Chapter 3

Optimal and Efficient Designs for p = 2, 3, and for $p \geq 4$, k = 2

3.1. Optimal and Efficient Designs for p = 3

To find families of optimal designs and efficient designs, we derive the following inequality

$$g(s_{d0}; p, b, k) \geq pbk\left(\frac{1}{bks_{d0} - s_{d0}^2} + \frac{(p-1)^2}{bk\big(2bk(p-2) - (p-3)s_{d0}\big)}\right)$$

$$= pbk\big(g^*(s_{d0}; p, b, k)\big), \text{ say,} \tag{3.1}$$

and the equality holds when $(2bk - s_{d0})/pb$ and s_{d0}/b are integers.

For $p = 3$, $g^*(s_{d0};3,b,k) = (bks_{d0} - s_{d0}^2)^{-1} + 2(bk)^{-2}$, and by taking the derivative of $g^*(s_{d0};3,b,k)$ with respect to s_{d0}, the minimum value of $g^*(s_{d0};3,b,k)$ is achieved at $s_{d0} = s_0 = bk/2$, and $g^*(bk/2;3,b,k) = 6/(bk)^2$. In the following, the problems of finding and constructing families of A-optimal type S_0 block designs having $s_0 = bk/2$ are investigated.

A type S_0 block design $S_0(3, b, k, g_0, g_1, \lambda_0, \lambda_1)$ with $s_0 = bk/2$, has the following values for $s_1, g_0, g_1, \lambda_0, \lambda_1$, and

$$s_1 = bk/2, \quad g_0 = g_1 = bk/6, \text{ and}$$

$$\lambda_0 = \lambda_1 = \begin{cases} bk^2/4, & \text{if } k \text{ is even,} \\ bk^2/4 - b/12, & \text{if } k \text{ is odd.} \end{cases}$$

For these designs to exist, $s_0, s_1, g_0, g_1, \lambda_0, \lambda_1$ must all be integers, and the possible combinations of the values of such *b* and *k* are as follows.

(I) $b = 0 \pmod 3$, $k = 0 \pmod 2$,

(II) $k = 0 \pmod 6$,

(III) $b = 0 \pmod 6$, $k = 1 \pmod 2$.

In cases (I) and (II), k is an even number, and both $(2bk - s_0)/3b = k/2$ and $s_0/b = k/2$ are integers, hence, the minimum values of $g^*(s_{d0};3,b,k)$ or $g(s_{d0};3,b,k)$ can be achieved by the corresponding type S_0 block designs. Then by Theorem 2.3, these designs are A-optimal in their respective classes.

Lemma 3.1. For $b = 0 \pmod 3$, $k = 0 \pmod 2$, that is, $b = 3u$, $k = 2q$, where u, $q \geq 1$ are integers, a type S_0 block design $S_0(3, 3u, 2q, uq, uq, 3uq^2, 3uq^2)$ exists, and is A-optimal in $D(3+1, 3u, 2q)$.

Lemma 3.2. For $k = 0 \pmod 6$, that is, $b = u$, $k = 6q$, where $u, q \geq 1$ are integers, a type S_0 block design $S_0(3, u, 6q, uq, uq, 9uq^2, 9uq^2)$ exists, and is A-optimal in $D(3+1, u, 6q)$.

The optimal designs in Lemma 3.1 and 3.2 can be constructed by using the following two initiate designs d_A and d_B in Examples 3.1 and 3.2, respectively. These two designs are A-optimal by the above two lemmas.

Example 3.1. For $b = 3$, $k = 2$, that is, $u = q = 1$, the following design d_A with columns as blocks is a $S_0(3, 3, 2, 1, 1, 3, 3)$, and is A-optimal in $D(3+1,3,2)$.

$$d_A: \begin{array}{ccc} (0,1) & (0,2) & (0,3) \\ (2,3) & (1,3) & (1,2) \end{array}.$$

Example 3.2. For $b = 1$, $k = 6$, that is, $u = q = 1$, the following design d_B with columns as blocks is a $S_0(3, 1, 6, 1, 1, 9, 9)$, and is A-optimal in $D(3+1,1,6)$.

$$d_B: \begin{array}{c} (0,1) \\ (0,2) \\ (0,3) \\ (1,2) \\ (1,3) \\ (2,3) \end{array}.$$

Case (I): For $b = 0 \pmod 3$, $k = 0 \pmod 2$, that is, $b = 3u, k = 2q$, where u, $q \geq 1$ are integers, an A-optimal $S_0(3, 3u, 2q, uq, uq, 3uq^2, 3uq^2)$ design d_I with columns as blocks can be constructed by repeating d_A u times in the column direction and q times in the row direction, and is illustrated in the following.

$$d_I: \overbrace{\begin{array}{cccc} d_A & d_A & \cdots & d_A \\ d_A & d_A & \cdots & d_A \\ \vdots & \vdots & \ddots & \vdots \\ d_A & d_A & \cdots & d_A \end{array}}^{u} \left.\vphantom{\begin{array}{c} d_A \\ d_A \\ \vdots \\ d_A \end{array}}\right\} q \qquad (3.2)$$

Example 3.3. For $b = 6$, $k = 6$, that is, $u = 2$, $q = 3$, the following design d_I is a $S_0(3, 6, 6, 6, 6, 54, 54)$ design, and is A-optimal in $D(3+1,6,6)$.

