Chapter 4

A-Optimal Designs for $p \geq k$

Due to the complexity of $g(s_{d0}; p,b,k)$, the problem of finding families of A-optimal designs for $p \geq k \geq 3$, and $p \geq 4$ is subdivided into $p > 2k - s_{d0}/b$ and $2k - s_{d0}/b \geq p \geq k \geq 3$, that is, when $[(2bk - s_{d0})/pb] = 0$ and $[(2bk - s_{d0})/pb] = 1$, respectively.

4.1. A-Optimal Designs for $[(2bk - s_{d0})/pb] = 0$

For $[(2bk - s_{d0})/pb] = 0$, or $p > 2k - s_{d0}/b$, one has $y_1 = 0$, $a(s_{d0}) = 2bk - s_{d0}$, and

$$g(s_{d0}; p,b,k) = \frac{(p-1)^2}{2b(k-1) - (p(k-1)+k)s_{d0}/pk + h(s_{d0})/pk} + \frac{p}{s_{d0} - h(s_{d0})/k}.$$

Let $\Lambda_1 = \{(0,z) : z = 1,\cdots,b\}$, $\Lambda_2 = \{(x,z) : x = 1,\cdots,[k/2]-1; z = 0,1,\cdots,b\}$, and $\Lambda = \Lambda_1 \cup \Lambda_2$. Let $s_{d0} = bx + z$, then $y_2 = [s_{d0}/b] = x$ and $h(s_{d0}) = s_{d0}(2y_2+1) - by_2(y_2+1) = (bx+z)(2x+1) - bx(x+1) = bx^2 + 2zx + z$. If we denote $g(s_{d0}; p,b,k)$ by $g(x,z)$, then

$$g(x,z) = \frac{pk(p-1)^2}{2pbk(k-1) - (p(k-1)+k)(bx+z) + bx^2 + 2zx + z}$$

$$+\frac{p}{(bx+z)-(bx^2+2zx+z)/k}$$

$$=pk\left(\frac{(p-1)^2}{2a_1-a_2(bx+z)+bx^2+2zx+z}+\frac{1}{k(bx+z)-(bx^2+2zx+z)}\right),$$

where $a_1=pbk(k-1)$, $a_2=p(k-1)+k$. Note that $g(x,0)=g(x-1,b)$ for $x\geq 1$. Then the s_0 value found by minimizing $g(s_{d0};p,b,k)$ over $1\leq s_{d0}\leq b[k/2]$ in (i) of Theorem 2.2 can be found by minimizing $g(x,z)$ over $(x,z)\in\Lambda$, instead, that is, $s_0=bt+s$, and $g(t,s)=\min_{(x,z)\in\Lambda} g(x,z)$. In the following, a similar procedure as that in Hedayat and Majumdar (1985) is followed to find families of A-optimal designs.

Lemma 4.1. For fixed value of *x*, $0\leq x\leq [k/2]-1$, there exists z_1 ($0\leq z_1\leq b$), a function of *x*, such that $g(x,z)$ decreases in *z* when $0\leq z\leq z_1$ and increases in *z* when $z_1\leq z\leq b$. If $z_1=0$, $g(x,z)$ increases in *z*, and if $z_1=b$, then $g(x,z)$ decreases in *z*.

Proof: Taking the derivative of $g(x,z)$ with respective to *z*, one has

$$\frac{\partial}{\partial z}g(x,z)=pk\left(\frac{-(p-1)^2(2x+1-a_2)}{\left(2a_1-a_2(bx+z)+bx^2+2zx+z\right)^2}\right.$$

$$\left.-\frac{k-2x-1}{\left(k(bx+z)-(bx^2+2zx+z)\right)^2}\right),$$

and the sign of $\partial g(x,z)/\partial z$ is the same as the sign of

$$-(p-1)^2(2x+1-a_2)\left(k(bx+z)-(bx^2+2zx+z)\right)^2$$

$$-(k-2x-1)\left(2a_1-a_2(bx+z)+bx^2+2zx+z\right)^2$$

$= \Phi(z)$, say.

Now let

$$\varphi_2(x) = -(p-1)^2(2x+1-a_2)(k-2x-1)^2 - (k-2x-1)(2x+1-a_2)^2$$

$$= p(a_2-2x-1)(k-2x-1)\big((p-2)(k-2x-1)-(k-1)\big),$$

$$\varphi_1(x) = -b(p-1)^2(2x+1-a_2)(k-2x-1)x(k-x)$$

$$-(k-2x-1)(2x+1-a_2)(2a_1-a_2bx+bx^2)$$

$$= pb(a_2-2x-1)(k-2x-1)\big(x(k-x)(p-2)+(k-1)(2k-x)\big),$$

$$\varphi_0(x) = \big((p-1)b\big)^2(2x+1-a_2)\big(x(k-x)\big)^2 + (k-2x-1)(2a_1-a_2bx+bx^2)^2$$

$$= b^2\Big((p-1)^2(2x+1-a_2)\big(x(k-x)\big)^2$$

$$+(k-2x-1)\big(p(k-1)(2k-x)+x(x-k)\big)^2\Big).$$

Then $\Phi(z)$ can be rewritten as $\Phi(z) = \varphi_2(x)z^2 + 2\varphi_1(x)z - \varphi_0(x)$. For x in $[0,[k/2]-1]$, one can see that

$$k-2x-1 \ge k-2[k/2]+1 \ge k-k+1 = 1, \qquad (4.1)$$

$$a_2-2x-1 = pk-p+k-2x-1 > k-2x-1 \ge 1. \qquad (4.2)$$

And through some straightforward calculation one can show that $x(k-x)(p-2)$ $+(k-1)(2k-x) > 0$ and $(p-2)(k-2x-1)-(k-1) > 0$, hence $\varphi_1(x) > 0$, and $\varphi_2(x) > 0$. Now $\partial\Phi(z)/\partial z = 2z\varphi_2(x) + 2\varphi_1(x) \ge 0$ for $x \in [0,[k/2]-1]$, hence $\Phi(z)$ is increasing in *z*, therefore $\Phi(z)$ is either all negative, or increasing from negative to positive, or all positive for $0 \le z \le b$. Since the signs of $\partial g(x,z)/\partial z$ and $\Phi(z)$ are the same, the lemma is thus proved.

Lemma 4.2. Suppose $0 \le x \le [k/2]-1$. Then a necessary and sufficient condition for

(i) $g(x,0) = \min_{0 \le z \le b} g(x,z)$ is $g(x,0) \le g(x,1)$,

(ii) $g(x,b) = \min_{0 \le z \le b} g(x,z)$ is $g(x,b) \le g(x,b-1)$.

Proof: The necessary conditions for both cases are obvious, and the procedure of proving the sufficient condition for (ii) is similar to that of proving (i), and is thus omitted.

Now from the proof of Lemma 4.1, $g(x,0) \le g(x,1)$ implies that $\Phi(z) > 0$, that is, $g(x,z)$ is increasing in *z*, (i) is thus proved.

