第三章　生成函數的應用

以往我們解決生成函數的問題[2]都是利用微分

例如：求數列$0^2,1^2,2^2,\cdots\cdots$之生成函數

我們的做法是利用

$$\frac{1}{1-x}=1+x+x^2+\cdots\cdots+x^n+\cdots\cdots$$

兩邊同時對 x 微分　$\therefore$得到　$\frac{1}{(1-x)^2}=1+2x+\cdots\cdots+nx^{n-1}+\cdots\cdots$

兩邊同乘以 x　$\frac{x}{(1-x)^2}=x+2x^2+\cdots\cdots+nx^n+\cdots\cdots$

兩邊同時對 x 微分　$\frac{1+x}{(1-x)^3}=\sum_{i=1}^{\infty}i^2x^{i-1}$

在兩邊同乘 x　$\frac{x+x^2}{(1-x)^3}=\sum_{i=1}^{\infty}i^2x^i=\sum_{i=0}^{\infty}i^2x^i$

$\therefore$如果要求數列$0^k,1^k,2^k,\ldots\ldots$之生成函數必須以同樣的步驟做了 k 次可能會是一個很大的工程，但是我們可以利用 $n^k=b_{k,0}C_0^n+b_{k,1}C_1^n+b_{k,2}C_2^n+\cdots\cdots+b_{k,k}C_k^n$ 較簡便的算出數列$0^k,1^k,2^k,\ldots\ldots$之生成函數

再回來剛剛的例子，我們可以利用到$n^2=C_1^n+2C_2^n=C_{n-1}^n+2C_{n-2}^n=H_{n-1}^2+2H_{n-2}^3,n\geq 2$

$$\because\frac{1}{1-x}=1+x+\cdots\cdots+x^n+\cdots\cdots=\sum_{n\geq 0}x^n$$

$$\frac{1}{(1-\mathrm{x})^{\mathrm{m}}}=\sum_{n\geq 0}H_n^m x^n$$

$$\therefore \sum_{n\ge 0} n^2 x^n = \sum_{n\ge 0}(C_1^n + 2C_2^n)x^n$$
$$= \sum_{n\ge 0} C_1^n x^n + 2\sum_{n\ge 0} C_2^n x^n$$
$$= \sum_{n\ge 1} H_{n-1}^2 x^n + 2\sum_{n\ge 2} H_{n-2}^3 x^n$$
$$= x\sum_{n\ge 1} H_{n-1}^2 x^{n-1} + 2x^2 \sum_{n\ge 2} H_{n-2}^3 x^{n-2}$$
$$= x\sum_{l\ge 0} H_l^2 x^l + 2x^2 \sum_{l\ge 0} H_l^3 x^l$$
$$= \frac{x}{(1-x)^2} + \frac{2x^2}{(1-x)^3} = \frac{x+x^2}{(1-x)^3}$$

定理 5：數列$0^k, 1^k, 2^k, \ldots\ldots$的生成函數爲

$$b_{k,1}\cdot\frac{x}{(1-x)^2} + b_{k,2}\cdot\frac{x^2}{(1-x)^3} + \cdots\cdots + b_{k,k-1}\cdot\frac{x^{k-1}}{(1-x)^k} + b_{k,k}\cdot\frac{x^k}{(1-x)^{k+1}}$$

證明：

$$\because\ n^k = b_{k,1}C_1^n + b_{k,2}C_2^n + \cdots\cdots + b_{k,k}C_k^n$$
$$= b_{k,1}H_{n-1}^2 + b_{k,2}H_{n-2}^3 + \cdots\cdots + b_{k,k}H_{n-k}^{k+1} \qquad (\ C_k^n = C_{n-k}^n = H_{n-k}^{k+1})$$

$$\therefore \sum_{n\ge 0} n^k x^n = \sum_{n\ge 0}(b_{k,1}C_1^n + b_{k,2}C_2^n + \cdots\cdots + b_{k,k}C_k^n)x^n$$
$$= \sum_{n\ge 1} b_{k,1}H_{n-1}^2 x^n + \sum_{n\ge 2} b_{k,2}H_{n-2}^3 x^n + \cdots\cdots + \sum_{n\ge k-1} b_{k,k-1}H_{n-(k-1)}^k x^n + \sum_{n\ge k} b_{k,k}H_{n-k}^{k+1}x^n$$
$$= x\cdot\sum_{n\ge 1} b_{k,1}H_{n-1}^2 x^{n-1} + x^2\cdot\sum_{n\ge 2} b_{k,2}H_{n-2}^3 + \cdots\cdots + x^{k-1}\cdot\sum_{n\ge k-1} b_{k,k-1}H_{n-(k-1)}^k x^{n-(k-1)}$$
$$+ x^k\cdot\sum_{n\ge k} b_{k,k}H_{n-k}^{k+1}x^{n-k}$$
$$= (b_{k,1})\cdot x\sum_{l\ge 0} H_l^2 x^l + (b_{k,2})x^2\sum_{l\ge 0} H_l^3 x^l + \cdots\cdots + (b_{k,k-1})x^{k-1}\sum_{l\ge 0} H_l^k x^l + (b_{k,k})x^k\sum_{l\ge 0} H_l^{k+1}x^l$$
$$= b_{k,1}\cdot\frac{x}{(1-x)^2} + b_{k,2}\cdot\frac{x^2}{(1-x)^3} + \cdots\cdots + b_{k,k-1}\cdot\frac{x^{k-1}}{(1-x)^k} + b_{k,k}\cdot\frac{x^k}{(1-x)^{k+1}}$$

例如：如果我們要求數列$0, 0, 0, 6, 24, \cdots, n(n-1)(n-2).$之生成函數

$$\because a_n = n(n-1)(n-2) = C_3^n\cdot 3! = 6C_3^n$$

$$\therefore A(x) = \sum_{n\ge 0} a_n x^n = 6x^3\cdot\frac{1}{(1-x)^4}$$