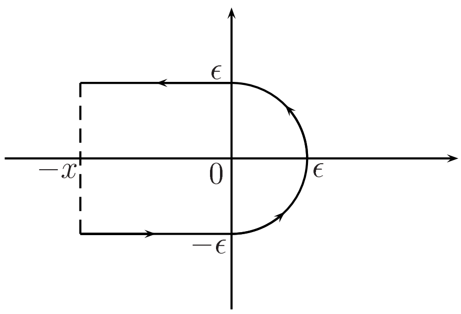

Figure 1: The contour $C_{\epsilon,x}$

Using both Eq. (11) and Theorem 4b of Sumner (1949), we derive the following lemma, which gives another version of inversion formula of a generalized Stieltjes transform.

Lemma 3.14 *If $c > 0$, $\alpha(x)$ is a normalized function of bounded variation in $(0,b)$ for all positive b, and Eq. (3.1) converges, then*

$$\lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{C_{\epsilon,x}} (z+x)^{c-1} F_c(z)\, dz = \frac{\alpha(x^+) + \alpha(x^-)}{2}$$

for any positive x, where $C_{\epsilon,x}$ is defined in Lemma 3.13.

In the following two theorems, we give another versions of inverse formula of a generalized Stieltjes transform for a generalized domain.

Theorem 3.15 *If $a \in \mathbb{R}$, $c > 0$, $f \in L[a,b]$ for all $b > a$, and $h(z) = \int_a^\infty (x+z)^{-c} f(x)\,dx$ converges, then*

$$\frac{f(x^+)+f(x^-)}{2} = \lim_{\epsilon\to 0^+} \frac{-1}{2\pi i}\int_{D_{\epsilon,x}} (z+x)^{c-1} h'(z)\,dz$$

for any $x > a$ at which both $f(x^+)$ and $f(x^-)$ exist, where $D_{\epsilon,x}$ is the contour which starts at the point $-x - i\epsilon$, proceeds along the straight line $Im\, z = -\epsilon$ to the point $-a - i\epsilon$, then along the semi-circle $|z+a| = \epsilon$, $Re\, z \geq -a$, to the point $-a + i\epsilon$, and finally along the line $Im\, z = \epsilon$ to the point $-x + i\epsilon$.

Proof. Consider the following equality:

$$h(z) = \int_a^\infty \frac{f(x)}{(x+z)^c}\,dx = \int_a^\infty \frac{f(x)}{(x-a+z+a)^c}\,dx. \tag{3.5}$$

Let $x_1 = x - a$ and $z_1 = z + a$, Eq. (3.5) can then be rewritten as

$$h(z) = \int_0^\infty \frac{f_1(x_1)}{(x_1+z_1)^c}\,dx_1 \equiv h_1(z_1) \tag{3.6}$$

where $f_1(x_1) = f(x_1 + a) = f(x)$. Since $f(x) \in L[a,b]$, it can be seen that $f_1 \in L[0, b-a]$. By Lemma 3.13 and Eq. (3.6), we have

$$\frac{f_1(x_1^+)+f_1(x_1^-)}{2} = \lim_{\epsilon\to 0^+} \frac{-1}{2\pi i}\int_{C_{\epsilon,x_1}} (z_1+x_1)^{c-1} h_1'(z_1)\,dz_1$$

or

$$\begin{aligned}\frac{f(x^+)+f(x^-)}{2} &= \lim_{\epsilon\to 0^+} \frac{-1}{2\pi i}\int_{C_{\epsilon,x-a}} (z_1+x-a)^{c-1} h_1'(z_1)\,dz_1 \\ &= \lim_{\epsilon\to 0^+} \frac{-1}{2\pi i}\int_{D_{\epsilon,x}} (z+x)^{c-1} h'(z)\,dz. \blacksquare\end{aligned}$$

The proof of the next theorem, which extends Lemma 3.14 to a generalized domain, is similar to that of Theorem 3.15 and we omit it.

Theorem 3.16 *If $a \in \mathbb{R}$, $c > 0$, $\alpha(x)$ is a normalized function of bounded variation in (a,b) for all $b > a$, and $h(z) = \int_a^\infty (x+z)^{-c}\,d\alpha(x)$ converges, then*

$$\frac{\alpha(x^+)+\alpha(x^-)}{2} = \lim_{\epsilon\to 0^+} \frac{1}{2\pi i}\int_{D_{\epsilon,x}} (z+x)^{c-1} h(z)\,dz$$

for any $x > a$, where $D_{\epsilon,x}$ is defined in Theorem 3.15.

Applying Theorems 3.15 and 3.16, we construct our inversion formulas of a univariate c-characteristic function in the following theorem.

