3 Mixed hypergraphs spanned by complete bipartite graphs

As mentioned in section 2, since K_5 is the only connected non-planar graph which is good, [3], we know that $K_{s,t}$ is bad for $3 \leq s \leq t$. So we consider $K_{2,t}$ first.

Theorem 3.1 *For $t \in \mathbb{N}$, $K_{2,t}$ is good.*

Proof. Let $A = \{a_1, a_2\}, B = \{b_1, b_2, \ldots, b_t\}$ be the bipartition of $G = K_{2,t}$, and $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be a mixed hypergraph spanned by G, $X = A \cup B$. Suppose c is a strict l-coloring of $\mathcal{H}$, $l \geq 4$, and $\{X_1, X_2, \ldots, X_l\}$ is the feasible partition with respect to c. Since a_1 and a_2 belong to at most two of $\{X_1, X_2, \ldots, X_l\}$, we assume that $a_1, a_2 \in X_1 \cup X_2$.

Let $\{Y_1, Y_2, \ldots, Y_{l-1}\}$ be a feasible partition with respect to c', where $Y_i = X_i$ for $1 \leq i \leq l-2$ and $Y_{l-1} = X_{l-1} \cup X_l$. We plan to show that c' is also a strict coloring of $\mathcal{H}$. For all $C \in \mathcal{C}$, there are two vertices $x_1, x_2 \in C$ belong to X_j for some $1 \leq j \leq l$, if $1 \leq j \leq l-1$, then x_1, x_2 belong to Y_j; if $j = l$, then x_1, x_2 belong to Y_{l-1}. Hence, C is colored properly by c'. If c' is not a strict coloring, then there exist $D \in \mathcal{D}$ which D is not colored properly, it means that $D \subseteq Y_i$ for some i. But c is a strict coloring, $D \nsubseteq Y_i, 1 \leq i \leq l-2$. Therefore, $D \subseteq Y_{l-1}$. Since $a_1, a_2 \in Y_1 \cup Y_2$ and $l \geq 4$, $Y_{l-1} \subseteq B$ and D must be a connected subgraph of G, such D can not exist. Hence, c' is a strict $(l-1)$-coloring of $\mathcal{H}$.

This is for $l \geq 4$, so we know that $\bar{\chi}(\mathcal{H}), \bar{\chi}(\mathcal{H}) - 1, \ldots, 4, 3 \in S(\mathcal{H})$. Hence, $\mathcal{H}$ has no gap. ❑

Consider the maximum gap of a mixed hypergraph spanned by $K_{s,t}, 3 \leq s \leq t$, we have the following theorem.

Theorem 3.2 *The maximum gap of a mixed hypergraph, $\mathcal{H}$, spanned by $K_{s,t}$ is s, $3 \leq s \leq t$.*

Proof. By the same way as in the proof of thearem 3.1, for any strict l-coloring, $l \geq s+2$, we can find a strict $(l-1)$-coloring by combining two members of a feasible partition. So we know that if $\mathcal{H}$ has a gap, it must be at most s.

Now we construct a mixed hypergraph to show that s is the best possible. Let $A = \{a_1, a_2, \ldots, a_s\}$ and $B = \{b_1, b_2, \ldots, b_t\}$ be the bipartition of $G = K_{s,t}$. Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be a mixed hypergraph spanned by G,

$$\mathcal{C} = \{a_1a_ib_i, a_ib_ib_1 | 2 \leq i \leq s\} \cup \{a_ib_ia_j, b_ia_jb_j | 2 \leq i < j \leq s\} \cup \{a_sb_i | s+1 \leq i \leq t\},$$

$$\mathcal{D} = \{a_1a_ib_i, a_ib_ib_1 | 2 \leq i \leq s\} \cup \{a_ib_j | 2 \leq i < j \leq s\} \cup \{a_1b_1\}.$$

Let c be a coloring of $\mathcal{H}$. Since a_1, a_2, b_2 form a $\mathcal{C}$-edge and a $\mathcal{D}$-edge at the same time, we have the following three cases:

Case 1: Let $c(a_1) = c(a_2) = 1, c(b_2) = 2$. Since a_1b_1 is a $\mathcal{D}$-edge and $a_2b_2b_1$ is a $\mathcal{C}$-edge, we have $c(b_1) = 2$. If $c(a_3) = c(b_3)$, then $a_1a_3b_3, a_3b_3b_1$ are $\mathcal{D}$-edges will imply $c(a_3) = c(b_3) \notin \{1, 2\}$ and force $c(a_3) = c(b_3) = 3$. But $a_2b_2a_3$ is a $\mathcal{C}$-edge and $a_2,\ b_2, a_3$ have different colors, this reaches a contradiction. Hence, $c(a_3) \neq c(b_3)$. Since $a_1a_3b_3, a_3b_3b_1 \in \mathcal{C}$ and $a_2b_3 \in \mathcal{D}$, we have $c(a_3) = 1, c(b_3) = 2$. Similarly, $c(a_i) = 1, c(b_i) = 2$ for all i, $4 \leq i \leq s$. Finally, $c(b_j) = c(a_s) = 1$ for all j, $s+1 \leq j \leq t$, this completes a 2-coloring of $\mathcal{H}$.

Case 2: Let $c(a_1) = c(b_2) = 1, c(a_2) = 2$. As in case 1, we get $c(b_1) = 2$, and $c(a_i) = 2, c(b_i) = 1$ for all i, $3 \leq i \leq s$. Set $c(b_j) = c(a_s) = 2$ for all j, $s+1 \leq j \leq t$, then c is also a 2-coloring of $\mathcal{H}$.

Case 3: Let $c(a_1) = 1, c(a_2) = c(b_2) = 2$. Since $a_1b_1, a_2b_2b_1 \in \mathcal{D}$, $c(b_1) \notin \{1, 2\}$. Set $c(b_1) = 0$. Since $a_1a_3b_3, a_3b_3b_1$ and $b_2a_3b_3$ are $\mathcal{C}$-edges and $\mathcal{D}$-edges at the same time, $c(a_3) = c(b_3) \notin \{0, 1, 2\}$. Set $c(a_3) = c(b_3) = 3$. Similarly, we can assume that $c(a_i) = c(b_i) = i$ for all i, $4 \leq i \leq s$, and $c(a_s) = c(b_j) = s$ for all j $s+1 \leq j \leq t$. We obtain a $(s+1)$-coloring.

Therefore, $S(\mathcal{H}) = \{2, s+1\}$ and $\mathcal{H}$ has a gap at s. So s is the maximum gap of a mixed hypergraph spanned by a complete bipartite graph, $K_{s,t}$. ❑

Actually, lemma 2.5 is a special case of this proof. Let $x_1 = a_1,\ x_2 = a_2,\ x_3 = b_2,\ x_4 = a_3,\ x_5 = b_3$ and $x_6 = b_1$. Hence, $\mathcal{H}_{2,4}$ is a mixed hypergraph spanned by $K_{3,3}$ and satisfies above proof.

In the proof of theorem 3.2, we constructed a mixed hypergraph whose feasible set is $\{2, s+1\}$, hence for each k, $3 \leq k \leq s$, k is a gap of this mixed hypergrpah. We have a quick result:

Corollary 3.3 *A mixed hypergraph which is spanned by $K_{s,t}$ for $3 \leq s \leq t$ has a gap at k if and only if $3 \leq k \leq s$.*