(0,1)	(0,2)	(0,3)	(0,1)	(0,2)	(0,3)
(2,3)	(1,3)	(1,2)	(2,3)	(1,3)	(1,2)
(0,1)	(0,2)	(0,3)	(0,1)	(0,2)	(0,3)
(2,3)	(1,3)	(1,2)	(2,3)	(1,3)	(1,2)
(0,1)	(0,2)	(0,3)	(0,1)	(0,2)	(0,3)
(2,3)	(1,3)	(1,2)	(2,3)	(1,3)	(1,2)

Case (II): For $k = 0 \pmod 6$, that is, $b = u$, $k = 6q$, where u, $q \geq 1$ are integers, an A-optimal $S_0(3, u, 6q, uq, uq, 9uq^2, 9uq^2)$ design d_{II} can be constructed as (3.2) by replacing d_A with d_B.

Example 3.4. For $b = 3$, $k = 6$, that is, $u = 3$, $q = 1$, the following design d_{II} is a $S_0(3, 3, 6, 3, 3, 27, 27)$ design, and is A-optimal in $D(3+1,3,6)$.

(0,1)	(0,1)	(0,1)
(0,2)	(0,2)	(0,2)
(0,3)	(0,3)	(0,3)
(1,2)	(1,2)	(1,2)
(1,3)	(1,3)	(1,3)
(2,3)	(2,3)	(2,3)

Remark: For $b = 3u$, $k = 6q$, no matter how differently the optimal designs, constructed by using the above two methods, might appear, they all are type S_0 block designs, and hence they all are A-optimal designs. For example, for $b = 3$, $k = 6$, by Lemmas 3.1 and 3.2, the following two designs d and d' both are A-optimal type S_0 block designs in $D(3+1,3,6)$.

$$
d: \begin{array}{ccc} (0,1) & (0,2) & (0,3) \\ (0,1) & (0,2) & (0,3) \\ (0,1) & (0,2) & (0,3) \\ (2,3) & (1,3) & (1,2) \\ (2,3) & (1,3) & (1,2) \\ (2,3) & (1,3) & (1,2) \end{array}, \quad d': \begin{array}{ccc} (0,1) & (0,1) & (0,1) \\ (0,2) & (0,2) & (0,2) \\ (0,3) & (0,3) & (0,3) \\ (1,2) & (1,2) & (1,2) \\ (1,3) & (1,3) & (1,3) \\ (2,3) & (2,3) & (2,3) \end{array}.
$$

For values of b and k in case (III), type S_0 block designs, though exist, can not be proved to be A-optimal by existing methods. Their efficiencies are thus investigated.

Case (III). For $b = 0 \pmod 6$, $k = 1 \pmod 2$, that is, $b = 6u, k = 2q+1$, where u, $q \geq 1$ are integers, a type S_0 block design d_{III} with $s_0 = s_1 = 3u(2q+1)$, $g_0 = g_1 = u(2q+1)$, and $\lambda_0 = \lambda_1 = u(6q^2 + 6q + 1)$, can be constructed by using the following two initiate designs d_1 for b = 6, k = 3, and d_2 for b = 6, k = 2, respectively, where

$$
d_1: \begin{array}{cccccc} (0,1) & (0,2) & (0,3) & (0,1) & (0,2) & (0,3) \\ (0,2) & (0,3) & (0,1) & (1,2) & (1,3) & (2,3) \\ (2,3) & (1,3) & (1,2) & (2,3) & (1,3) & (1,2) \end{array},
$$

$$
d_2: \begin{array}{cccccc} (0,1) & (0,2) & (0,3) & (0,1) & (0,2) & (0,3) \\ (2,3) & (1,3) & (1,2) & (2,3) & (1,3) & (1,2) \end{array},
$$

and

$$
d_{III}: \begin{array}{l} \overbrace{\begin{array}{cccc} d_1 & d_1 & \cdots & d_1 \end{array}}^{u} \\ \left.\begin{array}{cccc} d_2 & d_2 & \cdots & d_2 \\ d_2 & d_2 & \cdots & d_2 \\ \vdots & \vdots & \ddots & \vdots \\ d_2 & d_2 & \cdots & d_2 \end{array}\right\} q-1. \end{array} \tag{3.3}
$$

Example 3.5. For b = 6, k = 5, that is, $u=1$, $q=2$, the following design d_{III} is a S_0(3, 6, 5, 5, 5, 37, 37) design.

(0,1)	(0,2)	(0,3)	(0,1)	(0,2)	(0,3)
(0,2)	(0,3)	(0,1)	(1,2)	(1,3)	(2,3)
(2,3)	(1,3)	(1,2)	(2,3)	(1,3)	(1,2)
(0,1)	(0,2)	(0,3)	(0,1)	(0,2)	(0,3)
(2,3)	(1,3)	(1,2)	(2,3)	(1,3)	(1,2)

Example 3.6. For b = 12, k = 7, that is, $u=2$, $q=3$, the following design d_{III} is a S_0(3, 12, 7, 14, 14, 146, 146) design.