Theorem 3.17 *Let X be a continuous random variable on (a,b), $g(t;X,c)$ be the univariate c-characteristic function of X, and $f(x)$ and $F(x)$ be the PDF and the CDF of X, respectively. Then,*

$$f(x) = \lim_{\epsilon\to 0^+} \frac{1}{2\pi i}\left\{\int_a^x \left[(x-y+i\epsilon)^{c-1}G'(z)|_{-y+i\epsilon} - (x-y-i\epsilon)^{c-1}G'(z)|_{-y-i\epsilon}\right] dy - \int_{-\pi/2}^{\pi/2} (x-a+\epsilon e^{i\theta})^{c-1}G'(z)|_{-a+\epsilon e^{i\theta}} i\epsilon e^{i\theta}\, d\theta\right\}, \quad (3.7)$$

$$F(x) = \lim_{\epsilon\to 0^+} \frac{1}{2\pi i}\left\{\int_a^x \left[(x-y-i\epsilon)^{c-1}G(-y-i\epsilon) - (x-y+i\epsilon)^{c-1}G(-y+i\epsilon)\right] dy + \int_{-\pi/2}^{\pi/2} (x-a+\epsilon e^{i\theta})^{c-1}G(-a+\epsilon e^{i\theta}) i\epsilon e^{i\theta}\, d\theta\right\}, \quad (3.8)$$

where $G(z) = z^{-c}g(i/z;X,c)$. In particular, when $c=1$, we have

$$f(x) = \lim_{\epsilon\to 0^+} \frac{1}{2\pi i}\left[\frac{g\left(\frac{i}{-x-i\epsilon};X,1\right)}{-x-i\epsilon} - \frac{g\left(\frac{i}{-x+i\epsilon};X,1\right)}{-x+i\epsilon}\right]. \quad (3.9)$$

Proof. Let

$$G(z) = z^{-c}g(i/z;X,c) = \int_a^b \frac{f(x)}{(z+x)^c}\, dx.$$

By Theorem 3.15, we have

$$f(x) = \lim_{\epsilon\to 0^+} \frac{-1}{2\pi i}\int_{D_{\epsilon,x}} (z+x)^{c-1}G'(z)\, dz$$

for any $x \in (a,b)$. Now, the contour $D_{\epsilon,x}$ can be divided into three parts, say $D_1 = \{z \mid z = y-i\epsilon,\ -x \le y \le -a\}$, $D_2 = \{z \mid z = -a+\epsilon e^{i\theta},\ -\pi/2 \le \theta \le \pi/2\}$, and $D_3 = \{z \mid z = -y+i\epsilon,\ a \le y \le x\}$. Then Eq. (3.7) can be obtained by

$$\begin{aligned}
&\lim_{\epsilon\to 0^+} \frac{-1}{2\pi i}\int_{D_{\epsilon,x}} (z+x)^{c-1}G'(z)\, dz \\
&= \lim_{\epsilon\to 0^+} \frac{-1}{2\pi i}\left\{\int_{-x}^{-a} (y+x-i\epsilon)^{c-1}G'(z)|_{y-i\epsilon}\, dy + \int_a^x (x-y+i\epsilon)^{c-1}G'(z)|_{-y+i\epsilon}(-1)\, dy \right.\\
&\quad \left. + \int_{-\pi/2}^{\pi/2} (-a+x+\epsilon e^{i\theta})^{c-1}G'(z)|_{-a+\epsilon e^{i\theta}} i\epsilon e^{i\theta}\, d\theta\right\} \\
&= \lim_{\epsilon\to 0^+} \frac{1}{2\pi i}\left\{\int_a^x \left((x-y+i\epsilon)^{c-1}G'(z)|_{-y+i\epsilon} - (x-y-i\epsilon)^{c-1}G'(z)|_{-y-i\epsilon}\right) dy \right.\\
&\quad \left. - \int_{-\pi/2}^{\pi/2} (x-a+\epsilon e^{i\theta})^{c-1}G'(z)|_{-a+\epsilon e^{i\theta}} i\epsilon e^{i\theta}\, d\theta\right\}.
\end{aligned}$$

When $c = 1$, by Eq. (3.7), we have

$$\begin{aligned} f(x) &= \lim_{\epsilon\to 0^+} \frac{1}{2\pi i}\left\{\int_a^x \left(G'(z)|_{-y+i\epsilon} - G'(z)|_{-y-i\epsilon}\right) dy - \int_{-\pi/2}^{\pi/2} G'(z)|_{-a+\epsilon e^{i\theta}} i\epsilon e^{i\theta}\, d\theta\right\} \\ &= \lim_{\epsilon\to 0^+} \frac{1}{2\pi i}[-G(-x+i\epsilon)+G(-x-i\epsilon)] \\ &= \lim_{\epsilon\to 0^+} \frac{1}{2\pi i}\left[-\frac{g\left(\frac{i}{-x+i\epsilon};X,1\right)}{-x+i\epsilon} + \frac{g\left(\frac{i}{-x-i\epsilon};X,1\right)}{-x-i\epsilon}\right]. \end{aligned}$$

On the other hand, Eq. (3.8) can be obtained by Theorem 3.16 and by considering

$$G(z) = z^{-c}g(i/z;X,c) = \int_a^b \frac{dF(x)}{(z+x)^c}. \blacksquare$$

Next, we give two examples to demonstrate the use of our inversion formulas (3.7) and (3.8). An application of the inversion formula (3.9) can be seen in Subsections 3.5 and 4.1.

