CHAPTER 4

Mathematic Model

So far, the competing behaviors of control subframe in the distributed scheduling of IEEE 802.16 mesh mode are presented. Next, we are going to propose a mathematical analysis to model the MSH-DSCH transmission behavior of IEEE 802.16 mesh mode. The delay time of MSH-DSCH transmission will be evaluated by our proposed mathematical model.

4.1. Markov Chain

We observed the behavior of distributed scheduling in the mesh mode in previous chapter. We found this behavior does not depend on all of past history. In other words, it is a “Time Homogeneous” and suitable for being modeled by stochastic process. The delay time in the period of MSH-DSCH transaction is what we are interested in this thesis, which the Markov Chain is easy using to observe it. Depends on Leonard Kleinrock’s description in his book “QUEUEING SYSTEMS VOLUME I: THEORY”, Markov processes may be used to describe the motion of a particle in some space. We consider discrete-time Markov chains, which permit the particle to occupy discrete positions and permit transitions between these positions to take place only at discrete times.[11]

Assume X_n is denoted as a state in our consequent Markov Chain model that a node stays at a certain time to transmit MSH-DSCH. Time unit is an opportunity. A set of random variable $\{X_n\}$ forms a Markov chain if the probability that the next state is X_{n+1} depends only upon the current state X_n and not upon any previous stations. Base on our analysis in previous chapter, the next state merely depends on the current competing result, neither on the last nor on all of past history. Thus we have a random sequence in which the dependency extends backwards one unit in time. If this node's Temp Xmt Time overlaps with its neighbors, it implies the competing is occurred with them. If it wins or there is no competition, it will set this Temp Xmt Time as its Next Xmt Time. If it loses, it will back one opportunity to run this behavior again until it wins. In order to simplify the notification, we assume integer 1,2,3 … represent each of certain state X_n , the physical concept of our proposed Markov Chain are depicted as Figure 4.1.

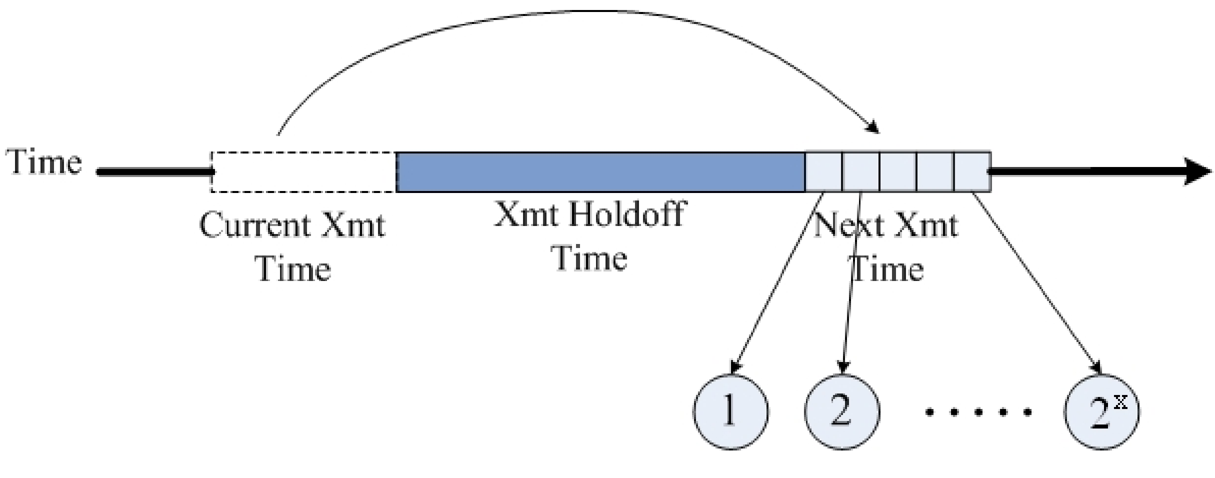

Figure 4.1: Each state corresponds to the Next Xmt Time-1

With this concept of Figure 4.1, we can model this behavior with a vertical chain as Figure 4.2. The states and transition definitions are defined as Table 4.1. From state 1 to state 2^x implies the time duration of one Next Xmt Time. Suppose we have N nodes totally, the probability which a node wins N-1 nodes can be expressed by formula (5).