(0,1)	(0,2)	(0,3)	(0,1)	(0,2)	(0,3)	(0,1)	(0,2)	(0,3)	(0,1)	(0,2)	(0,3)
(0,2)	(0,3)	(0,1)	(1,2)	(1,3)	(2,3)	(0,2)	(0,3)	(0,1)	(1,2)	(1,3)	(2,3)
(2,3)	(1,3)	(1,2)	(2,3)	(1,3)	(1,2)	(2,3)	(1,3)	(1,2)	(2,3)	(1,3)	(1,2)
(0,1)	(0,2)	(0,3)	(0,1)	(0,2)	(0,3)	(0,1)	(0,2)	(0,3)	(0,1)	(0,2)	(0,3)
(2,3)	(1,3)	(1,2)	(2,3)	(1,3)	(1,2)	(2,3)	(1,3)	(1,2)	(2,3)	(1,3)	(1,2)
(0,1)	(0,2)	(0,3)	(0,1)	(0,2)	(0,3)	(0,1)	(0,2)	(0,3)	(0,1)	(0,2)	(0,3)
(2,3)	(1,3)	(1,2)	(2,3)	(1,3)	(1,2)	(2,3)	(1,3)	(1,2)	(2,3)	(1,3)	(1,2)

As for the efficiency of a design $d \in D(p+1, b, k)$, the same definition is used as in Das, Gupta, and Kageyama (2002), that is,

$$
E(d) = \sum_{i=1}^{p} Var(\hat{\tau}_{d_{opt}i} - \hat{\tau}_{d_{opt}0}) \Big/ \sum_{i=1}^{p} Var(\hat{\tau}_{di} - \hat{\tau}_{d0}), \tag{3.4}
$$

where d_{opt} is an A-optimal design in $D(p+1,b,k)$. Now for $p=3$, and by applying inequality (3.1),

$$\sum_{i=1}^{p} Var(\hat{\tau}_{d_{opt}i} - \hat{\tau}_{d_{opt}0}) \geq (3bk)g*(bk/2;3,b,k)\sigma^2 = (18/bk)\sigma^2,$$

hence, $E(d) \geq (18/bk)\sigma^2 \Big/ \sum_{i=1}^{p} Var(\hat{\tau}_{di} - \hat{\tau}_{d0}) = e(d)$, say.

For design d_{III} in case (III), through some straightforward calculation one can see that $\sigma^{-2}\sum_{i=1}^{p} Var(\hat{\tau}_{d_{III}i} - \hat{\tau}_{d_{III}0}) = 3(2q+1)/4uq(q+1)$ and $e(d_{III}) = 1 - 1/(2q+1)^2$. For $q=1$, that is, $b=6u$, k = 3, $E(d_{III}) \geq$ 0.89, for $q \geq 2$, that is, $b=6u$, $k \geq$ 5 is an odd number, $E(d_{III}) \geq$ 0.96. One can thus conclude that the type S_0 block designs in case (III) are highly efficient designs in $D(3+1,6u,2q+1)$.

3.2. Efficient Designs for p = 2

For $p = 2$, then $g^*(s_{d0};2,b,k) = 1/(bks_{d0} - s_{d0}^2) + 1/bks_{d0}$, by taking the derivative of $g^*(s_{d0};2,b,k)$ with respect to s_{d0}, the minimum value of $g^*(s_{d0};2,b,k)$ is achieved at $s_{d0} = s_0 = bk\left(2-\sqrt{2}\right)$, and $g^*(s_{d0};2,b,k) = \left(3+2\sqrt{2}\right)/(bk)^2$. Now $bk\left(2-\sqrt{2}\right)$ is not an integer, the method we use in section 3.1 to find families of A-optimal designs for p = 3 cannot be applied here. Though $bk\left(2-\sqrt{2}\right)$ is not an integer, it is close to $bk/2$, hence, in the following, the efficiencies of type S_0 block designs having $s_0 = bk/2$ are investigated.

A type S_0 block design $S_0(2, b, k, g_0, g_1, \lambda_0, \lambda_1)$ with $s_0 = bk/2$, has the following values for $s_1, g_0, g_1, \lambda_0, \lambda_1$, and

$$s_1 = 3bk/4, \quad g_0 = bk/4, \quad g_1 = bk/2,$$

$$\lambda_0 = \begin{cases} 3bk^2/8, & \text{if } k \text{ is even,} \\ b(3k^2-1)/8, & \text{if } k \text{ is odd,} \end{cases}$$

$$\lambda_1 = \begin{cases} 9bk^2/16, & \text{if } k = 0 \pmod 4, \\ 9bk^2/8 - h(s_1), & \text{if } k \text{ is even, and } k \neq 0 \pmod 4, \\ b(9k^2+1)/8 - h(s_1), & \text{otherwise,} \end{cases}$$

where $h(\cdot)$ is as in chapter 2. For these designs to exist, $s_0, s_1, g_0, g_1, \lambda_0, \lambda_1$ must all be integers, and the possible combinations of the values of such b and k are as follows.