Example 3.1 Consider a random variable X on $(0,1)$ and suppose that

$$g(t;X,c) = \left(\frac{2}{1+\sqrt{1-it}}\right)^{2c}, \text{ for all } |t|<1.$$

Let $G(z) = z^{-c}g(i/z;X,c) = 4^c(2z+1-2z\sqrt{1+1/z})^c$. Then

$$G'(z) = -4^c c\frac{(2z+1-2z\sqrt{1+1/z})^c}{z\sqrt{1+1/z}}.$$

Now, we apply the inversion formula (3.7) to derive $f(x)$, the PDF of X. Notice that

$$\begin{aligned} &\lim_{\epsilon\to 0^+}(x-y+i\epsilon)^{c-1}G'(z)|_{-y+i\epsilon} \\ &= -4^c c\lim_{\epsilon\to 0^+}(x-y+i\epsilon)^{c-1}\frac{(2(-y+i\epsilon)+1-2(-y+i\epsilon)\sqrt{1+1/(-y+i\epsilon)})^c}{(-y+i\epsilon)\sqrt{1+1/(-y+i\epsilon)}} \\ &= -4^c c(x-y)^{c-1}\frac{(-2y+1+2y\sqrt{1/y-1}e^{-i\pi/2})^c}{-y\sqrt{1/y-1}e^{-i\pi/2}} \\ &= -4^c c(x-y)^{c-1}\frac{(1-2y-2i\sqrt{y-y^2})^c}{i\sqrt{y}\sqrt{1-y}} \\ &= \frac{-4^c c(x-y)^{c-1}}{i\sqrt{y}\sqrt{1-y}}e^{ic\arg(1-2y-2i\sqrt{y-y^2})}. \end{aligned}$$

Similarly,

$$\lim_{\epsilon\to 0^+}(x-y-i\epsilon)^{c-1}G'(z)|_{-y-i\epsilon} = \frac{-4^c c(x-y)^{c-1}}{-i\sqrt{y}\sqrt{1-y}}e^{ic\arg(1-2y+2i\sqrt{y-y^2})}.$$

Therefore, the sum of the first two terms on the right hand side of Eq. (3.7) is

$$\begin{aligned}
&\frac{-4^c c}{2\pi i}\int_0^x \frac{(x-y)^{c-1}}{\sqrt{y}\sqrt{1-y}}\left(\frac{e^{ic\arg(1-2y-2i\sqrt{y-y^2})}}{i} - \frac{e^{ic\arg(1-2y+2i\sqrt{y-y^2})}}{-i}\right)dy \\
&= \frac{4^c c}{\pi}\int_0^x (x-y)^{c-1}\frac{\cos(c\arccos(1-2y))}{\sqrt{y}\sqrt{1-y}}\,dy \\
&= \frac{4^c c}{\pi}\int_0^v (\frac{\cos u-\cos v}{2})^{c-1}\cos(cu)\,du \qquad (3.10)\\
&= \frac{2c}{\sqrt{\pi}}\frac{\Gamma(c)}{\Gamma(c+1/2)}\sin^{2c-1}v \qquad (3.11)\\
&= \frac{x^{c-1/2}(1-x)^{c-1/2}}{B(c+1/2,c+1/2)}. \qquad (3.12)
\end{aligned}$$

Identity (3.10) holds by letting $y=\sin^2(u/2)$ and $v=2\arcsin\sqrt{x}$. Identity (3.11) can be obtained from Eqs. (2.15), (2.22), and (2.23). Since $x=\sin^2(v/2)$, we have

$$\sin^{2c-1}v = (2\sin(v/2)\cos(v/2))^{2c-1} = 2^{2c-1}x^{c-1/2}(1-x)^{c-1/2}.$$

Identity (3.12) can be derived by using Eq. (2.20).

The last term of the right hand side of Eq. (3.7) is

$$\begin{aligned}
&\lim_{\epsilon\to 0^+}\int_{-\pi/2}^{\pi/2}(x+\epsilon e^{i\theta})^{c-1}G'(z)|_{\epsilon e^{i\theta}}\,i\epsilon e^{i\theta}\,d\theta \\
&= \lim_{n\to\infty}\int_{-\pi/2}^{\pi/2}(x+\epsilon_n e^{i\theta})^{c-1}G'(z)|_{\epsilon_n e^{i\theta}}\,i\epsilon_n e^{i\theta}\,d\theta \\
&= -4^c c\lim_{n\to\infty}\int_{-\pi/2}^{\pi/2}(x+\epsilon_n e^{i\theta})^{c-1}\frac{(2\epsilon_n e^{i\theta}+1-2\epsilon_n e^{i\theta}\sqrt{1+1/\epsilon_n e^{-i\theta}})^c}{\epsilon_n e^{i\theta}\sqrt{1+1/\epsilon_n e^{-i\theta}}}\,d\theta
\end{aligned}$$

where $0<\epsilon_n<1/2$ and $\epsilon_{n+1}<\epsilon_n$ for all n. Since

$$\begin{aligned}
&\left|(x+\epsilon_n e^{i\theta})^{c-1}\frac{(2\epsilon_n e^{i\theta}+1-2\epsilon_n e^{i\theta}\sqrt{1+1/\epsilon_n e^{-i\theta}})^c}{\epsilon_n e^{i\theta}\sqrt{1+1/\epsilon_n e^{-i\theta}}}\right| \\
&\le (|x+\epsilon_n e^{i\theta}|)^{c-1}\frac{(\sqrt{|\epsilon_n e^{i\theta}+1|}+\sqrt{\epsilon_n})^{2c}}{\sqrt{|\epsilon_n e^{i\theta}+1|}}\sqrt{\epsilon_n} \\
&\le \left(x+\frac{1}{2}\right)^{c-1}\frac{(\sqrt{3/2}+\sqrt{1/2})^{2c}}{1/2}, \qquad (3.13)
\end{aligned}$$

and Eq. (3.13) is integrable on $(-\pi/2, \pi/2)$, we then have, by the Lebesgue Dominated Convergence Theorem,