Lemma 4.3. (i) Suppose $0 < t \le [k/2]-1$, then $g(t,0) \le g(t,1)$ implies

$$g(x,0) \le g(x,1) \quad \text{for} \quad t \le x \le [k/2]-1,$$

(ii) Suppose $0 < t \le [k/2]-1$, then $g(t,b) \le g(t,b-1)$ implies

$$g(x,b) \le g(x,b-1) \quad \text{for} \quad 0 \le x \le t.$$

Proof: (i) Let $h_1(x) = 2a_1 + bx(x-a_2)$, $h_2(x) = bx(k-x)$,

$h_3(x) = 2a_1 + bx(x-a_2) - (a_2 - 2x - 1)$ and $h_4(x) = bx(k-x) + (k-2x-1)$,

then

$$g(x,0) - g(x,1)$$

$$= \frac{pk\Big(\big((p-1)^2 h_2(x) + h_1(x)\big)h_3(x)h_4(x) - \big((p-1)^2 h_4(x) + h_3(x)\big)h_1(x)h_2(x)\Big)}{h_1(x)h_2(x)h_3(x)h_4(x)}$$

$$= \frac{pk\big((k-2x-1)h_1(x)h_3(x) - (p-1)^2(a_2 - 2x - 1)h_2(x)h_4(x)\big)}{h_1(x)h_2(x)h_3(x)h_4(x)}$$

$= pkf_1(x)/(h_1(x)h_2(x)h_3(x)h_4(x))$, say.

Hence $g(x,0) \le g(x,1)$ if and only if $f_1(x) \le 0$, $f_1(x)$ is a fourth degree polynomial in *x*, and

$$f_1(0) = 2a_1(k-1)(2a_1 - a_2 + 1)$$

$$= 2pbk(k-1)^3(2pbk - p - 1) > 0,$$

$$f_1\big((k-1)/2\big) = -\big(b(k-1)(k+1)(a_2 - k)(p-1)^2/4\big)\big(b(k-1)(k+1)/4\big)$$

$$= -(1/16)p\big((p-1)b(k+1)\big)^2(k-1)^3 < 0.$$

Now

$$h_3(x) > 2a_1 + bx(x - a_2) - b(a_2 - 2x - 1)$$

$$= 2a_1 - b(x+1)(a_2 - x - 1)$$

$$\ge b\big(x^2 + (k-1)(3pk - p - k)/2\big) \ge 0,$$

and by (4.2), one has $h_1(x) \ge h_3(x) > 0$. By (4.1), one has $h_4(x) \ge h_2(x) > 0$.

Observe that

(a) $h_1(x)$ is convex in *x*, and the minimum value of $h_1(x)$ occurs at $x = a_2/2 > (k-1)/2$.

(b) $h_3(x)$ is convex in *x*, and the minimum value of $h_3(x)$ occurs at $x = a_2/2 - 1/b > (k-1)/2$.

(c) $k - 2x - 1 \ge 0$ and is decreasing in *x* for $0 \le x \le (k-1)/2$.

(d) $\partial\big((a_2 - 2x - 1)h_2(x)\big)/\partial x$

$= b\big(6x^2 - 2\big((p+1)(k-1) + 2k\big)x + (p+1)k(k-1)\big)$, and the minimum value of $\partial\big((a_2-2x-1)h_2(x)\big)/\partial x$ occurs at

$x=\big((p+1)(k-1)+2k\big)/6>(k-1)/2$, and

$$\frac{\partial}{\partial x}\big((a_2-2x-1)h_2(x)\big)\Big|_{x=(k-1)/2}=(k-1)(p-k/2-1/2)>0.$$

Hence $(a_2-2x-1)h_2(x)$ is increasing in x for $0\le x\le (k-1)/2$.

(e) $h_4(x)$ is concave in *x*, and the maximum value of $h_4(x)$ occurs at $x=k/2-1/b\ge (k-1)/2$.

For $0\le x\le (k-1)/2$, by (a), (b), (c), and the fact that $h_1(x)$, $h_3(x)>0$, one can see that $(k-2x-1)h_1(x)h_3(x)$ is decreasing in *x*; by (d), (e), and the fact that a_2-2x-1, $h_2(x)$, $h_4(x)>0$, one can see that $(a_2-2x-1)h_2(x)h_4(x)$ is increasing in *x*. Hence $f_1(x)$ is decreasing in *x* for $0\le x\le (k-1)/2$.

Now, since $f_1(0)>0$ and $f_1\big((k-1)/2\big)<0$, there exists *t* such that $f_1(t)\le 0$, then $f_1(x)\le 0$ for $t\le x\le [k/2]-1$, and the result follows.

(ii) Let $h_5(x)=2a_1-a_2(bx+b-1)+bx^2+2(b-1)x+b-1$,

$h_6(x)=k(bx+b-1)-\big(bx^2+2(b-1)x+b-1\big)$, then

$$g(x,b)-g(x,b-1)$$

$$=\frac{pk\big((p-1)^2h_2(x+1)+h_1(x+1)\big)h_5(x)h_6(x)}{h_1(x+1)h_2(x+1)h_5(x)h_6(x)}$$

$$-\frac{pk\big((p-1)^2h_6(x)+h_5(x)\big)h_1(x+1)h_2(x+1)}{h_1(x+1)h_2(x+1)h_5(x)h_6(x)}$$

$$=\frac{pk\big((p-1)^2(a_2-2x-1)h_2(x+1)h_6(x)-(k-2x-1)h_1(x+1)h_5(x)\big)}{h_1(x+1)h_2(x+1)h_5(x)h_6(x)}$$

$$= pkf_2(x)/\big(h_1(x+1)h_2(x+1)h_5(x)h_6(x)\big), \text{ say.}$$

Note that

$$h_1(x+1) > 2a_1 + b(x+1)(x+1-a_2) - b(a_2 - 2x - 3)$$

$$= 2a_1 - 2b(x+1)(a_2 - x - 2)$$

$$\geq b\big((x+1)^2 + (k-1)\big(3pk - p - k - 2(k-2)/(k-1)\big)/2\big) \geq 0,$$

$$h_2(x+1) = b(x+1)(k-x-1) > 0,$$

$$h_5(x) = h_1(x+1) + (a_2 - 2x - 1) > 0,$$

$$h_6(x) = h_2(x+1) - (k - 2x - 1) > (k-x-1)\big(b(x+1) - 1\big) > 0.$$

Hence $g(x,b) \leq g(x,b-1)$ if and only if $f_2(x) \leq 0$. $f_2(x)$ is a fourth degree polynomial in *x*, and

$$f_2(0) = (p+1)b(b-1)(p-1)^2(k-1)^3$$

$$- b\big(2pk - (p+1)\big)\big(2pbk - (b-1)(p+1)\big)(k-1)^3$$

$$< b(b-1)(k-1)^3\big((p+1)(p-1)^2 - \big(p(2k-1)-1\big)^2\big)$$

$$= b(b-1)\big(\big(p - 3 - (2k-1)^2\big)p^2(k-1)^3 + (4k+1)p - 2\big) < 0$$

if $p \leq 2 + (2k-1)^2$,

$$f_2\big((k-1)/2\big) = (a_2 - k)(p-1)^2\big(b(k+1)(k-1)/4\big)^2$$

$$= (1/16)p(k-1)\big((p-1)b(k-1)(k+1)\big)^2 > 0.$$

Now through some straightforward calculation, one observe that

(f) $h_6(x)$ is concave in *x*, and the maximum value of $h_6(x)$ occurs at $x = (k-2)/2 + 1/b > (k-2)/2$.

(g) $\partial\big((a_2 - 2x - 1)h_2(x+1)\big)/\partial x$

$$= b\big(6x^2 - 2\big(2(k-1) + (k-1)(p+1) - 2\big)x + (k-1)\big((p+1)(k-2) - 2\big)\big),$$

and the minimum value of $\partial\big((a_2 - 2x - 1)h_2(x+1)\big)/\partial x$ occurs at

$x = (pk - p + 3k - 5)/6 > (k-1)/2$, and

$$\frac{\partial}{\partial x}\big((a_2 - 2x - 1)h_2(x+1)\big)\Big|_{x=(k-3)/2} = (k-1)(p - k/2) + (17-k)/2 > 0 .$$

Hence $(a_2 - 2x - 1)h_2(x+1)$ is increasing in *x* for $0 \le x \le (k-3)/2$.