(I) $k = 0 \pmod 4$,

(II) $b = 0 \pmod 4$, $k = 1 \pmod 4$,

(III) $b = 0 \pmod 2$, $k = 2 \pmod 4$,

(IV) $b = 0 \pmod 4$, $k = 3 \pmod 4$.

Using the same efficiency definition in (3.4), for p = 2, and $d \in D(2+1, b, k)$, the lower bound to the efficiency is obtained as follows.

$$e(d) = \frac{2\left(3+2\sqrt{2}\right)\sigma^2 / bk}{\sum_{i=1}^{p} Var(\hat{\tau}_{di} - \hat{\tau}_{d0})}.$$

For designs $\tilde{d}_I$, $\tilde{d}_{II}$, $\tilde{d}_{III}$, and $\tilde{d}_{IV}$ in cases (I), (II), (III), and (IV), respectively, through some straightforward calculation, one has

(I) $\sigma^{-2} \sum_{i=1}^{p} Var(\hat{\tau}_{\tilde{d}_I i} - \hat{\tau}_{\tilde{d}_I 0}) = g(s_0; p, b, k) = 12 / bk$, and

$$E(\tilde{d}_I) \geq e(\tilde{d}_I) = \left(3+2\sqrt{2}\right)/6 = 0.9714,$$

(II) $\sigma^{-2} \sum_{i=1}^{p} Var(\hat{\tau}_{\tilde{d}_{II} i} - \hat{\tau}_{\tilde{d}_{II} 0}) = g(s_0; p, b, k) = 12 / (bk - b/k)$, and

$$E(\tilde{d}_{II}) \geq e(\tilde{d}_{II}) = \left(3+2\sqrt{2}\right)\left(1 - 1/k^2\right)/6,$$

(III) $\sigma^{-2} \sum_{i=1}^{p} Var(\hat{\tau}_{\tilde{d}_{III} i} - \hat{\tau}_{\tilde{d}_{III} 0}) = g(s_0; p, b, k) = 8 / bk + 4/(bk - 2b/k))$, and

$$E(\tilde{d}_{III}) \geq e(\tilde{d}_{III}) = \left(3+2\sqrt{2}\right)/\left(4 + 2/(1 - 2/k^2)\right),$$

(IV) $\sigma^{-2} \sum_{i=1}^{p} Var(\hat{\tau}_{\tilde{d}_{IV} i} - \hat{\tau}_{\tilde{d}_{IV} 0}) = g(s_0; p, b, k) = 12 / (bk - b/k)$, and

$$E(\tilde{d}_{IV}) \geq e(\tilde{d}_{IV}) = \left(3+2\sqrt{2}\right)\left(1 - 1/k^2\right)/6.$$

All of the above lower bounds to the efficiencies are greater than or equal to 0.9325, the only two exceptions are in case (III), when k = 2, and $E(\tilde{d}_{III}) \geq 0.7286$, and in case (IV), when k = 3, and $E(\tilde{d}_{IV}) \geq 0.8635$. For $k = 2$, $E(\tilde{d}_{III})$ is reexamined by using the function $g(s_{d0}; p, b, k)$, instead of $g^*(s_{d0}; p, b, k)$. For $k = 2$,

$$y_1 = \left\lceil \frac{2bk - s_{d0}}{2b} \right\rceil = 1, \quad a(s_{d0}) = 8b - 3s_{d0} \text{ for } 0 < s_{d0} < 2b,$$

$$y_2 = \begin{cases} 0, \text{ if } 0 < s_{d0} < b, \\ 1, \text{ if } b \leq s_{d0} < 2b, \end{cases}$$

$$h(s_{d0}) = \begin{cases} s_{d0}, & \text{if } 0 < s_{d0} < b, \\ 3s_{d0} - 2b, & \text{if } b \leq s_{d0} < 2b. \end{cases}$$

Hence

$$g(s_{d0}; 2, b, 2) = \begin{cases} 8/s_{d0}, & \text{if } 0 < s_{d0} < b, \\ 4/(2b - s_{d0}) + 4/(3s_{d0} - 2b), & \text{if } b \leq s_{d0} < 2b. \end{cases}$$

Furthermore, $g(s_{d0}; 2, b, 2)$ is decreasing in s_{d0} when $0 < s_{d0} < 2\sqrt{3}b/3$, and is increasing in s_{d0} when $2\sqrt{3}b/3 \leq s_{d0} < 2b$. Therefore, the minimum value of $g(s_{d0}; 2, b, 2)$ occurs at $s_{d0} = \left[2\sqrt{3}b/3\right]$ or $\left[2\sqrt{3}b/3\right] + 1$. And $E(\tilde{d}_{III}) \geq \min\Big(g\big(\left[2\sqrt{3}b/3\right]; 2, b, 2\big), g\big(\left[2\sqrt{3}b/3\right] + 1; 2, b, 2\big)\Big) \Big/ \sigma^2 \sum_{i=1}^{p} Var(\hat{\tau}_{di} - \hat{\tau}_{d0})$. By using a computer, one has $E(\tilde{d}_{III}) = 1$ for $b = 2$, that is, $\tilde{d}_{III}$ is an A-optimal design for $b = 2$ and $k = 2$, and $E(\tilde{d}_{III}) \geq 0.933$ for a practical range of *b*, that is, $4 \leq b \leq 50$. One can therefore conclude that the type S_0 block designs in the above four cases are highly efficient designs in $D(2 + 1, b, k)$ for $k \geq 3$.