$$\begin{aligned}
&-4^c c \lim_{n\to\infty} \int_{-\pi/2}^{\pi/2} (x+\epsilon_n e^{i\theta})^{c-1} \frac{(2\epsilon_n e^{i\theta} + 1 - 2\epsilon_n e^{i\theta}\sqrt{1+1/\epsilon_n e^{-i\theta}})^c}{\epsilon_n e^{i\theta}\sqrt{1+1/\epsilon_n e^{-i\theta}}}\, d\theta \\
&= -4^c c \int_{-\pi/2}^{\pi/2} \lim_{n\to\infty} (x+\epsilon_n e^{i\theta})^{c-1} \frac{(2\epsilon_n e^{i\theta} + 1 - 2\epsilon_n e^{i\theta}\sqrt{1+1/\epsilon_n e^{-i\theta}})^c}{\epsilon_n e^{i\theta}\sqrt{1+1/\epsilon_n e^{-i\theta}}}\, d\theta \\
&= 0.
\end{aligned}$$

Therefore, by the formula (3.7),

$$f(x) = \frac{x^{c-1/2}(1-x)^{c-1/2}}{B(c+1/2, c+1/2)}.$$

That is, X has the beta distribution with parameters $c+1/2$ and $c+1/2$. ■

Example 3.2 Consider a random variable X on $(-1,1)$ and suppose that

$$g(t; X, c) = \left[\frac{2}{1+\sqrt{1+t^2}}\right]^c.$$

Now, let $G(z) = z^{-c} g(i/z; X, c) = 2^c z^{-c}(1+\sqrt{1-z^{-2}})^{-c}$. We apply the inversion formula (3.8) to obtain $f(x)$, the PDF of X. Notice that

$$\begin{aligned}
&\lim_{\epsilon\to 0^+} (x-y-i\epsilon)^{c-1} G(-y-i\epsilon) \\
&= 2^c \lim_{\epsilon\to 0^+} (x-y-i\epsilon)^{c-1}(-y-i\epsilon)^{-c}(1+\sqrt{1-(-y-i\epsilon)^{-2}})^{-c} \\
&= \begin{cases} 2^c(x-y)^{c-1} y^{-c} e^{i(-c)(-\pi)}(1+\sqrt{y^{-2}-1}e^{i\pi/2})^{-c}, & \text{if } y>0, \\ 2^c(x-y)^{c-1}(-y)^{-c} e^{i(-c)0}(1+\sqrt{y^{-2}-1}e^{-i\pi/2})^{-c}, & \text{if } y<0, \end{cases} \\
&= 2^c(x-y)^{c-1}(-y-i\sqrt{1-y^2})^{-c} \\
&= 2^c(x-y)^{c-1} e^{i(-c)\arg(-y-i\sqrt{1-y^2})}.
\end{aligned}$$

Similarly,

$$\lim_{\epsilon\to 0^+} (x-y+i\epsilon)^{c-1} G(-y+i\epsilon) = 2^c(x-y)^{c-1} e^{i(-c)\arg(-y+i\sqrt{1-y^2})}.$$

Therefore, the sum of the first two terms on the right hand side of Eq. (3.8) is

$$\begin{aligned}
&\frac{2^c}{2\pi i}\int_{-1}^{x}(x-y)^{c-1}\left(e^{i(-c)\arg(-y-i\sqrt{1-y^2})}-e^{i(-c)\arg(-y+i\sqrt{1-y^2})}\right)dy \\
&= \frac{2^c}{\pi}\int_{-1}^{x}(x-y)^{c-1}\sin(c\arccos(-y))\,dy \qquad (3.14)\\
&= \frac{2^c}{\pi}\int_{0}^{\arccos(-x)}(x+\cos u)^{c-1}\sin(cu)\sin u\,du \text{ (by letting } u=\arccos(-y))\\
&= \frac{2^c}{\pi}\int_{0}^{\arccos(-x)}(x+\cos u)^{c}\cos(cu)\,du \qquad (3.15)\\
&= \frac{2^c}{\pi}\int_{0}^{v}(-\cos v+\cos u)^{c}\cos(cu)\,du \text{ (by letting } v=\arccos(-x))\\
&= \frac{2^{2c}}{B(1/2,c+1/2)}\int_{0}^{(1-\cos v)/2}t^{c-1/2}(1-t)^{c-1/2}\,dt \qquad (3.16)\\
&= \frac{2^{2c}}{B(1/2,c+1/2)}\int_{0}^{(1+x)/2}t^{c-1/2}(1-t)^{c-1/2}\,dt. \qquad (3.17)
\end{aligned}$$

Identity (3.14) is derived since

$$\begin{aligned}
&e^{i(-c)\arg(-y-i\sqrt{1-y^2})}-e^{i(-c)\arg(-y+i\sqrt{1-y^2})}\\
&= \begin{cases} e^{i(-c)(-\arccos(-y))}-e^{i(-c)(\arccos(-y))}, & \text{if } y>0,\\ e^{i(-c)(-\arccos(-y))}-e^{i(-c)(\arccos(-y))}, & \text{if } y<0,\end{cases}\\
&= e^{ic\arccos(-y)}-e^{-ic\arccos(-y)}\\
&= 2i\sin(c\arccos(-y)).
\end{aligned}$$

Identity (3.15) can be obtained using integration by parts. Identity (3.16) follows from Eqs (2.15), (2.22), and (2.21). Identity (3.17) holds as $v=\arccos(-x)$.