Oppositely, the probability of a node loses them can be expressed by formula (6).

$$\text{Win} = \text{Prob}_{\text{N-1}} \tag{5}$$

$$\text{Lose} = 1 - \text{Prob}_{\text{N-1}} \tag{6}$$

Table 4.1: The Notation definitions in the Markov Chain

Notation	Description
Integers in the state	The state probability that the transmission time backs to certain opportunity
Prob	The transition probability to indicate the probability that the node wins.
X	Exponent of Xmt Holdoff Time
N	The number of nodes

For example, if our observed node loses, it transfers from state 1 to state 2, the transition probability is $1 - \text{Prob}_{\text{N-1}}$. If it wins, it stays at state 1, the transition probability is $\text{Prob}_{\text{N-1}}$.

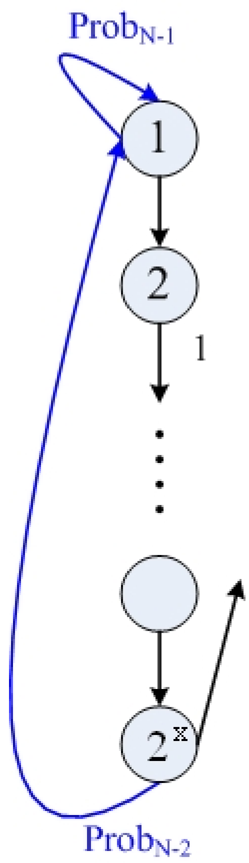

Figure 4.2: One vertical chain

In order to model it easily, we assume that as long as the node lose this competition, it does not back one opportunity. It has to back a length of Next Xmt Time. That's why the transition probabilities during the inter-states are always 1 in Figure 4.2. Thus, the state transfers to the second vertical chain are shown as Figure 4.4. If this node loses again, it will back a length of Next Xmt Time again to the third vertical chain …etc.

