2 Maximum gaps of mixed hypergraphs with n vertices

Definition 2.1 *We say that a mixed hypergraph, $\mathcal{H}$, is spanned by a simple graph, G, if $\mathcal{H}$ and G have the same vertex set, and each $\mathcal{C}$-edge and $\mathcal{D}$-edge of $\mathcal{H}$ is a connected subgraph of G.*

Observation 2.2 *A mixed hypergraph with n vertices is spanned by the complete graph, K_n.*

Definition 2.3 *A mixed hypergraph, $\mathcal{H}$, is good if it has no gap, and we also say that it is gap-free. Otherwise, $\mathcal{H}$ is bad. If each mixed hypergraph spanned by a simple graph, G, is good, then G is good. Otherwise, G is bad.*

We know that K_5 is the only connected non-planar graph which is good, [3]. Hence, for any $n \geq 6$, K_n is bad. In order to find the maximum gap of a mixed hypergraph with n vertices, we have the following lemma:

Lemma 2.4 *Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where $X = \{x_1, x_2, \ldots, x_n\}$. If $\bar{\chi}(\mathcal{H}) \geq n-1$, then $\mathcal{H}$ is good.*

Proof. Since $\bar{\chi}(\mathcal{H}) \leq n$, we have two cases:

First, if $\bar{\chi}(\mathcal{H}) = n$, then $\mathcal{C} = \phi$. Let c_1 be a strict k-coloring of $\mathcal{H}$, $k < n$, and $\{X_1, X_2, \ldots, X_k\}$ be the feasible partition with respect to c_1. We may assume that $|X_1| > 1$ and $x_1 \in X_1$, then we define $Y_1 = X_1 - \{x_1\}$, $Y_i = X_i$ for $2 \leq i \leq k$, $Y_{k+1} = \{x_1\}$, then we plan to show that $\{Y_1, Y_2, \ldots, Y_{k+1}\}$ is another feasible partition of X with respect to a coloring, say c_1'.

If c_1' is not a strict coloring, then there exist a $C \in \mathcal{C}$ which is polychromatic or a $D \in \mathcal{D}$ which is monochromatic. Since $\mathcal{C} = \phi$, we only consider the other situation. If there is a D-edge which the vertices in it get the same Y_i, then they get the same X_i, such implies c_1 is not a proper coloring of $\mathcal{H}$. This contradiction implies that for any $k \in S(\mathcal{H})$, $k < n$, then $k+1 \in S(\mathcal{H})$. Hence, $\mathcal{H}$ is no gap.

Second, we consider the case $\bar{\chi}(\mathcal{H}) = n-1$. Let c_2 be the strict $(n-1)$-coloring of $\mathcal{H}$. Suppose $c_2(x_i) = i$, for $1 \le i \le n-1$, and $c_2(x_n) = n-1$, then for all $C \in \mathcal{C}$, we get $x_{n-1}, x_n \in C$. Let $\mathcal{H}' = (X', \mathcal{C}', \mathcal{D}')$, where $X' = X - \{x_1, x_2\}$, $\mathcal{C}' = \{C \in \mathcal{C} \mid x_1, x_2 \notin C\}$, $\mathcal{D}' = \{D \in \mathcal{D} \mid x_1, x_2 \notin D\}$, then $\mathcal{C}' = \phi$. By the same arguments with case 1, $S(\mathcal{H}')$ has no gap and $\bar{\chi}(\mathcal{H}') = n-2$. Set $\chi(\mathcal{H}') = t$, then $S(\mathcal{H}') = \{t, t+1, \ldots, n-2\}$. For any coloring of $\mathcal{H}'$ using k colors, we add x_{n-1}, x_n to a new color, it becomes a coloring of $\mathcal{H}$ using $k+1$ colors. Hence, $t+1, t+2, \ldots, n-1 \in S(\mathcal{H})$. Therefore, if $\mathcal{H}$ has a gap, it must be less than $t+1$, but if there is a coloring of $\mathcal{H}$ using less than t colors, then there would be a coloring of $\mathcal{H}'$ using less than t colors, and it is impossible because $\chi(\mathcal{H}') = t$. Hence, $\mathcal{H}$ has no gap. ❑

Next, we construct a mixed hypergraph with 6 vertices which has a gap.

