4 Gaps of (l,m)-uniform mixed hypergraphs

Definition 4.1 *A mixed hypergraph, $\mathcal{H}$, is (l,m)-uniform if each $\mathcal{C}$-edge consists of l vertices and each $\mathcal{D}$-edge consists of m vertices. If $l = m = r$, then we say that $\mathcal{H}$ is r-uniform.*

Definition 4.2 *We say that a mixed hypergraph is a bihypergraph if $\mathcal{C} = \mathcal{D}$.*

Definition 4.3 *Let $\mathcal{H}$ be a mixed hypergraph, and e be an edge with two vertices u, v, contraction of e is the replacement of u and v with a new vertex whose incident edges are the edges other than e that were incident u or v.*

Theorem 4.4 *Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ where $|C| = l$ for all $C \in \mathcal{C}$. If $l = 2$ then $\mathcal{H}$ has no gap. If $l > 2$ and let k be a gap of $\mathcal{H}$, then $k \geq l$.*

Proof. If $l = 2$, then for all $C \in \mathcal{C}$, by contraction we can use a new vertex to replace C. Then we have a new mixed hypergraph $\mathcal{H}' = (X', \mathcal{C}', \mathcal{D}')$, and $S(\mathcal{H}) = S(\mathcal{H}')$. Because $\mathcal{C}' = \phi$, $\mathcal{H}'$ has no gap. Therefore, $\mathcal{H}$ has no gap.

If $l > 2$, let c be a strict t-coloring of $\mathcal{H}$, $t < l - 1$, and $\{X_1, X_2, \ldots, X_t\}$ be the feasible partition with respect to c, $|X_t| \neq 1$. Choose $a \in X_t$, and let $\{Y_1, Y_2, \ldots, Y_{t+1}\}$ be the feasible partition with respect to c', where $Y_i = X_i$ for $1 \leq i \leq t - 1$, $Y_t = X_t - \{a\}$, and $Y_{t+1} = \{a\}$.Then we prove that c' is a strict $(t + 1)$-coloring of $\mathcal{H}$:

For all $D \in \mathcal{D}$, there are $d_1, d_2 \in D$ such that $d_1 \in X_i, d_2 \in X_j$ for $i \neq j$. Therefore $d_1 \in Y_{i'}, d_2 \in Y_{j'}$ for $i' \neq j'$. For all $C \in \mathcal{C}$, since $|C| = l$ and $t + 1 < l$, by Pigeonhole Principle, there are $c_1, c_2 \in C$ such that $c_1, c_2 \in Y_i$ for some i. So c' is a strict $(t + 1)$-coloring of $\mathcal{H}$. Thus, $k \geq l$. ❑

Theorem 4.5 *Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ where $|D| = m \geq 3$ for all $D \in \mathcal{D}$. Let $n = |X|$ and $s = \lceil \frac{m}{2} \rceil$, then we can rewrite $n = sh + m - 1 - s, sh + m - 2 - s, \ldots,$ or $sh + m - 2s$, $h \in \mathbb{N}$, up to n modulo s. If $\mathcal{H}$ has a gap at k, then $k < h$.*

Proof. Let c be a strict t-coloring of $\mathcal{H}$, $t \geq h+1$, and $\{X_1, X_2, \ldots, X_t\}$ be the feasible partition with respect to c.

If m is even, then $m = 2s$, and $n = sh, sh+1, \ldots$, or $sh+s-1$. If m is odd, then $m = 2s-1$, and $n = sh-1, sh, \ldots$, or $sh+s-2$. Then by Pigeonhole Principle, there exists X_i such that $|X_i| = q \leq s+1$, for some $1 \leq i \leq t$, and there exist X_j such that $|X_j| < m-q$, for some $j \neq i, 1 \leq j \leq t$. Set $|X_t| = q \leq s+1$ and $|X_{t-1}| < m-q$, then $|X_t \cup X_{t-1}| < m$.

Let $Y_i = X_i$ for $1 \leq i \leq t-2$, $Y_{t-1} = X_{t-1} \cup X_t$, and $\{Y_1, Y_2, \ldots, X_{t-1}\}$ be the feasible partition with respect to c'. For all $C \in \mathcal{C}$, there are two members of C have the same X_i, then these two vertices have the same Y_i. For all $D \in \mathcal{D}$, since $D \nsubseteq X_i$ for $1 \leq i \leq t$, $D \nsubseteq Y_i$ for $1 \leq i \leq t-2$, and $|D| = m$ and $|Y_{t-1}| = |X_{t-1} \cup X_t| < m$, so $D \nsubseteq Y_{t-1}$. Hence, c' is a strict $(t-1)$-coloring of $\mathcal{H}$. Therefore, if $\mathcal{H}$ has a gap at k, then $k < h$. ❑

Theorem 4.6 *Let $\mathcal{H}$ be a (l,m)-uniform mixed hypergraph where $n, s,$ and h are defined as above. If $s \leq l$, then $\mathcal{H}$ has a gap at k if and only if $l \leq k < h$.*

Proof. By theorem 4.4 and theorem 4.5, we know that if $\mathcal{H}$ has a gap, then $l \leq k < h$. Then we prove the converse, that is for $l \leq k < h$, we can find a (l,m)-uniform mixed hypergraph has a gap at k.