The last term of the right hand side of Eq. (3.8) is

$$\begin{aligned}
&\lim_{\epsilon\to 0^+}\int_{-\pi/2}^{\pi/2}(x+1+\epsilon e^{i\theta})^{c-1}G(1+\epsilon e^{i\theta})i\epsilon e^{i\theta}\,d\theta\\
&= 2^c\lim_{n\to\infty}\int_{-\pi/2}^{\pi/2}(x+1+\epsilon e^{i\theta})^{c-1}(1+\epsilon e^{i\theta})^{-c}(1+\sqrt{1-(1+\epsilon e^{i\theta})^{-2}})^{-c}\,d\theta\\
&= 0.
\end{aligned}$$

The last identity can be determined by using the Lebesgue Dominated Convergence Theorem. Therefore, the PDF of X is

$$f(x)=F'(x)=\frac{(1-x^2)^{c-1/2}}{B(1/2,c+1/2)},\ \ -1<x<1. \blacksquare$$

Theorem 3.17 provides useful inversion formulas of a univariate c-characteristic function in determining the explicit form of PDF and CDF for any random variable when its univariate c-characteristic function is known.

Further discussion can be seen in Jiang and Kuo (2008b).

In the next section, we will apply our inversion formulas to study the distribution of $\xi_\mu(h)$, which is defined in Eq. (1.1). In the following subsection, we give a relation between the univariate c-characteristic function and the Fourier transformation. Using this relation, we provide a different method to obtain the PDF of a random variable when its univariate c-characteristic function is specified.

3.4 Density construction through Fourier transformation

Chung (1974, p. 178, exercise 5) stated that for any probability measure whose support is in a finite interval, then its distribution function is uniquely determined by its moments of all orders. By Lemma 3.4, we have

$$\phi(t) = \sum_{n=0}^{\infty} \frac{i^n t^n}{n!} E(X^n) = \sum_{n=0}^{\infty} \frac{t^n}{n!(c,n)} \frac{d^n}{dt^n} g(t; X, c)\bigg|_{t=0},$$

where $\phi(t)$ denotes the traditional characteristic function (the Fourier transformation). Following the well-known inversion formula of $\phi(t)$, we have the next theorem.

Theorem 3.18 *Let $g(t; X, c)$ be the univariate c-characteristic function of X, then the PDF of X can be expressed as*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \sum_{n=0}^{\infty} \frac{t^n}{n!(c,n)} \frac{d^n}{dt^n} g(t; X, c)\bigg|_{t=0} dt.$$

Example 3.3 Consider the univariate c-characteristic function in Example 3.2. By Eq. (2.30), we have

$$g(t; X, c) = {}_2F_1(c/2, (c+1)/2; c+1; -t^2) = \sum_{k=0}^{\infty} \frac{(c/2, k)((c+1)/2, k)}{k!(c+1, k)} (-t^2)^k.$$

Notice that

$$\left.\frac{d^n}{dt^n}g(t;X,c)\right|_{t=0} = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \dfrac{n!(c/2,n/2)((c+1)/2,n/2)(-1)^{n/2}}{(n/2)!(c+1,n/2)}, & \text{if } n \text{ is even.} \end{cases}$$

By Theorem 3.18, the PDF of X can be expressed as

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!(c,2n)}\frac{(2n)!(c/2,n)((c+1)/2,n)(-1)^n}{(n)!(c+1,n)}\,dt \\ &= \frac{\Gamma(c+1)}{\pi}\int_0^{\infty}\cos(tx)\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{2^{2n}n!\Gamma(c+1+n)}\,dt \\ &= \frac{2^c\Gamma(c+1)}{\pi}\int_0^{\infty}\cos(xt)\frac{J_c(t)}{t^c}\,dt \\ &= \frac{1}{B(c+1/2,1/2)}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2}-c;\frac{1}{2};x^2\right) \\ &= \frac{(1-x^2)^{c-1/2}}{B(c+1/2,1/2)}. \end{aligned}$$

The second identity can be obtained by Eqs. (2.26) and (2.27). The third identity follows Eq. (2.2). The fourth identity is derived by Eq. (2.18). The last identity can be obtained by Eq. (2.23). ■

3.5 Distributions of linear combinations of the components of a Dirichlet random vector

In the following lemma, which is given by Jiang, Dickey, and Kuo (2004), we have the multivariate c-characteristic function expression of a Dirichlet distribution.

Lemma 3.19 *Suppose that* $\boldsymbol{X} \sim Dir(\boldsymbol{b})$ *where* $\boldsymbol{b}' = (b_1, \ldots, b_L)$. *Then*

$$g(\boldsymbol{t};\boldsymbol{X},b_+) = \prod_{j=1}^{L}(1-it_j)^{-b_j},$$

where $|\boldsymbol{t}| < 1$.

The following corollary can be obtained by Lemmas 3.9 and 3.19.