The efficient type S_0 block designs can be constructed by using the same techniques as in section 3.1.

Case (I): For $k = 0 \pmod 4$, that is, $b = u$, $k = 4q$, where $u, q \ge 1$ are integers. By replacing d_A with $\tilde{d}_1$ in (3.2), an efficient $S_0(2, u, 4q, uq, 2uq, 6uq^2, 9uq^2)$ design $\tilde{d}_I$ can thus be constructed, and

$$\tilde{d}_1: \begin{array}{c} (0,1) \\ (0,2) \\ (1,2) \\ (1,2) \end{array}.$$

Example 3.7. For b = 2, k = 4, that is, u = 2, q = 1, the following design $\tilde{d}_I$ is a S_0(2, 2, 4, 2, 4, 12, 18) design, and $e(\tilde{d}_I) = 0.9714$.

$$\begin{array}{cc} (0,1) & (0,1) \\ (0,2) & (0,2) \\ (1,2) & (1,2) \\ (1,2) & (1,2) \end{array}$$

Case (II): For $b = 0 \pmod 4$, $k = 1 \pmod 4$, that is, $b = 4u$, $k = 4q+1$, where u, $q \ge 1$ are integers. By replacing d_1 with $\tilde{d}_{21}$, and d_2 with $\tilde{d}_{22}$ in (3.3), an efficient $S_0(2, 4u, 4q+1, u(4q+1), 2u(4q+1), u(24q^2+12q+1), 2u(18q^2+9q+1))$ design $\tilde{d}_{II}$ can thus be constructed, and

$$\tilde{d}_{21}: \begin{array}{cccc} (0,1) & (0,2) & (0,1) & (0,2) \\ (0,2) & (0,1) & (0,2) & (0,1) \\ (0,2) & (0,1) & (1,2) & (1,2) \\ (1,2) & (1,2) & (1,2) & (1,2) \\ (1,2) & (1,2) & (1,2) & (1,2) \end{array},$$

$$\tilde{d}_{22}: \begin{array}{cccc} (0,1) & (0,2) & (0,1) & (0,2) \\ (0,2) & (0,1) & (0,2) & (0,1) \\ (1,2) & (1,2) & (1,2) & (1,2) \\ (1,2) & (1,2) & (1,2) & (1,2) \end{array}.$$

Example 3.8. For b = 8, k = 9, that is, $u = q = 2$, the following design $\tilde{d}_{II}$ is a S_0(2, 8, 9, 18, 36, 242, 364) design, and $e(\tilde{d}_{II}) = 0.9594$.

(0,1)	(0,2)	(0,1)	(0,2)	(0,1)	(0,2)	(0,1)	(0,2)
(0,2)	(0,1)	(0,2)	(0,1)	(0,2)	(0,1)	(0,2)	(0,1)
(0,2)	(0,1)	(1,2)	(1,2)	(0,2)	(0,1)	(1,2)	(1,2)
(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)
(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)
(0,1)	(0,2)	(0,1)	(0,2)	(0,1)	(0,2)	(0,1)	(0,2)
(0,2)	(0,1)	(0,2)	(0,1)	(0,2)	(0,1)	(0,2)	(0,1)
(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)
(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)

Example 3.9. For b = 8, k = 13, that is, $u = 2$, $q = 3$, the following design $\tilde{d}_{II}$ is a S_0(2, 8, 13, 26, 52, 506, 760) design, and $e(\tilde{d}_{II}) = 0.9657$.

(0,1)	(0,2)	(0,1)	(0,2)	(0,1)	(0,2)	(0,1)	(0,2)
(0,2)	(0,1)	(0,2)	(0,1)	(0,2)	(0,1)	(0,2)	(0,1)
(0,2)	(0,1)	(1,2)	(1,2)	(0,2)	(0,1)	(1,2)	(1,2)
(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)
(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)
(0,1)	(0,2)	(0,1)	(0,2)	(0,1)	(0,2)	(0,1)	(0,2)
(0,2)	(0,1)	(0,2)	(0,1)	(0,2)	(0,1)	(0,2)	(0,1)
(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)
(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)
(0,1)	(0,2)	(0,1)	(0,2)	(0,1)	(0,2)	(0,1)	(0,2)
(0,2)	(0,1)	(0,2)	(0,1)	(0,2)	(0,1)	(0,2)	(0,1)
(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)
(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)

Case (III): For $b = 0$ (mod 2), $k = 2$ (mod 4), that is, $b = 2u$, $k = 4q - 2$, where $u, q \geq 1$ are integers. By replacing d_1 with $\tilde{d}_{31}$, and d_2 with $\tilde{d}_{32}$ in (3.3), an efficient $S_0(2,\ 2u, 4q-2,\ u(2q-1),\ 2u(2q-1),\ 3u(2q-1)^2,\ 2u(9q^2-9q+2))$

design $\tilde{d}_{III}$ can thus be constructed, and

$$\tilde{d}_{31}: \begin{array}{cc} (0,1) & (0,2) \\ (1,2) & (1,2) \end{array},$$

$$\tilde{d}_{32}: \begin{array}{cc} (0,1) & (0,2) \\ (0,2) & (0,1) \\ (1,2) & (1,2) \\ (1,2) & (1,2) \end{array}.$$