If m is even and $n \equiv b \pmod{s}$, $0 \leq b \leq s-1$. Let A and A_i, $0 \leq i \leq s-1$, are sets of vertices,

$$A = \left\{ \begin{array}{cccc} a_{1,1} & a_{1,2} & \cdots & a_{1,h} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,h} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s,1} & a_{s,2} & \cdots & a_{s,h} \end{array} \right\},$$

$A_0 = \phi$, and $A_i = \{a_{i,h+1}\},\ 1 \leq i \leq s-1$. Then $X = A \cup (\bigcup_{i=0}^{b} A_i)$.

If m is odd and $n \equiv b \pmod{s}$, $-1 \le b \le s-2$. Let A' and A'_i, $0 \le i \le s-1$, are sets of vertices,

$$A = \left\{ \begin{array}{ccccc} a_{1,1} & a_{1,2} & \cdots & a_{1,h-1} & a_{1,h} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,h-1} & a_{2,h} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{s-1,1} & a_{s-1,2} & \cdots & a_{s-1,h-1} & a_{s-1,h} \\ a_{s,1} & a_{s,2} & \cdots & a_{s,h-1} & \end{array} \right\},$$

$A_0 = \phi$, $A_1 = \{a_{s,h}\}$, and $A_i = \{a_{i-1,h+1}\}$, $2 \le i \le s-1$. Then $X = A \cup (\bigcup_{i=0}^{b+1} A_i)$.

Let $B_j = \{a_{1,j}, a_{2,j}, \ldots, a_{s,j}\} \cap X$, $1 \le j \le h+1$, and $\mathcal{C} = \{C \subseteq X \mid |C| = l$, at least two elements of C are in some B_j, $1 \le j \le h+1\}$, and $\mathcal{D} = \{D \subseteq X \mid |D| = m$, at least two elements of D are in some B_j, $1 \le j \le h+1\}$.

Let c be a coloring of $\mathcal{H}$. If there exist $c(a_{i_1,j}) \ne c(a_{i_2,j})$ for some $i_1 \ne i_2$ and $1 \le j \le h+1$, then because $a_{i_1,j}, a_{i_2,j}$ and any other $l-2$ vertices form a $\mathcal{C}$-edge, c is at most $(l-1)$-coloring. Suppose $c(a_{i,j}) = i$ for all i, j, then c is a strict s-coloring, and by the proof of theorem 4.4, $s, s+1, \ldots, l-1 \in S(\mathcal{H})$. If $c(a_{1,j}) = c(a_{2,j}) = \cdots = c(a_{s,j})$ for all $1 \le j \le h+1$. Since $B_{j_1} \cup B_{j_2}$, $1 \le j_1, j_2 \le h$, contains at lease one $\mathcal{D}$-edge, $c(a_{1,j_1}) \ne c(a_{1,j_2})$. Hence, c is a strict h-coloring or a strict $h+1$-coloring. Let $c(a_{i,j}) = j$ for all i, j, then c is a strict $(h+1)$-coloring, let $c(a_{i,j}) = j$ for $1 \le j \le h$, and $c(a_{i,h+1}) = h$, then c is a strict h-coloring. Therefore, $\mathcal{H}$ has gaps at k, $l \le k < h$. ❑

Theorem 4.7 *The minimum number of vertices of a bad (l, m)-uniform mixed hypergraph is $s(l-1) + m$, $s = \lceil \frac{m}{2} \rceil$.*

Proof. Let $\mathcal{H}$ is a bad (l, m)-uniform mixed hypergraph. By theorem 4.6, the gap is between $l-1$ and h. To find the minimum number of vertices, it implies that $h = l+1$. Hence, by theorem 4.5, minimum n is $sh + m - 2s = s(l+1) + m - 2s = s(l-1) + m$. ❑

Finally, we consider r-uniform mixed hypergraphs. Since $l = m = r$, $s = \lceil \frac{m}{2} \rceil = \lceil \frac{r}{2} \rceil \le r = l$. Hence, all facts of (l, m)-uniform can be generalized to r-uniform.

Corollary 4.8 *By theorem 4.6, a r-uniform mixed hypergraph has a gap at k if and only if $r \leq k < h$ where h is as above.*

In the proof of theorem 4.6, we consider r-uniform mixed hypergraphs, the hypergraph we constructed becomes a r-uniform bihypergraph. Hence, we have another corollary about r-uniform bihypergraphs.

Corollary 4.9 *A r-uniform bihypergraph has a gap at k if and only if $r \leq k < h$.*

And we also can find the minimum number of vertices of a r-uniform mixed hypergraph (or a r-uniform bihypergraph) which has gaps.

Corollary 4.10 *Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be a r-uniform mixed hypergraph (or r-uniform bihypergraph). If $\mathcal{H}$ has a gap, then the minimum number of vertices is $|X| = n = s(r-1)+r$ where $s = \lceil \frac{r}{2} \rceil$.*