(h) $h_1(x+1)$ is convex in *x*, and the minimum value of $h_1(x+1)$ occurs at

$x = (a_2 - 2)/2 > (k-1)/2$.

(i) $h_5(x)$ is convex in *x*, and the minimum value of $h_5(x)$ occurs at

$x = (a_2 - 2)/2 + 1/b > (k-1)/2$.

For $0 \le x \le (k-3)/2$, by (f), (g), and the fact that $a_2 - 2x - 1$, $h_2(x+1)$, $h_6(x) > 0$, one can see that $(a_2 - 2x - 1)h_2(x+1)h_6(x)$ is increasing in *x*; for $0 \le x \le (k-1)/2$, by (c), (h), (i), and the fact that $k - 2x - 1$, $h_1(x+1)$, $h_5(x) > 0$, one can see that $(k - 2x - 1)h_1(x+1)h_5(x)$ is decreasing in *x*. Hence $f_2(x)$ is increasing in *x* for $0 \le x \le (k-3)/2$.

Moreover, let $h_7(x) = (p-1)^2(a_2 - 2x - 1)h_2(x+1)h_6(x)$, $h_8(x) = (k - 2x - 1)h_1(x+1)h_5(x)$, then $f_2(x) = h_7(x) - h_8(x)$, and

$$f_2\big((k-2)/2\big) - f_2\big((k-3)/2\big)$$
$$= h_7\big((k-2)/2\big) - h_7\big((k-3)/2\big) + h_8\big((k-3)/2\big) - h_8\big((k-2)/2\big).$$

Since $h_8(x)$ is decreasing in x and is nonnegative for $0 \le x \le (k-1)/2$, one can obtain that $h_8((k-3)/2) - h_8((k-2)/2) > 0$.

Note that

$$h_7((k-2)/2) = (a_2 - k + 1)(p-1)^2 h_2(k/2) h_6((k-2)/2)$$

$$= (1/4)b(p(k-1)+1)(p-1)^2 k^2 (bk^2/4 - 1),$$

$$h_7((k-3)/2) = (a_2 - k + 2)(p-1)^2 h_2((k-1)/2) h_6((k-3)/2)$$

$$= (1/4)b(p(k-1)+2)(p-1)^2 (k^2-1)(b(k^2-1)/4 - 2),$$

one thus has

$$h_7((k-2)/2) - h_7((k-3)/2)$$

$$= (1/4)b(p-1)^2 \big(k^2 (p(k-1)+1)(bk^2/4 - b(k^2-1)/4 + 1)$$

$$+ (b(k^2-1)/4 - 2)(p(k-1) - (k^2-1) + 1)\big)$$

$$\geq (1/4)b(b/4+1)(p-1)^2 \big(k^2 (p(k-1)+1)$$

$$+ (k(k-1)/2 + 1)(b(k^2-1)/4 - 2)\big) > 0$$

since $p > 2k - x - z/b > 2k - (k-3)/2 - 1 = (3k+1)/2$, that is, $f_2((k-2)/2) - f_2((k-3)/2) > 0$. Hence $f_2(x)$ is increasing in x for $0 \leq x \leq (k-2)/2$.

Now since $f_2(0) < 0$ and $f_2((k-1)/2) > 0$, there exists t such that $f_2(t) \leq 0$, then $f_2(x) \leq 0$ for $0 \leq x \leq t$, and the result follows.

Lemma 4.4. (i) For $1 \leq t \leq [k/2] - 1$, then $g(t,0) = \min_{(x,z)\in\Lambda} g(x,z)$ if and only if

$$g(t,0) \leq g(t,1) \quad \text{and} \quad g(t,0) \leq g(t-1, b-1),$$

(ii) For $t = [k/2]$, then $g([k/2],0) = \min_{(x,z)\in\Lambda} g(x,z)$ if and only if

$$g([k/2],0) \leq g([k/2]-1, b-1).$$

Proof: (i) If $g(t,0) \le g(t,1)$, by Lemma 4.2(i),

$$g(t,0) = \min_{z=0,1,\cdots,b} g(t,z) < g(t,b) = g(t+1,0);$$

and by Lemma 4.3(i), $g(t+1,0) \le g(t+1,1)$. Then $g(t,0)$ is the minimum of $g(x,z)$ for $x = t,\cdots,[k/2]-1$, and $z = 0,1,\cdots,b$.

Similarly, if $g(t,0) = g(t-1,b) \le g(t-1,b-1)$, by Lemma 4.2(ii),

$$g(t-1,b) = \min_{z=0,1,\cdots,b} g(t-1,z) < g(t-1,0) = g(t-2,b);$$

and by Lemma 4.3(ii), $g(t-2,b) \le g(t-2,b-1)$. Then $g(t,0)$ is the minimum of $g(x,z)$ for $x = 0,1,\cdots,t-1$, and $z = 0,1,\cdots,b$. Therefore, $g(t,0)$ $= \min_{(x,z)\in\Lambda} g(x,z)$. The necessary is obvious, and (i) is thus proved.

(ii) can be proved by using a similar procedure as that in the previous proof, hence is omitted.

Lemma 4.1 to 4.4 can be used to find families of A-optimal type S_0 block designs with the control line appearing in *t* crosses in each block.

Theorem 4.5. Let $1 \le t \le [k/2]-1$. A type S_0 block design $S_0(p, b, k, g_0, g_1, \lambda_0, \lambda_1)$ having $s_0 = tb$, $s_1 = b(2k-t)/p$, $g_0 = tb/p$, $g_1 = 2b(k-t)/p(p-1)$, $\lambda_0 = tb(2k-t)/p$, $\lambda_1 = b(2k-t)(2k-t-1)/p(p-1)$, if exists, is A-optimal when p, b, k, and t satisfy

$$b\Big((k-2t-1)\big(p(k-1)(2k-t)-t(k-t)\big)^2-(a_2-2t-1)\big(t(p-1)(k-t)\big)^2\Big)$$
$$\le p(a_2-2t-1)(k-2t-1)\big(t(p-2)(k-t)+(k-1)(2k-t)\big), \qquad (4.3)$$

and

$$b\Big((a_2 - 2t + 1)\big(t(p-1)(k-t)\big)^2 - (k - 2t + 1)\big(p(k-1)(2k-t) - t(k-t)\big)^2\Big)$$

$$\leq p(a_2 - 2t + 1)(k - 2t + 1)\big(t(p-2)(k-t) + (k-1)(2k-t)\big). \qquad (4.4)$$

For $t = [k/2]$, only inequality (4.4) needs to be satisfied.

Proof: By Lemma 4.4, $g(t,0) = \min_{(x,z)\in\Lambda} g(x,z)$ if and only if $g(t,0) \leq g(t,1)$ and $g(t,0) \leq g(t-1, b-1)$. And in the proof of Lemma 4.3, it is shown that $g(t,0) \leq g(t,1)$ if and only if $f_1(t) \leq 0$, and $g(t,0) \leq g(t-1, b-1)$ if and only if $f_2(t-1) \leq 0$. Inequality (4.3) is an expression for $f_1(t) \leq 0$, and inequality (4.4) is an expression for $f_2(t-1) \leq 0$.