Example 3.10. For b = 4, k = 6, that is, $u = q = 2$, the following design $\tilde{d}_{III}$ is a S_0(2, 4, 6, 6, 12, 54, 80) design, and $e(\tilde{d}_{III}) = 0.9527$.

$$\begin{array}{cccc} (0,1) & (0,2) & (0,1) & (0,2) \\ (1,2) & (1,2) & (1,2) & (1,2) \\ (0,1) & (0,2) & (0,1) & (0,2) \\ (0,2) & (0,1) & (0,2) & (0,1) \\ (1,2) & (1,2) & (1,2) & (1,2) \\ (1,2) & (1,2) & (1,2) & (1,2) \end{array}$$

Case (IV): For $b = 0 \text{ (mod 4)}, k = 3 \text{ (mod 4)}$, that is, $b = 4u, k = 4q - 1$, where $u, q \geq 1$ are integers. By replacing d_1 with $\tilde{d}_{41}$, and d_2 with $\tilde{d}_{22}$ (as in Case(II)) in (3.3), an efficient $S_0(2, 4u, 4q-1, u(4q-1), 2u(4q-1), u(24q^2-12q+1), 2u(18q^2-9q+1))$ design $\tilde{d}_{IV}$ can thus be constructed, and

$$\tilde{d}_{41}: \begin{array}{cccc} (0,1) & (0,2) & (0,1) & (0,2) \\ (0,2) & (0,1) & (1,2) & (1,2) \\ (1,2) & (1,2) & (1,2) & (1,2) \end{array}.$$

Example 3.11. For b = 8, k = 7, that is, $u = q = 2$, the following design $\tilde{d}_{IV}$ is a S_0(2, 8, 7, 14, 28, 146, 220) design, and $e(\tilde{d}_{IV}) = 0.9516$.

(0,1)	(0,2)	(0,1)	(0,2)	(0,1)	(0,2)	(0,1)	(0,2)
(0,2)	(0,1)	(1,2)	(1,2)	(0,2)	(0,1)	(1,2)	(1,2)
(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)
(0,1)	(0,2)	(0,1)	(0,2)	(0,1)	(0,2)	(0,1)	(0,2)
(0,2)	(0,1)	(0,2)	(0,1)	(0,2)	(0,1)	(0,2)	(0,1)
(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)
(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)

3.3. Optimal and Efficient Designs for $p \geq 4$, $k = 2$

For $p \geq 4$, and $k = 2$, one can obtain that

$$a(s_{d0}) = 4b - s_{d0}, \text{ and}$$

$$h(s_{d0}) = \begin{cases} s_{d0}, & \text{if } 0 < s_{d0} < b, \\ 3s_{d0} - 2b, & \text{if } b \leq s_{d0} < 2b, \end{cases}$$

therefore

$$g(s_{d0}; p, b, 2) = \begin{cases} 2p\left(\dfrac{1}{s_{d0}} + \dfrac{(p-1)^2}{4bp - (p+1)s_{d0}}\right), & \text{if } 0 < s_{d0} < b, \\ 2p\left(\dfrac{1}{2b - s_{d0}} + \dfrac{(p-1)^2}{2b(2p-1) - (p-1)s_{d0}}\right), & \text{if } b \leq s_{d0} < 2\,b. \end{cases}$$

Theorem 3.3. (i) For $4 \leq p \leq 9$, $k = 2$, a type S_0 block design $S_0(p, b, 2, g_0, g_1, \lambda_0, \lambda_1)$ with $s_{d0} = b$, $s_1 = 3b/p$, $g_0 = b/p$, $g_1 = 2b/p(p-1)$, $\lambda_0 = 3b/p$, and $\lambda_1 = 6b/p(p-1)$, if exists, is A-optimal in $D(p+1, b, 2)$;

(ii) For $p \geq 10$, $k = 2$, a type S_0 block design $S_0(p, b, 2, g_0, g_1, \lambda_0, \lambda_1)$ with $s_{d0} = s_0$, $s_1 = (4b - s_0)/p$, $g_0 = s_0/p$, $g_1 = 2(2b - s_0)/p(p-1)$, $\lambda_0 = 3s_0/p$, and $\lambda_1 = 6(2b - s_0)/p(p-1)$, if exists, is A-optimal in $D(p+1, b, 2)$ where s_0 is obtained by

$$g(s_0; p, b, 2) = \min\left(g([s_0^*]; p, b, 2),\ g([s_0^*] + 1; p, b, 2)\right),$$

$$s_0^* = \frac{4b}{p-3}\left(\frac{p-1}{\sqrt{p+1}} - 1\right).$$

Proof: For $0 < s_{d0} < b$, taking the derivative of $g(s_{d0}; p, b, 2)$ with respect to

s_{d0}, one has

$$\frac{d}{ds_{d0}} g(s_{d0}; p, b, 2) = \frac{2p^2 l_1(s_{d0})}{\left(4bp-(p+1)s_{d0}\right)^2 s_{d0}^2}, \text{ where}$$