Corollary 3.20 *Suppose that* $\boldsymbol{X} = (X_1, \ldots, X_L)' \sim Dir(\boldsymbol{b})$ *and let* $Y = \sum_{j=1}^{L} a_j X_j$. *Then*

$$g(t; Y, b_+) = \prod_{j=1}^{L} (1 - ita_j)^{-b_j}.$$

Theorem 3.21 *Suppose that* $\boldsymbol{X} = (X_1, \ldots, X_L)' \sim Dir(\boldsymbol{b})$ *where* $\boldsymbol{b} = (b_1, \ldots, b_L)'$ *with* $b_+ = 1$. *Define a random variable* $X = \sum_{j=1}^{L} a_j X_j$ *for some real numbers* $a_1, \ldots, a_L$ *with* $a_1 < a_2 < \cdots < a_L$. *Then the PDF of* X *can be expressed as*

$$f(x) = \frac{1}{\pi}\left(\prod_{j=1}^{L} |x - a_j|^{-b_j}\right) \sin\left(\pi \sum_{k=1}^{m} b_k\right), \ x \in (a_m, a_{m+1}), \ 1 \le m \le L - 1.$$

Proof. By Corollary 3.20 and using the inversion formula (3.9), we have

$$\begin{aligned}
f(x) &= \lim_{\epsilon \to 0^+} \frac{1}{2\pi i}\left[\frac{\prod_{j=1}^{L}(1 + \frac{a_j}{-x-i\epsilon})^{-b_j}}{-x - i\epsilon} - \frac{\prod_{j=1}^{L}(1 + \frac{a_j}{-x+i\epsilon})^{-b_j}}{-x + i\epsilon}\right] \\
&= \lim_{\epsilon \to 0^+} \frac{1}{2\pi i}\left[\prod_{j=1}^{L}(a_j - x - i\epsilon)^{-b_j} - \prod_{j=1}^{L}(a_j - x + i\epsilon)^{-b_j}\right] \\
&= \frac{1}{2\pi i}\left[\prod_{j=1}^{m}(x - a_j)^{-b_j} e^{ib_j\pi} \prod_{j=m+1}^{L}(a_j - x)^{-b_j} - \prod_{j=1}^{m}(x - a_j)^{-b_j} e^{-ib_j\pi} \prod_{j=m+1}^{L}(a_j - x)^{-b_j}\right] \\
&= \frac{1}{2\pi i}\prod_{j=1}^{m}(x - a_j)^{-b_j} \prod_{j=m+1}^{L}(a_j - x)^{-b_j}\left(e^{i\pi(b_1 + \cdots + b_m)} - e^{-i\pi(b_1 + \cdots + b_m)}\right) \\
&= \frac{1}{\pi}\left(\prod_{j=1}^{L} |x - a_j|^{-b_j}\right) \sin\left(\pi \sum_{k=1}^{m} b_k\right), \ x \in (a_m, a_{m+1}), \ 1 \le m \le L - 1. \ \blacksquare
\end{aligned}$$

In the following corollary, we remove the restriction $a_1 < \cdots < a_L$.

Corollary 3.22 *Suppose that* $\boldsymbol{u} \sim D(\boldsymbol{b})$, *where* $\sum_{j=1}^{L} b_j = 1$. *Define* $V = \sum_{j=1}^{L} a_j u_j$ *for some real numbers* $a_1, \ldots, a_L$. *Set* $\{d_1, \ldots, d_k\} = \{a_1, \ldots, a_L\}$ *with* $d_1 < d_2 < \cdots < d_k$ *and* $d_j = a_{j_1} = a_{j_2} = \cdots = a_{j_{n_j}}$ *where* $\{j_1, j_2, \ldots, j_{n_j}\} \subseteq \{1, 2, \ldots, L\}$ *for* $1 \le j \le k$, *and* $b_j^* = \sum_{n=j_1}^{j_{n_j}} b_n$. *Then the PDF of* V *can be expressed as*

$$f(v) = \frac{1}{\pi}\left(\prod_{j=1}^{k} |v - d_j|^{-b_j^*}\right) \sin\left(\pi \sum_{j=1}^{n} b_j^*\right),$$

where $v \in (d_n, d_{n+1})$ *and* $1 \le n \le k - 1$.

Next, we apply Lemma 3.4 to give the moment representation of X.

Theorem 3.23 *Under the assumptions of Theorem 3.21, the nth moment $E(X^n)$ is recursively by*

$$E(X^n) = \frac{1}{n}\sum_{k=0}^{n-1}\left(\sum_{j=1}^{L} b_j a_j^{n-k}\right) E(X^k),$$

where $E(X^0) \equiv 1$. In particular,

$$E(X) = \sum_{j=1}^{L} b_j a_j \text{ and } Var(X) = \frac{\sum_{j=1}^{L} b_j a_j^2 - (\sum_{j=1}^{L} b_j a_j)^2}{2}.$$

Proof. First, we claim that the nth derivative of $g(t; X, 1)$ with respect to variable t can be expressed as

$$g^{(n)}(t; X, 1) = \sum_{k=0}^{n-1}\binom{n-1}{k}\left(\sum_{j=1}^{L}\frac{(n-k-1)!b_j(ia_j)^{n-k}}{(1-ita_j)^{n-k}}\right) g^{(k)}(t; X, 1), \qquad (3.18)$$

for all positive integer n, where $g^{(0)}(t; X, 1) \equiv g(t; X, 1)$. We use the mathematical induction to show that Eq. (3.18) is true.