Lemma 2.5 *Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where $X = \{x_1, x_2, \ldots, x_6\}$,*

$$\mathcal{C} = \{x_1x_2x_3,\ x_2x_3x_6,\ x_1x_4x_5,\ x_4x_5x_6,\ x_2x_3x_4,\ x_3x_4x_5\},\ \textit{and}$$

$$\mathcal{D} = \{x_1x_2x_3,\ x_2x_3x_6,\ x_1x_4x_5,\ x_4x_5x_6,\ x_2x_5,\ x_1x_6\}.$$

Then $S(\mathcal{H}) = \{2, 4\}$ has a gap at 3. And we use $\mathcal{H}_{2,4}$ to denote this mixed hypergraph.

Proof. We consider x_1, x_2, and x_3 first, since they form a $\mathcal{C}$-edge and a $\mathcal{D}$-edge, two of them must get the same color and the third one must have another color. Hence we have three cases:

Case 1: Let $c(x_1) = c(x_2) = 1,\ c(x_3) = 2$. Since $x_2x_3x_6$ is a $\mathcal{C}$-edge and $\mathcal{D}$-edge at the same time and x_1x_6 is a $\mathcal{D}$-edge, we have $c(x_6) = 2$. And $x_2x_3x_4,\ x_1x_4x_5$ and $x_4x_5x_6$ are $\mathcal{C}$-edges which implies $c(x_4), c(x_5) \in \{1, 2\}$. Because $x_1x_4x_5,\ x_4x_5x_6$ and x_2x_5 are $\mathcal{D}$-edges, $c(x_5) = 2$ and $c(x_4) = 1$. Hence, c is a 2-coloring.

Case 2: If $c(x_1) = c(x_3) = 1,\ c(x_2) = 2$, then we also have $c(x_6) = 2$. As in case 1, c is a 2-coloring.

Case 3: Let $c(x_1) = 1,\ c(x_2) = c(x_3) = 2$. Since $x_2x_3x_6$ and x_1x_6 are $\mathcal{D}$-edges, we have $c(x_6) \notin \{1, 2\}$. Set $c(x_6) = 3$. Then $x_1x_4x_5,\ x_4x_5x_6$ and $x_3x_4x_5$ are $\mathcal{C}$-edges which implies $c(x_4) = c_(x_5) \notin \{1, 2, 3\}$. Hence, c is a 4-coloring.

Therefore, $S(\mathcal{H}) = \{2, 4\}$ and $\mathcal{H}$ has a gap at 3. ❑

In lemma 2.4, we have proved that when $\bar{\chi}(\mathcal{H}) = n$ or $\bar{\chi}(\mathcal{H}) = n - 1$, $\mathcal{H}$ is gap-free. Hence, if there is a gap of $\mathcal{H}$, $\bar{\chi}(\mathcal{H})$ must be less than or equal to $n - 2$. Then we have the following two theorems.

Theorem 2.6 *The minimum number of vertices in a mixed hypergraph with a gap in its feasible set is* 6*.*

Proof. We know that $\mathcal{H}_{2,4}$ is a mixed hypergraph with 6 vertices and $S(\mathcal{H}_{2,4}) = \{2, 4\}$. Thus it suffices to show that there is no gap when $n < 6$.

Let $\mathcal{H}$ be a mixed hypergraph with less than 6 vertices. By lemma 2.4, if $\mathcal{H}$ has a gap at k, $\bar{\chi}(\mathcal{H})$ must be less than 4, then $k < 3$. But 1 can not be a gap, so we consider that if $k = 2$.