4.1.1. A-Optimal Designs for $t = 1$

For $t=1$, inequality (4.3) becomes

$$b(k-1)^2\left(4(k-3)(pk-1)-(p-1)^2\right)$$
$$\leq (k-3)(p+2k-3)\left(p(k-1)+k-3\right), \tag{4.5}$$

and inequality (4.4) becomes

$$b\left((p+1)(p+4k-3)-4pk^2\right)\leq (p+1)(p+2k-3)\,. \tag{4.6}$$

Now since that the right hand side of inequalities (4.5) and (4.6) are both nonnegative, the A-optimality of a type S_0 block design $S_0(p, b, k, g_0, g_1, \lambda_0, \lambda_1)$ having $s_0 = b$, $s_1 = b(2k-1)/p$, $g_0 = b/p$, $g_1 = 2b(k-1)/p(p-1)$, $\lambda_0 = b(2k-1)/p$, $\lambda_1 = 2b(2k-1)(k-1)/p(p-1)$ can be observed by showing that the following two inequalities are satisfied.

$$4(k-3)(pk-1)\leq (p-1)^2\text{, and} \tag{4.7}$$

$$(p+1)(p+4k-3)\leq 4pk^2\,. \tag{4.8}$$

Applying inequalities (4.7) and (4.8), one can obtain families of A-optimal type S_0 block designs.

Theorem 4.6. For $p \geq 2k$, $k \geq 3$, a type S_0 block design $S_0(p, b, k, g_0, g_1, \lambda_0, \lambda_1)$ having $s_0 = b,$ $s_1 = b(2k-1)/p,$ $g_0 = b/p,$ $g_1 = 2b(k-1)/p(p-1)$, $\lambda_0 = b(2k-1)/p$, $\lambda_1 = 2b(k-1)(2k-1)/p(p-1)$, if exists, is A-optimal when p and k satisfy (i) for k = 3, $6 \leq p \leq 25$, (ii) for $k \geq 4$, $4k(k-3)+2 \leq p \leq 4k(k-1)+1$.

Proof: For $k = 3$, by Theorem 4.5, it suffices to find p satisfies inequality (4.8) and $p \geq 2k$, and the intersection is $6 \leq p \leq 25$.

For $k \geq 4$, let $q_1(p) = (p-1)^2 - 4(k-3)(pk-1)$ and $q_2(p) = (p+1)(p+4k-3) - 4pk^2$. Observe that

$$q_1(0) = 4k - 11 > 0, \quad q_1(1) = 4k(4-k) - 12 < 0,$$

$$q_1\big(4k(k-3)+1\big) = 4k(4-k) - 12 < 0, \text{ and } q_1\big(4k(k-3)+2\big) = 4k - 11 > 0;$$

$$q_2(0) = 4k - 3 > 0, \quad q_2(1) = 4k(2-k) - 4 < 0,$$

$$q_2\big(4k(k-1)+1\big) = 4k(2-k) - 4 < 0, \text{ and } q_2\big(4k(k-1)+2\big) = 4k - 3 > 0.$$

Hence for $4k(k-3)+2 \leq p \leq 4k(k-1)+1$, both inequalities (4.7) and (4.8) are satisfied, and the theorem follows.

Some families of A-optimal type S_0 block designs and their construction methods are given.

Family 1. For $p = 6$, $b = 0 \text{ (mod 30)}$, and $k = 3$, that is, $b = 30u$, where $u \geq 1$ is an integer, a type S_0 block design $S_0(6, 30u, 3, 5u, 4u, 25u, 20u)$ is A-optimal in $D(6+1, 30u, 3)$, and can be constructed by repeating the following S_0(6, 30, 3, 5, 4, 25, 20) design u times in the row direction.

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(2,3)	(3,4)	(4,5)	(5,6)	(6,1)	(1,2)
(4,5)	(5,6)	(6,1)	(1,2)	(2,3)	(3,4)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(2,6)	(3,1)	(4,2)	(5,3)	(6,4)	(1,5)
(4,5)	(5,6)	(6,1)	(1,2)	(2,3)	(3,4)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(3,6)	(4,1)	(5,2)	(6,3)	(1,4)	(2,5)
(4,2)	(5,3)	(6,4)	(1,5)	(2,6)	(3,1)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(2,5)	(3,6)	(4,1)	(5,2)	(6,3)	(1,4)
(3,4)	(4,5)	(5,6)	(6,1)	(1,2)	(2,3)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(3,5)	(4,6)	(5,1)	(6,2)	(1,3)	(2,4)
(4,2)	(5,3)	(6,4)	(1,5)	(2,6)	(3,1)

Family 2. For $p = 7$, $b = 0 \pmod{21}$, and $k = 3$, that is, $b = 21u$, where $u \geq 1$ is an integer, a type S_0 block design $S_0(7, 21u, 3, 3u, 2u, 15u, 10u)$ is A-optimal in $D(7+1, 21u, 3)$, and can be constructed by repeating the following S_0(7, 21, 3, 3, 2, 15, 10) design u times in the row direction.

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
(7,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,1)
(6,3)	(7,4)	(1,5)	(2,6)	(3,7)	(4,1)	(5,2)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
(5,4)	(6,5)	(7,6)	(1,7)	(2,1)	(3,2)	(4,3)
(7,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,1)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
(6,3)	(7,4)	(1,5)	(2,6)	(3,7)	(4,1)	(5,2)
(5,4)	(6,5)	(7,6)	(1,7)	(2,1)	(3,2)	(4,3)

Family 3. For $p = 8$, $b = 0 \pmod{56}$, and $k = 3$, that is, $b = 56u$, where $u \geq 1$ is an integer, a type S_0 block design $S_0(8, 56u, 3, 7u, 4u, 35u, 20u)$ is A-optimal in $D(8+1, 56u, 3)$, and can be constructed by repeating the following S_0(8, 56, 3, 7, 4, 35, 20) design u times in the row direction.

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)
(2,6)	(3,7)	(4,8)	(5,1)	(6,2)	(7,3)	(8,4)	(1,5)
(3,4)	(4,5)	(5,6)	(6,7)	(7,8)	(8,1)	(1,2)	(2,3)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)
(5,6)	(6,7)	(7,8)	(8,1)	(1,2)	(2,3)	(3,4)	(4,5)
(4,7)	(5,8)	(6,1)	(7,2)	(8,3)	(1,4)	(2,5)	(3,6)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)
(3,8)	(4,1)	(5,2)	(6,3)	(7,4)	(8,5)	(1,6)	(2,7)
(4,5)	(5,6)	(6,7)	(7,8)	(8,1)	(1,2)	(2,3)	(3,4)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)
(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,1)	(8,2)	(1,3)
(7,8)	(8,1)	(1,2)	(2,3)	(3,4)	(4,5)	(5,6)	(6,7)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)
(6,8)	(7,1)	(8,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)
(3,7)	(4,8)	(5,1)	(6,2)	(7,3)	(8,4)	(1,5)	(2,6)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)
(2,7)	(3,8)	(4,1)	(5,2)	(6,3)	(7,4)	(8,5)	(1,6)
(3,5)	(4,6)	(5,7)	(6,8)	(7,1)	(8,2)	(1,3)	(2,4)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)
(2,5)	(3,6)	(4,7)	(5,8)	(6,1)	(7,2)	(8,3)	(9,4)
(6,8)	(7,1)	(8,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)

Family 4. For $p = 9$, $b = 0 \pmod{18}$, and $k = 3$, that is, $b = 18u$, where $u \geq 1$ is an integer, a type S_0 block design $S_0(9, 18u, 3, 2u, u, 10u, 5u)$ is A-optimal in $D(9+1, 18u, 3)$, and can be constructed by repeating the following S_0(9, 18, 3, 2, 1, 10, 5) design u times in the row direction.