For $n = 1$,

$$\begin{aligned} g^{(1)}(t; X, 1) &= \frac{d}{dt}\exp\left(\sum_{j=1}^{L}(-b_j)\ln(1-ita_j)\right) \\ &= \exp\left(\sum_{j=1}^{L}(-b_j)\ln(1-ita_j)\right)\left(\sum_{j=1}^{L}\frac{(-b_j)(-ia_j)}{1-ita_j}\right) \\ &= g^{(0)}(t; X, 1)\sum_{j=1}^{L}\frac{b_j(ia_j)}{1-ita_j}. \end{aligned}$$

Hence, Eq. (3.18) holds for $n = 1$. Assume that Eq. (3.18) holds for $n = m$, i.e.,

$$g^{(m)}(t; X, 1) = \sum_{k=0}^{m-1}\binom{m-1}{k}\left(\sum_{j=1}^{L}\frac{(m-k-1)!b_j(ia_j)^{m-k}}{(1-ita_j)^{m-k}}\right) g^{(k)}(t; X, 1).$$

Let $n = m+1$, we have

$$
\begin{aligned}
& g^{(m+1)}(t;X,1) \\
= & \frac{d}{dt} g^{(m)}(t;X,1) \\
= & \sum_{k=0}^{m-1} \binom{m-1}{k} \left(\sum_{j=1}^{L} \frac{(m-k)! b_j (ia_j)^{m-k+1}}{(1-ita_j)^{m-k+1}} \right) g^{(k)}(t;X,1) \\
& + \sum_{k=0}^{m-1} \binom{m-1}{k} \left(\sum_{j=1}^{L} \frac{(m-k-1)! b_j (ia_j)^{m-k}}{(1-ita_j)^{m-k}} \right) g^{(k+1)}(t;X,1) \\
= & \left(\sum_{j=1}^{L} \frac{m! b_j (ia_j)^{m+1}}{(1-ita_j)^{m+1}} \right) g^{(0)}(t;X,1) \\
& + \sum_{k=1}^{m-1} \binom{m-1}{k} \left(\sum_{j=1}^{L} \frac{(m-k)! b_j (ia_j)^{m-k+1}}{(1-ita_j)^{m-k+1}} \right) g^{(k)}(t;X,1) \\
& + \sum_{k=0}^{m-2} \binom{m-1}{k} \left(\sum_{j=1}^{L} \frac{(m-k-1)! b_j (ia_j)^{m-k}}{(1-ita_j)^{m-k}} \right) g^{(k+1)}(t;X,1) \\
& + \left(\sum_{j=1}^{L} \frac{b_j (ia_j)}{1-ita_j} \right) g^{(m)}(t;X,1) \\
= & \left(\sum_{j=1}^{L} \frac{m! b_j (ia_j)^{m+1}}{(1-ita_j)^{m+1}} \right) g^{(0)}(t;X,1) \\
& + \sum_{k=1}^{m-1} \binom{m-1}{k} \left(\sum_{j=1}^{L} \frac{(m-k)! b_j (ia_j)^{m-k+1}}{(1-ita_j)^{m-k+1}} \right) g^{(k)}(t;X,1) \\
& + \sum_{k=1}^{m-1} \binom{m-1}{k-1} \left(\sum_{j=1}^{L} \frac{(m-k)! b_j (ia_j)^{m-k+1}}{(1-ita_j)^{m-k+1}} \right) g^{(k)}(t;X,1) \\
& + \left(\sum_{j=1}^{L} \frac{b_j (ia_j)}{1-ita_j} \right) g^{(m)}(t;X,1) \\
= & \left(\sum_{j=1}^{L} \frac{m! b_j (ia_j)^{m+1}}{(1-ita_j)^{m+1}} \right) g^{(0)}(t;X,1) \\
& + \sum_{k=1}^{m-1} \binom{m}{k} \left(\sum_{j=1}^{L} \frac{(m-k)! b_j (ia_j)^{m-k+1}}{(1-ita_j)^{m-k+1}} \right) g^{(k)}(t;X,1) \\
& + \left(\sum_{j=1}^{L} \frac{b_j (ia_j)}{1-ita_j} \right) g^{(m)}(t;X,1) \\
= & \sum_{k=0}^{m} \binom{m}{k} \left(\sum_{j=1}^{L} \frac{(m-k)! b_j (ia_j)^{m-k+1}}{(1-ita_j)^{m-k+1}} \right) g^{(k)}(t;X,1).
\end{aligned}
$$

The fifth identity is obtained by $\binom{m-1}{k} + \binom{m-1}{k-1} = \binom{m}{k}$. Therefore, Eq. (3.18) holds for all n.

Next, by Lemma 3.4, the nth moment of X is

$$\begin{aligned}
E(X^n) &= \frac{1}{i^n n!} g^{(n)}(0; X, 1) \\
&= \frac{1}{i^n n!} \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\sum_{j=1}^{L} (n-k-1)! b_j (ia_j)^{n-k} \right) g^{(k)}(0; X, 1) \\
&= \frac{1}{i^n n!} \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-k-1)!} \left(\sum_{j=1}^{L} (n-k-1)! b_j (ia_j)^{n-k+1} \right) \left(i^k k! E(X^k) \right) \\
&= \frac{1}{n} \sum_{k=0}^{n-1} \left(\sum_{j=1}^{L} b_j a_j^{n-k} \right) E(X^k). \blacksquare
\end{aligned}$$

Corollary 3.24 *Under the assumptions of Corollary 3.22, the nth moment* $E(V^n)$ *is recursively by*

$$E(V^n) = \frac{1}{n} \sum_{m=0}^{n-1} \left(\sum_{j=1}^{k} b_j^* d_j^{n-m} \right) E(V^m),$$

where $E(V^0) \equiv 1$*. In particular,*

$$E(V) = \sum_{j=1}^{k} b_j^* d_j \text{ and } Var(V) = \frac{\sum_{j=1}^{k} b_j^* d_j^2 - (\sum_{j=1}^{k} b_j^* d_j)^2}{2}.$$