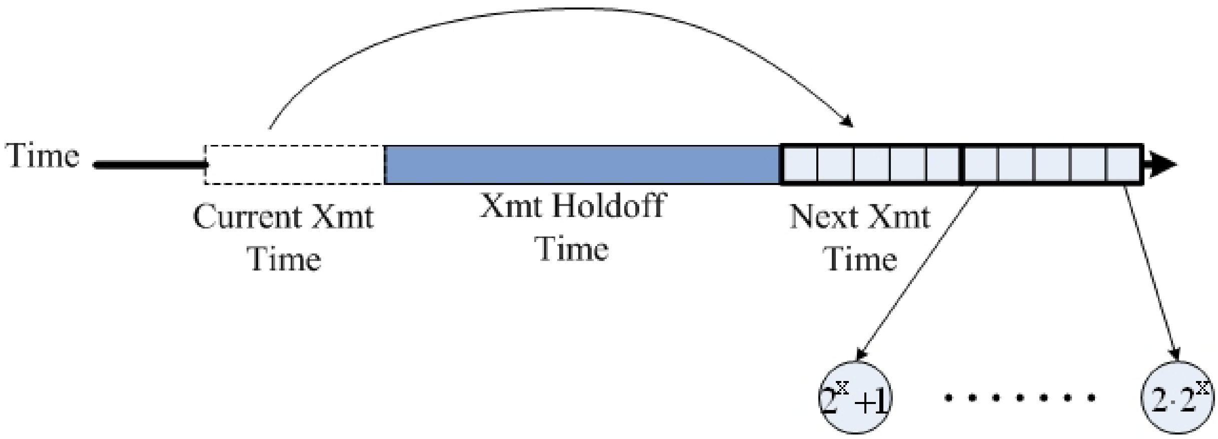

Figure 4.3: Each state corresponds to the Next Xmt Time-2

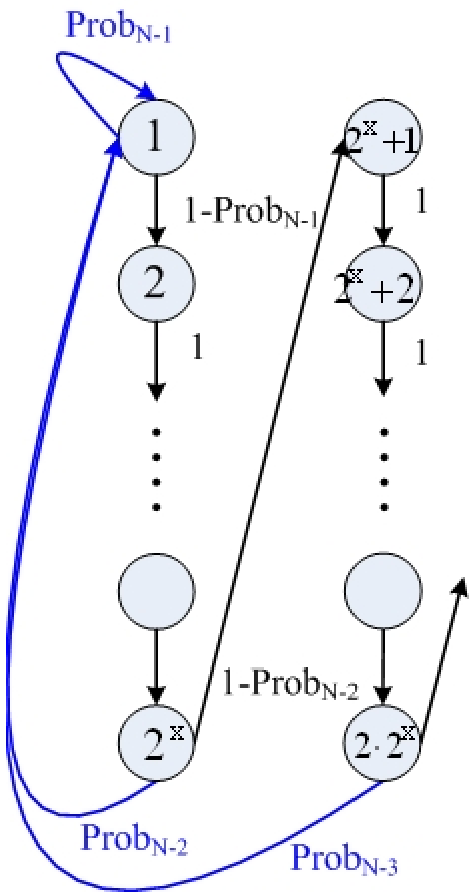

Figure 4.4 : Two vertical chains

At last, a Markov chain is organized as Figure 4.5.

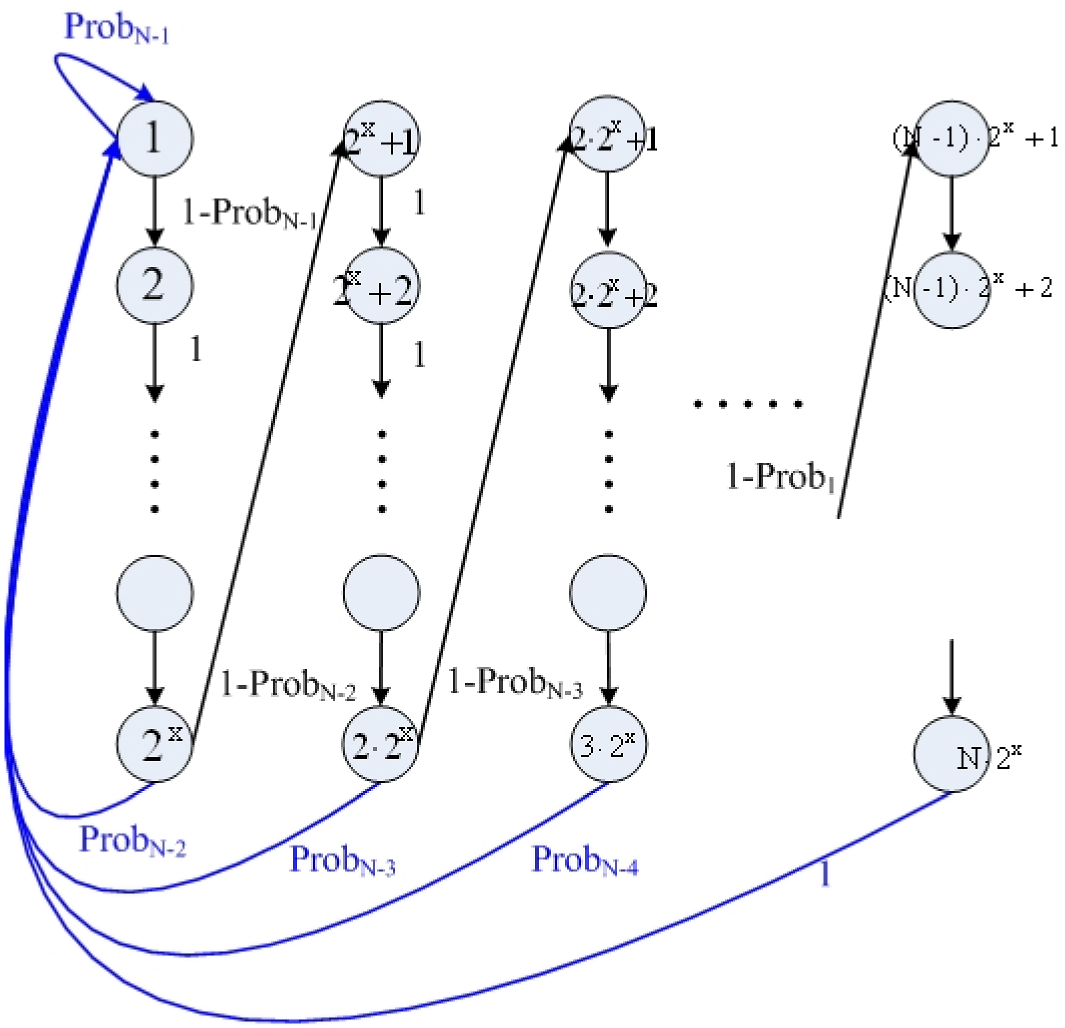

Figure 4.5: The Markov Chain

4.2. Mathematical Evaluation

The Markov Chain we proposed presents the variations of state transitions. We hope to induce an equation to evaluate the average delay time. The dependency of delay time relates to a probability of win. Before evaluating the delay time, we have to induce this probability formula initially. Then the expected value, the average delay time we target, is the product of probability and time.