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)
(2,3)	(3,4)	(4,5)	(5,6)	(6,7)	(7,8)	(8,9)	(9,1)	(1,2)
(4,6)	(5,7)	(6,8)	(7,9)	(8,1)	(9,2)	(1,3)	(2,4)	(3,5)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)
(8,3)	(9,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,1)	(7,2)
(4,7)	(5,8)	(6,9)	(7,1)	(8,2)	(9,3)	(1,4)	(2,5)	(3,6)

Family 5. For $p = 11$, $b = 0 \pmod{55}$, and $k = 3$, that is, $b = 55u$, where $u \geq 1$ is an integer, a type S_0 block design $S_0(11, 55u, 3, 5u, 2u, 25u, 10u)$ is A-optimal in $D(11+1, 55u, 3)$, and can be constructed by repeating the following S_0(11, 55, 3, 5, 2, 25, 10) design *u* times in the row direction.

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)
(3,4)	(4,5)	(5,6)	(6,7)	(7,8)	(8,9)	(9,10)	(10,11)	(11,1)	(1,2)	(2,3)
(5,9)	(6,10)	(7,11)	(8,1)	(9,2)	(10,3)	(11,4)	(1,5)	(2,6)	(3,7)	(4,8)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)
(2,7)	(3,8)	(4,9)	(5,10)	(6,11)	(7,1)	(8,2)	(9,3)	(10,4)	(11,5)	(1,6)
(3,10)	(4,11)	(5,1)	(6,2)	(7,3)	(8,4)	(9,5)	(10,6)	(11,7)	(1,8)	(2,9)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)
(2,11)	(3,1)	(4,2)	(5,3)	(6,4)	(7,5)	(8,6)	(9,7)	(10,8)	(11,9)	(1,10)
(6,9)	(7,10)	(8,11)	(9,1)	(10,2)	(11,3)	(1,4)	(2,5)	(3,6)	(4,7)	(5,8)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)
(5,10)	(6,11)	(7,1)	(8,2)	(9,3)	(10,4)	(11,5)	(1,6)	(2,7)	(3,8)	(4,9)
(8,11)	(9,1)	(10,2)	(11,3)	(1,4)	(2,5)	(3,6)	(4,7)	(5,8)	(6,9)	(7,10)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)
(4,6)	(5,7)	(6,8)	(7,9)	(8,10)	(9,11)	(10,1)	(11,2)	(1,3)	(2,4)	(3,5)
(7.8)	(8,9)	(9,10)	(10,11)	(11,1)	(1,2)	(2,3)	(3,4)	(4,5)	(5,6)	(6,7)

4.1.2. A-Optimal Designs for $t \geq 2$

We now extend the result to the general t value by using the same procedure as before.

Now since that the right hand side of the inequalities (4.3) and (4.4) are both nonnegative, the inequalities are satisfied if and only if the left hand sides are less than or equal to 0. Let the left hand side of inequality (4.3) be $bq_3(p)$, and the left hand side of inequality (4.4) be $bq_4(p)$, then

$$q_3(p) = (k-1)\big(t(k-t)\big)^2 p^2 - \Big((k-2t-1)\big((k-1)(2k-t)\big)^2 + (k+2t-1)\big(t(k-t)\big)^2\Big)p$$
$$+ 2t(k-1)(k-t)(2k-t)(k-2t-1) - (k-4t-1)\big(t(k-t)\big)^2,$$
$$q_4(p) = (k-1)\big(t(k-t)\big)^2 p^2 - \Big((k-2t+1)\big((k-1)(2k-t)\big)^2 + (k+2t-3)\big(t(k-t)\big)^2\Big)p$$
$$+ 2t(k-1)(k-t)(2k-t)(k-2t+1) - (k-4t+3)\big(t(k-t)\big)^2.$$

The conditions that $q_3(p) \geq 0$, and $q_4(p) \leq 0$ are given in the following Theorem 4.7.

Theorem 4.7. For $p \geq 2k-t+1$, a type S_0 block design $S_0(p, b, k, g_0, g_1, \lambda_0, \lambda_1)$ having $s_0 = tb,\ s_1 = b(2k-t)/p,\ g_0 = tb/p,\ g_1 = 2b(k-t)/p(p-1)$, $\lambda_0 = tb(2k-t)/p$, $\lambda_1 = b(2k-t)(2k-t-1)/p(p-1)$, if exists, is A-optimal when p, k, and t satisfy

(i) for $k = 2t, 2t+1$,

$$2k-t+1 \leq p \leq \frac{(k-2t+1)\big((k-1)(2k-t)\big)^2 + 2(t-1)\big(t(k-t)\big)^2}{(k-1)\big(t(k-t)\big)^2},$$

(ii) for $k \geq 2t+2$, $\dfrac{(k-2t-1)\big((k-1)(2k-t)\big)^2 + 2(k-t)^2 t^3}{(k-1)\big(t(k-t)\big)^2} + 1 \leq p$

$$\leq \frac{(k-2t+1)\big((k-1)(2k-t)\big)^2 + 2(t-1)\big(t(k-t)\big)^2}{(k-1)\big(t(k-t)\big)^2}.$$

Proof: For $k = 2t, 2t+1$, by Theorem 4.5, it suffices to find p satisfies $q_4(p) \leq 0$ and $p \geq 2k-t+1$, and the intersection is as shown above.

For $k \geq 2t+2$, notice that both $q_3(p)$ and $q_4(p)$ are convex in *p*, and both have two real positive roots, say r_1, r_2, and r_3, r_4, respectively. Observe that

$$q_3(0) = 2t(k-1)(k-t)(2k-t)(k-2t-1) - (k-4t-1)\big(t(k-t)\big)^2 > 0,$$

$$\begin{aligned} q_3(1) &= (k-1)\big(t(k-t)\big)^2 - \Big((k-2t-1)\big((k-1)(2k-t)\big)^2 + (k+2t-1)\big(t(k-t)\big)^2\Big) \\ &\quad + 2t(k-1)(k-t)(2k-t)(k-2t-1) - (k-4t-1)\big(t(k-t)\big)^2 \\ &= -2(k-t)^2 t^3 + (k-1)(2k-t)(k-2t-1)\big(k(3t+2-2k) - t - 2t^2\big) \\ &\quad - (k-4t-1)\big(t(k-t)\big)^2 < 0, \end{aligned}$$

$$\begin{aligned} & q_3\left(\frac{(k-2t-1)\big((k-1)(2k-t)\big)^2 + 2(k-t)^2 t^3}{(k-1)\big(t(k-t)\big)^2}\right) \\ &= \Big((k-2t-1)\big((k-1)(2k-t)\big)^2 + 2(k-t)^2 t^3\Big)\Big((k-2t-1)\big((k-1)(2k-t)\big)^2 \\ &\quad + 2(k-t)^2 t^3 - (k-2t-1)\big((k-1)(2k-t)\big)^2 \\ &\quad - (k+2t-1)\big(t(k-t)\big)^2\Big)\Big/\Big((k-1)\big(t(k-t)\big)^2\Big) \\ &\quad + 2t(k-1)(k-t)(2k-t)(k-2t-1) - (k-4t-1)\big(t(k-t)\big)^2 \\ &= (k-1)(2k-t)(k-2t-1)\big(k(3t+2-2k) - t - 2t^2\big) \\ &\quad - (k-2t-1)\big(t(k-t)\big)^2 < 0, \end{aligned}$$