For general $b_+ > 0$, we may use the inversion formula (3.8) or (3.9) to address the problem. In the following, we consider three special cases, say $W_k = \sum_{j=1}^{k} a_j X_j$ for $k = 2, 3, 4$, where $(X_1, \ldots, X_k) \sim Dir(1/2, \ldots, 1/2)$ and $a_1 < a_2 < \cdots < a_k$. The case of $k = 2$ has been addressed before. The PDF can be expressed as

$$\frac{1}{\pi} \left(\prod_{j=1}^{2} |w - a_j|^{-1/2} \right) \sin \left(\pi \sum_{k=1}^{2} \frac{1}{2} \right) = \frac{1}{\pi \sqrt{(w - a_1)(a_2 - w)}}, \ w \in (a_1, a_2).$$

By Corollary (3.20), we have

$$g(t; W_3, 3/2) = (1 - ita_1)^{-1/2} (1 - ita_2)^{-1/2} (1 - ita_3)^{-1/2}.$$

In addition,

$$G(z) = z^{-3/2} g(i/z; Y_3, 3/2) = (a_1 + z)^{-1/2} (a_2 + z)^{-1/2} (a_3 + z)^{-1/2}.$$

Using the inversion formula (3.9), the CDF of W_3 is given by

$$F(w) = \begin{cases} \displaystyle\int_{a_1}^{w} (1/\pi)(w-y)^{1/2}(y-a_1)^{-1/2}(a_2-y)^{-1/2}(a_3-y)^{-1/2}\,dy, & \text{for } a_1 < w < a_2, \\ \displaystyle\int_{a_1}^{a_2} (1/\pi)(w-y)^{1/2}(y-a_1)^{-1/2}(a_2-y)^{-1/2}(a_3-y)^{-1/2}\,dy, & \text{for } a_2 < w < a_3. \end{cases}$$

Hence, the PDF of W_3 can be expressed as

$$\begin{aligned} f(w) &= \begin{cases} \displaystyle\int_{a_1}^{w} (2\pi)^{-1}(w-y)^{-1/2}(y-a_1)^{-1/2}(a_2-y)^{-1/2}(a_3-y)^{-1/2}\,dy \\ \displaystyle\int_{a_1}^{a_2} (2\pi)^{-1}(w-y)^{-1/2}(y-a_1)^{-1/2}(a_2-y)^{-1/2}(a_3-y)^{-1/2}\,dy \end{cases} \\ &= \begin{cases} \dfrac{1}{\pi\sqrt{(a_3-w)(a_2-a_1)}} F\left(\dfrac{\pi}{2}, \sqrt{\dfrac{(a_3-a_2)(w-a_1)}{(a_3-w)(a_2-a_1)}}\right), & \text{for } a_1 < w < a_2, \\ \dfrac{1}{\pi\sqrt{(w-a_1)(a_3-a_2)}} F\left(\dfrac{\pi}{2}, \sqrt{\dfrac{(a_3-w)(a_2-a_1)}{(w-a_1)(a_3-a_2)}}\right), & \text{for } a_2 < w < a_3, \end{cases} \end{aligned}$$

where $F(\cdot,\cdot)$ denotes the elliptic integral of the first kind, see (Gradshteyn and Ryzhik, 2000, pp. 851-852). Both the two expressions in last identity above can be obtained by (Gradshteyn and Ryzhik, 2000, formula 3.147.2, p. 272).

Following the similar procedure above and using (Gradshteyn and Ryzhik, 2000, formulas 3.147.2 and 3.147.6, p. 272), we derive the PDF of W_4 as

$$f(w) = \begin{cases} \dfrac{2}{\pi\sqrt{(a_3-a_1)(a_4-a_2)}} F\left(\arcsin\sqrt{\dfrac{(w-a_1)(a_4-a_2)}{(a_4-w)(a_2-a_1)}}, \sqrt{\dfrac{(a_4-a_3)(a_2-a_1)}{(a_4-a_2)(a_3-a_1)}}\right), \\ \qquad \text{for } a_1 < w < a_2, \\ \dfrac{2}{\pi\sqrt{(a_3-a_1)(a_4-a_2)}} F\left(\dfrac{\pi}{2}, \sqrt{\dfrac{(a_4-a_3)(a_2-a_1)}{(a_4-a_2)(a_3-a_1)}}\right), \quad \text{for } a_2 < w < a_3, \\ \dfrac{2}{\pi\sqrt{(a_3-a_1)(a_4-a_2)}} F\left(\dfrac{\pi}{2}, \sqrt{\dfrac{(a_4-a_3)(a_2-a_1)}{(a_4-a_2)(a_3-a_1)}}\right) \\ \quad -\dfrac{2}{\pi\sqrt{(a_3-a_1)(a_4-a_2)}} F\left(\arcsin\sqrt{\dfrac{(w-a_3)(a_4-a_2)}{(w-a_2)(a_4-a_3)}}, \sqrt{\dfrac{(a_4-a_3)(a_2-a_1)}{(a_4-a_2)(a_3-a_1)}}\right), \\ \qquad \text{for } a_3 < w < a_4. \end{cases}$$