If $k = 2$ is a gap of $\mathcal{H}$, then $1 \in S(\mathcal{H})$, then $\mathcal{D} = \phi$. There is no gap when $\mathcal{D} = \phi$. Therefore, if $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ has a gap, then $|X| \geq 6$. ❑

Theorem 2.7 *The maximum gap of a mixed hypergraph with n vertices is $n - 3$, $n \geq 6$.*

By lemma 2.4, the proof of above theorem is completed if we can find a mixed hypergraph with a gap at $n-3$. To find this mixed hypergraph, we need some definitions and theorems.

Definition 2.8 *A mixed hypergraph, $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a clique if $|X| = n$, $\mathcal{C} = \phi$, and for any pair of vertices forms a $\mathcal{D}$-edge. Denote that $\mathcal{H} = \mathcal{H}_{(n)}$.*

Observation 2.9 *By definition, we have* $S(\mathcal{H}_{(n)}) = \{n\}$.

Definition 2.10 *Let* $\mathcal{H}_1 = (X_1, \mathcal{C}_1, \mathcal{D}_1)$ *and* $\mathcal{H}_2 = (X_2, \mathcal{C}_2, \mathcal{D}_2)$ *be two mixed hypergraphs. We say that* $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ *is the join of* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *if* $X = X_1 \cup X_2, \mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, *and* $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup R$, *where* R *is the all pairs consisting of one vertex from* X_1 *and one from* X_2. *Denote that* $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$.

Jiang et. al. (2003) have a lemma about join.

Lemma 2.11 *[2] Let* $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$, *then* $S(\mathcal{H}) = \{i + j | i \in S(\mathcal{H}_1), j \in S(\mathcal{H}_2)\}$.

Definition 2.12 *Let* $\mathcal{H}_1$ *be a mixed hypergraph, a mixed hypergraph is a shifting of* $\mathcal{H}_1$ *if it is a join of* $\mathcal{H}_1$ *and a clique.*

Observation 2.13 *Let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be two mixed hypergraphs, and* $\mathcal{H}_1$ *be the join of* $\mathcal{H}_2$ *and* $\mathcal{H}_{(n)}$ *(i.e.* $\mathcal{H}_1$ *is a shifting of* $\mathcal{H}_2$*), then* $S(\mathcal{H}_1) = \{i + n | i \in S(\mathcal{H}_2)\}$.

Finally, we take $\mathcal{H}_{2,4}$ again, it has 6 vertices and a gap at 3. For all $n > 6$, let $\mathcal{H}_n = \mathcal{H}_{2,4} \cup \mathcal{H}_{(n-6)}$, then $\mathcal{H}_n$ is a mixed hypergraph with n vertices which has a gap at $n - 3$. Then we complete the proof of theorem 2.7.

Therefore, we can deduce the following theorem from above way.

Theorem 2.14 *For any* $n \geq 6, 3 \leq k \leq n-3$, *there is a mixed hypergraph with* n *vertices has a gap at* k. *In fact, the feasible set is* $\{k - 1, k + 1\}$.

Proof. We take $\mathcal{H}_{2,4}$ first, then there are 6 vertices and $S(\mathcal{H}_{2,4}) = \{2, 4\}$. For any $3 \leq k \leq n - 3$, let $\mathcal{H}' = (X', \mathcal{C}', \mathcal{D}') = \mathcal{H}_{2,4} \cup \mathcal{H}_{(k-3)}$, then there are $k + 3$ vertices and $S(\mathcal{H}') = \{k - 1, k + 1\}$.

Then choose a vertex, a, in X'. Let $A = \{x_1, x_2, \ldots, x_{n-k-3}\}$ and $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ where $X = X' \cup A$, $\mathcal{D} = \mathcal{D}'$, and $\mathcal{C} = \mathcal{C}' \cup \{\{a, x_i\} | x_i \in A\}$. Hence, all vertices in A get the same color with a. Then $|X| = n$ and $S(\mathcal{H}) = \{k - 1, k + 1\}$. ❑

Finally, according to theorem 2.7, we get the following corollary.

Corollary 2.15 *For all $k \geq 3$, the minimum number of vertices of a mixed hypergraph which has a gap at k is $k+3$.*