$l_1(s_{d0}) = (p+1)(p-3)s_{d0}^2 + 8b(p+1)s_{d0} - 16pb^2$, and $dg(s_{d0}; p, b, 2)/ds_{d0} = 0$ if and only if $l_1(s_{d0}) = 0$. Now the positive root of $l_1(s_{d0})$ is

$$s_{d0} = \frac{4b}{p-3}\left(\frac{p-1}{\sqrt{p+1}} - 1\right) = s_0^*, \text{ say,}$$

and $0 < s_0^* < b$ for $p \geq 10$, $b < s_0^* < 2b$, for $4 \leq p \leq 9$. Hence, for $p \geq 10$, $l_1(s_{d0}) < 0$ if $0 < s_{d0} < s_0^*$, and $l_1(s_{d0}) > 0$ if $s_{d0} > s_0^*$, therefore, $g(s_{d0}; p, b, 2)$ is decreasing in s_{d0} whenever $0 < s_{d0} < s_0^*$, and is increasing in s_{d0} whenever $s_{d0} > s_0^*$. As for $4 \leq p \leq 9$, $l_1(s_{d0}) < 0$, $0 < s_{d0} < b$, that is, $g(s_{d0}; p, b, 2)$ is decreasing in s_{d0} for $0 < s_{d0} < b$.

For $b \leq s_{d0} < 2b$, taking the derivative of $g(s_{d0}; p, b, 2)$ with respect to s_{d0}, one has

$$\frac{d}{ds_{d0}} g(s_{d0}; p, b, 2) = \frac{2pl_2(s_{d0})}{(2b-s_{d0})^2\left(2(2p-1)b-(p-1)s_{d0}\right)^2},$$

where $l_2(s_{d0}) = p(p-1)^2 s_{d0}^2 - 4(p-1)bp^2 s_{d0} + 4b^2\left((2p-1)^2 + (p-1)^3\right)$.

Since

$$16\left((p-1)pb\right)^2 - 16p\left((p-1)b\right)^2\left((2p-1)^2 + (p-1)^3\right)$$

$$= -16(pb)^2(p-1)^3 < 0,$$

one has $l_2(s_{d0}) > 0$, and thus $g(s_{d0}; p, b, 2)$ is increasing in s_{d0} for

$b \le s_{d0} < 2b$.

Combining the above results, for $4 \le p \le 9$, the minimum value of $g(s_{d0}; p, b, 2)$ occurs at $s_{d0} = b$, and for $p \ge 10$, the minimum value of $g(s_{d0}; p, b, 2)$ occurs at $s_{d0} = s_0$, and $s_0 = [s_0^*]$ or $[s_0^*]+1$ depends on which one has a smaller $g(s_{d0}; p, b, 2)$ value. The theorem is proved.

For $4 \le p \le 9$, some families of A-optimal type S_0 block designs are given, and their construction methods are illustrated.

Example 3.12. For $p = 5$, $b = 10u$, where $u \ge 1$ is an integer, an A-optimal S_0(5, 10*u*, 2, 2*u*, *u*, 6*u*, 3*u*) design can be constructed by repeating the following design *u* times in the row direction.

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(5,2)	(1,3)	(2,4)	(3,5)	(4,1)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(4,3)	(5,4)	(1,5)	(2,1)	(3,2)

Example 3.13. For $p = 7$, $b = 21u$, where $u \ge 1$ is an integer, an A-optimal S_0(7, 21*u*, 2, 3*u*, *u*, 9*u*, 3*u*) design can be constructed by repeating the following design *u* times in the row direction.

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
(7,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,1)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
(6,3)	(7,4)	(1,5)	(2,6)	(3,7)	(4,1)	(5,2)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
(5,4)	(6,5)	(7,6)	(1,7)	(2,1)	(3,2)	(4,3)

Example 3.14. For $p = 9$, b = 36u, where $u \geq 1$ is an integer, an A-optimal S_0(9, 36u, 2, 4u, u, 12u, 3u) design can be constructed by repeating the following design u times in the row direction.

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)
(9,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,1)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)
(8,3)	(9,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,1)	(7,2)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)
(7,4)	(8,5)	(9,6)	(1,7)	(2,8)	(3,9)	(4,1)	(5,2)	(6,3)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)
(6,5)	(7,6)	(8,7)	(9,8)	(1,9)	(2,1)	(3,2)	(4,3)	(5,4)

Example 3.15. For $p = 4$, b = 12u, where $u \geq 1$ is an integer, an A-optimal S_0(4, 12u, 2, 3u, 2u, 9u, 6u) design can be constructed by repeating the following design u times in the row direction.

(0,1)	(0,2)	(0,3)	(0,4)
(3,2)	(1,3)	(2,1)	(1,3)
(0,1)	(0,2)	(0,3)	(0,4)
(2,4)	(3,4)	(1,4)	(2,1)
(0,1)	(0,2)	(0,3)	(0,4)
(3,4)	(1,4)	(2,4)	(3,2)

Example 3.16. For $p = 6$, b = 30u, where $u \geq 1$ is an integer, an A-optimal S_0(6, 30u, 2, 5u, 2u, 15u, 6u) design can be constructed by repeating the following design u times in the row direction.