$$q_3\left(\frac{(k-2t-1)\left((k-1)(2k-t)\right)^2+2(k-t)^2t^3}{(k-1)\left(t(k-t)\right)^2}+1\right)>0 \quad \text{since}$$

$$\frac{(k-2t-1)\left((k-1)(2k-t)\right)^2+2(k-t)^2t^3}{(k-1)\left(t(k-t)\right)^2}+1$$

$$=\frac{(k-2t-1)\left((k-1)(2k-t)\right)^2+(k+2t-1)\left(t(k-t)\right)^2}{(k-1)\left(t(k-t)\right)^2}$$

$$>r_2=\frac{(k-2t-1)\left((k-1)(2k-t)\right)^2+(k+2t-1)\left(t(k-t)\right)^2}{(k-1)\left(t(k-t)\right)^2}$$

$$+\frac{\sqrt{\left((k-2t-1)\left((k-1)(2k-t)\right)^2+(k+2t-1)\left(t(k-t)\right)^2\right)^2+4(k-1)\left(t(k-t)\right)^3h_9(t,k)}}{2(k-1)\left(t(k-t)\right)^2}$$

, where $h_9(t,k)=t(k-t)(k-4t-1)-2(k-1)(2k-t)(k-2t-1)<0$,

then $q_3(p)\geq 0$ for $p\leq 0$, and $p\geq\dfrac{(k-2t-1)\left((k-1)(2k-t)\right)^2+2(k-t)^2t^3}{(k-1)\left(t(k-t)\right)^2}+1$.

Observe also that

$$q_4(0)=2t(k-1)(k-t)(2k-t)(k-2t+1)-(k-4t+3)\left(t(k-t)\right)^2>0,$$

$$q_4(1)=(k-1)\left(t(k-t)\right)^2-\left((k-2t+1)\left((k-1)(2k-t)\right)^2+(k+2t-3)\left(t(k-t)\right)^2\right)$$

$$+2t(k-1)(k-t)(2k-t)(k-2t+1)-(k-4t+3)\left(t(k-t)\right)^2$$

$$=2(1-t)\left(t(k-t)\right)^2+(k-1)(2k-t)(k-2t+1)\left(k(3t+2-2k)-t-2t^2\right)$$

$$-(k-4t+3)\left(t(k-t)\right)^2<0,$$

$$q_4\left(\frac{(k-2t+1)\left((k-1)(2k-t)\right)^2+2(t-1)\left(t(k-t)\right)^2}{(k-1)\left(t(k-t)\right)^2}\right)$$

$$=\left((k-2t+1)\left((k-1)(2k-t)\right)^2+2(t-1)\left(t(k-t)\right)^2\right)\left((k-2t+1)\left((k-1)(2k-t)\right)^2\right.$$

$$+\,2(t-1)\big(t(k-t)\big)^2-(k-2t+1)\big((k-1)(2k-t)\big)^2$$

$$-\,(k+2t-3)\big(t(k-t)\big)^2\Big)\Big/\Big((k-1)\big(t(k-t)\big)^2\Big)$$

$$+\,2t(k-1)(k-t)(2k-t)(k-2t+1)-(k-4t+3)\big(t(k-t)\big)^2$$

$$=(k-1)(2k-t)(k-2t+1)\big(k(3t+2-2k)-2t-2t^2\big)$$

$$-\,(k-2t+1)\big(t(k-t)\big)^2<0\,,$$

$$q_4\left(\frac{(k-2t+1)\big((k-1)(2k-t)\big)^2+2(t-1)\big(t(k-t)\big)^2}{(k-1)\big(t(k-t)\big)^2}+1\right)>0 \quad \text{since}$$

$$\frac{(k-2t+1)\big((k-1)(2k-t)\big)^2+2(t-1)\big(t(k-t)\big)^2}{(k-1)\big(t(k-t)\big)^2}+1$$

$$=\frac{(k-2t+1)\big((k-1)(2k-t)\big)^2+(k+2t-3)\big(t(k-t)\big)^2}{(k-1)\big(t(k-t)\big)^2}$$

$$>r_4=\frac{(k-2t+1)\big((k-1)(2k-t)\big)^2+(k+2t-3)\big(t(k-t)\big)^2}{(k-1)\big(t(k-t)\big)^2}$$

$$+\frac{\sqrt{\Big((k-2t+1)\big((k-1)(2k-t)\big)^2+(k+2t-3)\big(t(k-t)\big)^2\Big)^2+4(k-1)\big(t(k-t)\big)^3h_{10}(t,k)}}{2(k-1)\big(t(k-t)\big)^2}$$

, where $h_{10}(t,k)=t(k-t)(k-4t-3)-2(k-1)(2k-t)(k-2t+1)<0$,

then $q_4(p)\le 0$ for $1\le p\le \dfrac{(k-2t+1)\big((k-1)(2k-t)\big)^2+2(t-1)\big(t(k-t)\big)^2}{(k-1)\big(t(k-t)\big)^2}$. The theorem follows.

Some families of A-optimal type S_0 block designs for t = 2, together with their construction methods are giving in the following.

A type S_0 block design $S_0(p, b, k, g_0, g_1, \lambda_0, \lambda_1)$ with $s_0 = 2b$, has the following values for $s_1, g_0, g_1, \lambda_0, \lambda_1$, and

$$s_1 = 2b(k-1)/p, \quad g_0 = 2b/p, \quad g_1 = 2b(k-2)/p(p-1),$$

$$\lambda_0 = 4b(k-1)/p, \quad \lambda_1 = 2b(k-1)(2k-3)/p(p-1).$$

For these designs to exist, $s_1, g_0, g_1, \lambda_0, \lambda_1$ must all be integers.

Family 6. For $p = 7$, $b = 0 \pmod{21}$, and $k = 4$, that is, $b = 21u$, where $u \geq 1$ is an integer, a type S_0 block design $S_0(7, 21u, 4, 6u, 2u, 12u, 15u)$ is A-optimal in $D(9+1, 36u, 4)$, and can be constructed by repeating the following S_0(7, 21, 4, 6, 2, 12, 15) design u times in the row direction.

(0,4)	(0,5)	(0,6)	(0,7)	(0,1)	(0,2)	(0,3)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
(7,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,1)
(6,3)	(7,4)	(1,5)	(2,6)	(3,7)	(4,1)	(5,2)

(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,1)	(0,2)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
(7,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,1)
(5,4)	(6,5)	(7,6)	(1,7)	(2,1)	(3,2)	(4,3)

(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,1)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
(6,3)	(7,4)	(1,5)	(2,6)	(3,7)	(4,1)	(5,2)
(5,4)	(6,5)	(7,6)	(1,7)	(2,1)	(3,2)	(4,3)

Family 7. For $p = 9$, $b = 0 \text{ (mod 36)}$, and $k = 5$, that is, $b = 36u$, where $u \geq 1$ is an integer, a type S_0 block design $S_0(9, 36u, 5, 8u, 3u, 64u, 28u)$ is A-optimal in $D(9+1, 36u, 5)$, and can be constructed by repeating the following S_0(9, 36, 5, 8, 3, 64, 28) design *u* times in the row direction.