4.2.1. Probability Theory and Assumptions

To begin with, we assume the number of nodes is N. Each observation of a node competes with neighbors is independent and represents one of two outcomes "competing" or "non-competing". So by using (7), we can get the competing probability $P_{competing}$. This is the binomial distribution we know. The P_c in the (7) is a probability that one node competes with one another node. More details can be retrieved from chapter 3.4 and Figure 3.6. $P_{competing}$ is different from P_c in our assumptions. P_c is the condition happened between one node and one node in a very short time. Nevertheless, $P_{competing}$ is the condition while at least one of the following events is happened: between one node and one node, or between one node and two nodes, or between one node and more another nodes. So formula (7) means one of the following situations is occurred: observed node competes with one neighbor, or observed node competes with two neighbors, or observed node competes with three neighbors …etc.

Table 4.2: Notations of equations

Notation	Description
Prob	Probability of $(\text{competing} \cap \text{win})$
$P_{competing}$	Probability of competing, at least one of more events happens
P_c	Probability of competing between one node to one of another node.

N	Number of nodes

$$P_{competing}$$
$$= C_0^{N-1} \cdot P_c^{\ 0} \cdot (1-P_c)^{N-1} + C_1^{N-1} \cdot P_c^{\ 1} \cdot (1-P_c)^{N-1-1} + C_2^{N-1} \cdot P_c^{\ 2} \cdot (1-P_c)^{N-1-2} + \dots \quad (7)$$

Moreover, the win probability should be an inverse proportion of the number of competing nodes. So we get (8) from (7).

$$\begin{aligned} Prob_{N-1} &= P_{competing \cap win} \\ &= \frac{1}{1} C_0^{N-1} \cdot P_c^{\ 0} \cdot (1-P_c)^{N-1} + \frac{1}{2} C_1^{N-1} \cdot P_c^{\ 1} \cdot (1-P_c)^{N-1-1} + \frac{1}{3} C_2^{N-1} \cdot P_c^{\ 2} \cdot (1-P_c)^{N-1-2} + \dots \\ &= \sum_{k=0}^{N-1} \frac{1}{k+1} \cdot C_k^{N-1} \cdot P_c^{\ k} \cdot (1-P_c)^{N-1-k} \end{aligned} \quad (8)$$

In opposition, the losing probability can be derived as (9).

$$1 - Prob_{N-1} = 1 - \left(\sum_{k=0}^{N-1} \frac{1}{k+1} \cdot C_k^{N-1} \cdot P_c^{\ k} \cdot (1-P_c)^{N-1-k} \right) \quad (9)$$

4.2.2. Delay Time

By the observation of Markov Chain (Figure 4.5), if the node wins at state 1, the transition probability of win can be expressed by using (10). If the node wins at the state 2^x that is the end of first vertical chain, the probability of win can be expressed by using (11). If the node wins at the state $2 \cdot 2^x$ that is the end of second vertical chain, the probability of win can be expressed by using (12)…etc.

$$\mathrm{Prob}_{\mathrm{N-1}} \tag{10}$$

$$(1-\mathrm{Prob}_{\mathrm{N-1}}) \cdot \mathrm{Prob}_{\mathrm{N-2}} \tag{11}$$

$$(1-\mathrm{Prob}_{\mathrm{N-1}}) \cdot (1-\mathrm{Prob}_{\mathrm{N-2}}) \cdot \mathrm{Prob}_{\mathrm{N-3}} \tag{12}$$

This probability distribution gives the trial number of the first success, so it is a geometric distribution. Substitute (8) and (9) into (10), (11) and (12), we can derive the probability (13), (14) and (15).