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(5,2)	(1,3)	(2,4)	(3,5)	(4,1)	(1,3)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(4,3)	(5,4)	(1,5)	(2,1)	(3,2)	(2,4)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(3,2)	(4,3)	(5,4)	(1,5)	(2,1)	(3,5)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(4,6)	(5,6)	(1,6)	(2,6)	(3,6)	(4,1)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(5,6)	(1,6)	(2,6)	(3,6)	(4,6)	(5,2)

Example 3.17. For $p = 8$, b = 56u, where $u \geq 1$ is an integer, an A-optimal S_0(8, 56u, 2, 7u, 2u, 21u, 6u) design can be constructed by repeating the following design u times in the row direction.

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)
(7,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,1)	(1,3)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)
(6,3)	(7,4)	(1,5)	(2,6)	(3,7)	(4,1)	(5,2)	(2,4)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)
(5,4)	(6,5)	(7,6)	(1,7)	(2,1)	(3,2)	(4,3)	(3,5)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)
(4,3)	(5,4)	(6,5)	(7,6)	(1,7)	(2,1)	(3,2)	(4,6)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)
(5,2)	(6,3)	(7,4)	(1,5)	(2,6)	(3,7)	(4,1)	(5,7)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)
(6,8)	(7,8)	(1,8)	(2,8)	(3,8)	(4,8)	(5,8)	(6,1)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)
(7,8)	(1,8)	(2,8)	(3,8)	(4,8)	(5,8)	(6,8)	(7,2)

As for $p \geq 10$, since

$$g(b;p,b,2) = 2p\left(\frac{1}{b} + \frac{(p-1)^2}{4pb-(p+1)b}\right) = \frac{2(p+1)p^2}{(2p-1)b}, \text{ and}$$

$$g(s_0^*;p,b,2) = \frac{(p-3)\left(2p\sqrt{p+1}+p-1\right)}{4\left(p-1-\sqrt{p+1}\right)b}, \text{ one has}$$

$E(d) \geq g^*(b;p,b,k)/g(b;p,b,k) = (3p-1)\left(p-1+\sqrt{p+1}\right)^2 / 4(p+1)p^2$

≥ 0.9529 for $10 \leq p \leq 30$. Therefore, one can still obtain high efficiency for $s_{d0} = b$ even if the A-optimal type S_0 block designs $S_0(p, b, 2, g_0, g_1, \lambda_0, \lambda_1)$ do not exist for $p \geq 10$ by our method. The construction methods for $p \geq 10$, and $k = 2$ are similar to Example 3.12 to 3.17, and list in the following.

Series 1. For p is odd, consider the initial blocks $\{(C,1), (0,2)\}$, $\{(C,1), (p-1,3)\}, \{(C,1), (p-2,4)\}, \cdots, \{(C,1), ((p+3)/2, (p+1)/2)\}$, Cyclically developing all these initial blocks, mod *p*, where *C* denotes the control line, 0 denotes the *p*th test line, and control line *C* is unchanged during the cyclical procedure, will yields a type S_0 block design $S_0(p, p(p-1)/2, 2, (p-1)/2, 1, 3(p-1)/2, 3)$. A type S_0 block design $S_0(p, up(p-1)/2, 2, u(p-1)/2, u, 3u(p-1)/2, 3u)$ can be constructed by repeating the

$S_0(p,\ p(p-1)/2,\ 2,\ (p-1)/2, 1,\ 3(p-1)/2,\ 3)$ design u times in the row direction.

Series 2. For p is even, consider the initial blocks $\{(C,1)\,,(0,2)\}$, $\{(C,1)\,,(p-2,3)\},\ldots,\{(C,1)\,,(p/2+1,\ p/2)\},\ \{(C,1)\,,(p/2,\ p/2-1)\}$, $\{(C,1),(p/2+1,\ p/2-2)\},\ldots,\{(C,1),(p-3,2)\},\ \{(C,1),(p-2\ ,\ p)\},\ \{(C,1),(0\ ,\ p)\}$. Cyclically developing all these initial blocks, mod $p-1$, where C denotes the control line, 0 denotes the (p – 1)th test line, and control line C and test line p are unchanged during the cyclical procedure, then add blocks $\{(C,p)\,,(1,3)\}$, $\{(C,p)\,,(2,4)\},\ldots,\{(C,p)\,,(p-4,\ p-2)\},\{(C,p)\,,(p-3,0)\},\{(C,p)\,,(p-2,1)\}$, $\{(C,p),(0,2)\}$, mod $p-1$, with control line C and test line p unchanged, will yields a type S_0 block design $S_0(p,\ p(p-1),\ 2,\ p-1,\ 2,\ 3(p-1),\ 6)$. A type S_0 block design $S_0(p,\ up(p-1),\ 2,\ u(p-1),\ 2u,\ 3u(p-1),\ 6u)$ can be constructed by repeating $S_0(p,\ p(p-1),\ 2,\ p-1,\ 2,\ 3(p-1),\ 6)$ design u times in the row direction.