(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,1)	(0,2)	(0,3)	(0,4)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)
(9,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,1)
(8,3)	(9,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,1)	(7,2)
(7,4)	(8,5)	(9,6)	(1,7)	(2,8)	(3,9)	(4,1)	(5,2)	(6,3)

(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,1)	(0,2)	(0,3)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)
(9,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,1)
(8,3)	(9,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,1)	(7,2)
(6,5)	(7,6)	(8,7)	(9,8)	(1,9)	(2,1)	(3,2)	(4,3)	(5,4)

(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,1)	(0,2)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)
(9,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,1)
(7,4)	(8,5)	(9,6)	(1,7)	(2,8)	(3,9)	(4,1)	(5,2)	(6,3)
(6,5)	(7,6)	(8,7)	(9,8)	(1,9)	(2,1)	(3,2)	(4,3)	(5,4)

(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,1)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)
(8,3)	(9,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,1)	(7,2)
(7,4)	(8,5)	(9,6)	(1,7)	(2,8)	(3,9)	(4,1)	(5,2)	(6,3)
(6,5)	(7,6)	(8,7)	(9,8)	(1,9)	(2,1)	(3,2)	(4,3)	(5,4)

Family 8. For $p=10$, $b=0 \pmod{45}$, and $k=5$, that is, $b = 45u$, where $u \geq 1$ is an integer, a type S_0 block design $S_0(10, 45u, 5, 9u, 3u, 72u, 28u)$ is A-optimal in $D(10+1, 45u, 5)$, and can be constructed by repeating the following $S_0(10, 45, 5, 9, 3, 72, 28)$ design u times in the row direction.

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)
(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,1)
(3,4)	(4,5)	(5,6)	(6,7)	(7,8)	(8,9)	(9,10)	(10,1)	(1,2)	(2,3)
(5,7)	(6,8)	(7,9)	(8,10)	(9,1)	(10,2)	(1,3)	(2,4)	(3,5)	(4,6)
(6,8)	(7,9)	(8,10)	(9,1)	(10,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)
(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,1)	(0,2)
(5,9)	(6,10)	(7,1)	(8,2)	(9,3)	(10,4)	(1,5)	(2,6)	(3,7)	(4,8)
(6,10)	(7,1)	(8,2)	(9,3)	(10,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)
(7,8)	(8,9)	(9,10)	(10,1)	(1,2)	(2,3)	(3,4)	(4,5)	(5,6)	(6,7)

(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,1)	(0,2)
(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,1)	(0,2)	(0,3)
(1,9)	(2,10)	(3,1)	(4,2)	(5,3)	(6,4)	(7,5)	(8,6)	(9,7)	(10,8)
(2,6)	(3,7)	(4,8)	(5,9)	(6,10)	(7,1)	(8,2)	(9,3)	(10,4)	(1,5)
(5,8)	(6,9)	(7,10)	(8,1)	(9,2)	(10,3)	(1,4)	(2,5)	(3,6)	(4,7)

(0,7)	(0,8)	(0,9)	(0,10)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(0,9)	(0,10)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)
(1,4)	(2,5)	(3,6)	(4,7)	(5,8)	(6,9)	(7,10)	(8,1)	(9,2)	(10,3)
(3,6)	(4,7)	(5,8)	(6,9)	(7,10)	(8,1)	(9,2)	(10,3)	(1,4)	(2,5)
(5,10)	(6,1)	(7,2)	(8,3)	(9,4)	(10,5)	(1,6)	(2,7)	(3,8)	(4,9)

(0,4)	(0,5)	(0,6)	(0,7)	(0,8)
(0,9)	(0,10)	(0,1)	(0,2)	(0,3)
(1,2)	(2,3)	(3,4)	(4,5)	(5,6)
(3,8)	(4,9)	(5,10)	(6,1)	(7,2)
(6,7)	(7,8)	(8,9)	(9,10)	(10,1)

Family 9. For $p = 11$, $b = 0 \pmod{55}$, and $k = 6$, that is, $b = 55u$, where $u \geq 1$ is an integer, a type S_0 block design $S_0(11, 55u, 6, 10u, 4u, 100u, 45u)$ is A-optimal in $D(11+1, 55u, 6)$, and can be constructed by repeating the following $S_0(11, 55, 6, 10, 4, 100, 45)$ design u times in the row direction.

(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)
(11,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,10)	(9,11)	(10,1)
(10,3)	(11,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,10)	(7,11)	(8,1)	(9,2)
(9,4)	(10,5)	(11,6)	(1,7)	(2,8)	(3,9)	(4,10)	(5,11)	(6,1)	(7,2)	(8,3)
(8,5)	(9,6)	(10,7)	(11,8)	(1,9)	(2,10)	(3,11)	(4,1)	(5,2)	(6,3)	(7,4)

(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)	(0,1)	(0,2)	(0,3)	(0,4)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)
(11,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,10)	(9,11)	(10,1)
(10,3)	(11,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,10)	(7,11)	(8,1)	(9,2)
(9,4)	(10,5)	(11,6)	(1,7)	(2,8)	(3,9)	(4,10)	(5,11)	(6,1)	(7,2)	(8,3)
(7,6)	(8,7)	(9,8)	(10,9)	(11,10)	(1,11)	(2,1)	(3,2)	(4,3)	(5,4)	(6,5)

(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)	(0,1)	(0,2)	(0,3)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)
(11,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,10)	(9,11)	(10,1)
(10,3)	(11,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,10)	(7,11)	(8,1)	(9,2)
(8,5)	(9,6)	(10,7)	(11,8)	(1,9)	(2,10)	(3,11)	(4,1)	(5,2)	(6,3)	(7,4)
(7,6)	(8,7)	(9,8)	(10,9)	(11,10)	(1,11)	(2,1)	(3,2)	(4,3)	(5,4)	(6,5)

(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)	(0,1)	(0,2)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)
(11,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,10)	(9,11)	(10,1)
(9,4)	(10,5)	(11,6)	(1,7)	(2,8)	(3,9)	(4,10)	(5,11)	(6,1)	(7,2)	(8,3)
(8,5)	(9,6)	(10,7)	(11,8)	(1,9)	(2,10)	(3,11)	(4,1)	(5,2)	(6,3)	(7,4)
(7,6)	(8,7)	(9,8)	(10,9)	(11,10)	(1,11)	(2,1)	(3,2)	(4,3)	(5,4)	(6,5)

(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)	(0,1)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)
(10,3)	(11,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,10)	(7,11)	(8,1)	(9,2)
(9,4)	(10,5)	(11,6)	(1,7)	(2,8)	(3,9)	(4,10)	(5,11)	(6,1)	(7,2)	(8,3)
(8,5)	(9,6)	(10,7)	(11,8)	(1,9)	(2,10)	(3,11)	(4,1)	(5,2)	(6,3)	(7,4)
(7,6)	(8,7)	(9,8)	(10,9)	(11,10)	(1,11)	(2,1)	(3,2)	(4,3)	(5,4)	(6,5)

4.2. A-Optimal Designs for $[(2bk - s_{d0})/pb] = 1$

For $[(2bk - s_{d0})/pb] = 1$, or $k \le p \le 2k - s_{d0}/b$, one has $y_1 = 1$, $a(s_{d0})$ $= 3(2bk - s_{d0}) - 2pb$, hence

$$g(s_{d0}; p,b,k) = \frac{pk(p-1)^2}{2pbk(k-3+p/k) - (p(k-3)+k)s_{d0} + h(s_{d0})}$$

$$+ \frac{p}{s_{d0} - h(s_{d0})/k}, \text{ and}$$

$$g(x,z) = \frac{pk(p-1)^2}{2pb(k(k-3)+p) - (p(k-3)+k)(bx+z) + bx^2 + 2zx + z}$$

$$+ \frac{p}{(bx+z) - (bx^2 + 2zx + z)/k}$$

$$= pk\left(\frac{(p-1)^2}{2a_1' - a_2'(bx+z) + bx^2 + 2zx + z} + \frac{1}{k(bx+z) - (bx^2 + 2zx + z)}\right),$$

where $a_1' = pb(k(k-3)+p)$, $a_2' = p(k-3)+k$, and $g(x,0) = g(x-1,b)$ for $x \ge 1$.