$$\mathrm{Prob}_{N\text{-}1} = \left(\sum_{k=0}^{N\text{-}1} \frac{1}{k+1} \cdot C_k^{N\text{-}1} \cdot P_c{}^k \cdot (1-P_c)^{N\text{-}1\text{-}k} \right) \tag{13}$$

$$(1-\mathrm{Prob}_{N\text{-}1}) \cdot \mathrm{Prob}_{N\text{-}2} = \left(1-\left(\sum_{k=0}^{N\text{-}1} \frac{1}{k+1} \cdot C_k^{N\text{-}1} \cdot P_c{}^k \cdot (1-P_c)^{N\text{-}1\text{-}k} \right)\right) \cdot \left(\sum_{k=0}^{N\text{-}2} \frac{1}{k+1} \cdot C_k^{N\text{-}2} \cdot P_c{}^k \cdot (1-P_c)^{N\text{-}2\text{-}k} \right) \tag{14}$$

$$(1-\mathrm{Prob}_{N\text{-}1}) \cdot (1-\mathrm{Prob}_{N\text{-}2}) \cdot \mathrm{Prob}_{N\text{-}3} = \left(1-\left(\sum_{k=0}^{N\text{-}1} \frac{1}{k+1} \cdot C_k^{N\text{-}1} \cdot P_c{}^k \cdot (1-P_c)^{N\text{-}1\text{-}k} \right)\right) \cdot \left(1-\left(\sum_{k=0}^{N\text{-}2} \frac{1}{k+1} \cdot C_k^{N\text{-}2} \cdot P_c{}^k \cdot (1-P_c)^{N\text{-}2\text{-}k} \right)\right) \cdot \left(\sum_{k=0}^{N\text{-}3} \frac{1}{k+1} \cdot C_k^{N\text{-}3} \cdot P_c{}^k \cdot (1-P_c)^{N\text{-}3\text{-}k} \right) \tag{15}$$

So far, each probability on the corresponding vertical chain has been derived. The expected value can be calculated by the summation of these probabilities and multiplied by time, Then formula (16) can be obtained. The unit of time in this formula is opportunity.

$$
\begin{aligned}
&E(\text{opportunity}) \\
&= \underbrace{\left(\sum_{k=0}^{N-1} \frac{1}{k+1} \cdot C_k^{N-1} \cdot P_c^{\ k} \cdot (1-P_c)^{N-1-k} \right)}_{\text{Win}} \cdot 2^x \\
&+ \\
&\left(1-\left(\sum_{k=0}^{N-1} \frac{1}{k+1} \cdot C_k^{N-1} \cdot P_c^{\ k} \cdot (1-P_c)^{N-1-k} \right)\right) \cdot \underbrace{\left(\sum_{k=0}^{N-2} \frac{1}{k+1} \cdot C_k^{N-2} \cdot P_c^{\ k} \cdot (1-P_c)^{N-2-k} \right)}_{\text{Win}} \cdot (2 \cdot 2^x) \\
&+ \\
&\left(1-\left(\sum_{k=0}^{N-1} \frac{1}{k+1} \cdot C_k^{N-1} \cdot P_c^{\ k} \cdot (1-P_c)^{N-1-k} \right)\right) \cdot \left(1-\left(\sum_{k=0}^{N-2} \frac{1}{k+1} \cdot C_k^{N-2} \cdot P_c^{\ k} \cdot (1-P_c)^{N-2-k} \right)\right) \\
&\cdot \underbrace{\left(\sum_{k=0}^{N-3} \frac{1}{k+1} \cdot C_k^{N-3} \cdot P_c^{\ k} \cdot (1-P_c)^{N-3-k} \right)}_{\text{Win}} \cdot (3 \cdot 2^x) \\
&+ \\
&\ldots.
\end{aligned}
\tag{16}
$$

Finally, we generalize our equation, as (17). In conclusion, the input parameters are N and x. It means the delay time is affected by the number of nodes and holdoff exponent.

$$
\begin{aligned}
&E[\text{opportunity}] \\
&= \sum_{j=2}^{N} \prod_{i=1}^{j-1} \left(1-\left(\sum_{k=0}^{N-(i-1)} \frac{1}{k+1} C_k^{N-(i-1)} P_c^{\ k} (1-P_c)^{N-(i-1)-k} \right)\right) . \left(\sum_{k=0}^{N-i} \frac{1}{k+1} C_k^{N-i} P_c^{\ k} (1-P_c)^{N-i-k} \right) (2^x \cdot j)
\end{aligned}
\tag{17}
$$

4.2.3. The Success Probability of MSH-DSCH Transmission

Except for delay time, we hope to know what the mean value of wining probability is that a node transmits MSH-DSCH. We know Markov Chain is more suitable to get the average probability of each state. If $\pi^{(k)}$ is the probability of certain state at certain time k in our proposed Markov Chain, $\pi^{(k-1)}$ is its probability of certain state at the time before k. P is the transition probability from the state of probability $\pi^{(k-1)}$ to the state of probability $\pi^{(k)}$ (Figure 4.6). Formula (18) and (19) are applied to evaluate the wining probability that a node transmit MSH-DSCH. These two formulas imply a recursive function and converge at ξ denoted as a convergence value. $\underline{\pi} = \{\pi_1, \pi_2, \pi_3 ...\}$ is a vector and each element $\pi_1, \pi_2, \pi_3, ...$ within the vector is denoted as the probabilities of corresponding state 1, 2, 3….in Figure 4.5. $\underline{\underline{P}}$ is a two dimension matrix which the size equals to $2^x \cdot (N-1) \times 2^x \cdot (N-1)$. For example x=2, N=5, the matrix $\underline{\underline{P}}$ will be shown as equation (20).

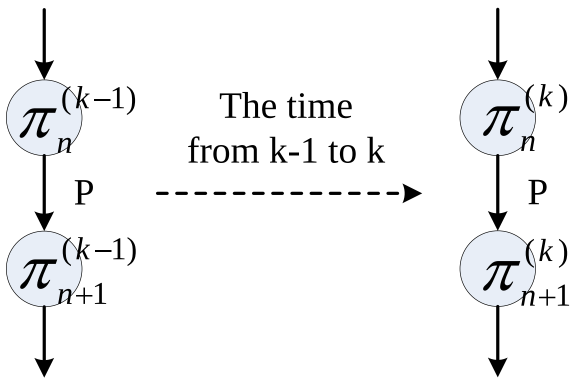

Figure 4.6: The transition probability of certain state

$$\underline{\pi}^{(k)} = \underline{\pi}^{(k-1)} \underline{\underline{P}} \tag{18}$$

$$\left| \underline{\pi}^{(k)} - \underline{\pi}^{(k-1)} \right| < \xi \tag{19}$$

```
p=[ P1   1-P1 0    0    0    0    0    0    0    0    0    0    0    0    0    0;
    0    0    1    0    0    0    0    0    0    0    0    0    0    0    0    0;
    0    0    0    1    0    0    0    0    0    0    0    0    0    0    0    0;
    P2   0    0    0    1-P2 0    0    0    0    0    0    0    0    0    0    0;
    0    0    0    0    0    1    0    0    0    0    0    0    0    0    0    0;
    0    0    0    0    0    0    1    0    0    0    0    0    0    0    0    0;
    0    0    0    0    0    0    0    1    0    0    0    0    0    0    0    0;
    P3   0    0    0    0    0    0    0    1-P3 0    0    0    0    0    0    0;
    0    0    0    0    0    0    0    0    0    1    0    0    0    0    0    0;
    0    0    0    0    0    0    0    0    0    0    1    0    0    0    0    0;
    0    0    0    0    0    0    0    0    0    0    0    1    0    0    0    0;
    P4   0    0    0    0    0    0    0    0    0    0    0    1-P4 0    0    0;
    0    0    0    0    0    0    0    0    0    0    0    0    0    1    0    0;
    0    0    0    0    0    0    0    0    0    0    0    0    0    0    1    0;
    0    0    0    0    0    0    0    0    0    0    0    0    0    0    0    1;
    1    0    0    0    0    0    0    0    0    0    0    0    0    0    0    0];
```

(20)

In order to simplify the expressions, P1 and 1-P1 within the matrix $\underline{\underline{P}}$ stand for the formula (5) and (6). With this same rule, we simplify other expressions as P2, 1-P2, P3, 1-P3… Then P(success) can be calculated by (21).

$$P(\text{success}) = \sum \text{Prob}_{\text{Win}} \cdot \pi_{\text{win}} \tag{21}$$