Applying the same procedure as that in section 4.1, that is, from Lemma 4.1 to Lemma 4.4, we have the following Lemma 4.8 to 4.11.

Lemma 4.8. For fixed value of *x*, $0 \le x \le [k/2] - 1$, there exists z_1 ($0 \le z_1 \le b$), a function of *x*, such that $g(x,z)$ decreases in *z* when $0 \le z \le z_1$ and increases in *z* when $z_1 \le z \le b$. If $z_1 = 0$, $g(x,z)$ increases in *z*, and if $z_1 = b$, then $g(x,z)$ decreases in *z*.

Proof: Taking the derivative of $g(x,z)$ with respective to z, one has

$$\frac{\partial}{\partial z} g(x,z) = pk\left(\frac{-(p-1)^2(2x+1-a_2')}{\left(2a_1' - a_2'(bx+z) + bx^2 + 2zx + z\right)^2} \right.$$

$$\left. - \frac{k-2x-1}{\left(k(bx+z) - (bx^2 + 2zx + z)\right)^2} \right)$$

and the sign of $\partial g(x,z)/\partial z$ is the same as the sign of

$$-(p-1)^2(2x+1-a_2')\left(k(bx+z) - (bx^2 + 2zx + z)\right)^2$$

$$-(k-2x-1)\left(2a_1' - a_2'(bx+z) + bx^2 + 2zx + z\right)^2$$

$= \Phi'(z)$, say.

Now let

$$\varphi_2'(x) = -(p-1)^2(2x+1-a_2')(k-2x-1)^2 - (k-2x-1)(2x+1-a_2')^2$$

$$= p(a_2' - 2x - 1)(k-2x-1)\left((p-2)(k-2x-1) - (k-3)\right),$$

$$\varphi_1'(x) = -b(p-1)^2(2x+1-a_2')(k-2x-1)x(k-x)$$

$$-(k-2x-1)(2x+1-a_2')(2a_1' - a_2'bx + bx^2)$$

$$= pb(a_2' - 2x - 1)(k-2x-1)\left(2p + (k-3)(2k-x) + (p-2)x(k-x)\right),$$

$$\varphi_0'(x) = (p-1)^2 b^2 (2x+1-a_2')(x(k-x))^2 + (k-2x-1)(2a_1' - a_2'bx + bx^2)^2$$

$$= b^2\left((p-1)^2(2x+1-a_2')\left(x(k-x)\right)^2\right.$$

$$\left. + (k-2x-1)\left(2p^2 + p(k-3)(2k-x) + x(x-k)\right)^2\right).$$

Then $\Phi'(z)$ can be rewritten as $\Phi'(z) = \varphi_2'(x)z^2 + 2\varphi_1'(x)z - \varphi_0'(x)$. For x in $[0,[k/2]-1]$, one can see that

$$a_2' - 2x - 1 = p(k-3) + k - 2x - 1 \geq k - 2x - 1 \geq 1. \qquad (4.9)$$

And through some straightforward calculation one can show that $x(k-x)(p-2)+(k-3)(2k-x)>0$ and $(p-2)(k-2x-1)-(k-3)>0$, hence $\varphi_1'(x)>0$, and $\varphi_2'(x)>0$. Now $\partial\Phi'(z)/\partial z = 2z\varphi_2'(x)+2\varphi_1'(x)\geq 0$ for $x\in[0,[k/2]-1]$, hence $\Phi'(z)$ is increasing in *z*, therefore $\Phi'(z)$ is either all negative, or increasing from negative to positive, or all positive for $0\leq z\leq b$. Since the signs of $\partial g(x,z)/\partial z$ and $\Phi'(z)$ are the same, the lemma is thus proved.

The proof of Lemma 4.9 is the same as Lemma 4.2, hence is omitted.

Lemma 4.9. For $0\leq x\leq [k/2]-1$. Then a necessary and sufficient condition for

(i) $g(x,0)=\min\limits_{0\leq z\leq b} g(x,z)$ is $g(x,0)\leq g(x,1)$,

(ii) $g(x,b)=\min\limits_{0\leq z\leq b} g(x,z)$ is $g(x,b)\leq g(x,b-1)$.

Lemma 4.10. (i) Suppose $0<t\leq [k/2]-1$, then $g(t,0)\leq g(t,1)$ implies

$$g(x,0)\leq g(x,1) \quad \text{for} \quad t\leq x\leq [k/2]-1,$$

(i) Suppose $0<t\leq [k/2]-1$, then $g(t,b)\leq g(t,b-1)$ implies

$$g(x,b)\leq g(x,b-1) \quad \text{for} \quad 0\leq x\leq t.$$

Proof: (i) Let $h_1'(x)=2a_1'+bx(x-a_2')$, and

$h_3'(x)=2a_1'+bx(x-a_2')-(a_2'-2x-1)$, then

$g(x,0)-g(x,1)$

$$=\frac{pk\left(\left((p-1)^2h_2(x)+h_1'(x)\right)h_3'(x)h_4(x)-\left((p-1)^2h_4(x)+h_3'(x)\right)h_1'(x)h_2(x)\right)}{h_1'(x)h_2(x)h_3'(x)h_4(x)}$$

$$= \frac{pk\big((k-2x-1)h_1'(x)h_3'(x)-(p-1)^2(a_2'-2x-1)h_2(x)h_4(x)\big)}{h_1'(x)h_2(x)h_3'(x)h_4(x)}$$

$= pkf_1'(x)/\big(h_1'(x)h_2(x)h_3'(x)h_4(x)\big)$, say.

Hence $g(x,0) \le g(x,1)$ if and only if $f_1'(x) \le 0$, $f_1'(x)$ is a fourth degree polynomial in *x*, and

$$f_1'(0) = 2a_1'(k-1)(2a_1'-a_2'+1)$$

$$= 2pb(k-1)\big(k(k-3)+p\big)\big(p(k-3)(2bk-1)+2bp^2-k+1\big) > 0\,,$$

$$f_1'\big((k-1)/2\big) = -\big(b(k-1)(k+1)(a_2'-k)(p-1)^2/4\big)\big(b(k-1)(k+1)/4\big)$$

$$= -(1/16)\,p\big((p-1)b(k+1)\big)^2(k-1)^3 < 0\,.$$

Now

$$h_3'(x) > 2a_1' + bx(x-a_2') - b(a_2'-2x-1)$$

$$= 2a_1' - b(x+1)(a_2'-x-1)$$

$$\ge b\big(x^2 + p(k-3)(3k-1)/2 + k(3k+1)/2\big) \ge 0\,,$$

and by (4.9), one has $h_1'(x) \ge h_3'(x) > 0$.

Observe that

(a) $h_1'(x)$ is convex in *x*, and the minimum value of $h_1'(x)$ occurs at $x = a_2'/2 > (k-1)/2$.

(b) $h_3'(x)$ is convex in *x*, and the minimum value of $h_3'(x)$ occurs at $x = a_2'/2 - 1/b > (k-1)/2$.

(c) $k-2x-1 \ge 0$ and is decreasing in *x* for $0 \le x \le (k-1)/2$.

(d) $\partial\big((a_2'-2x-1)h_2(x)\big)/\partial x$

$$= b\left(6x^2 - 2\left(p(k-3)+3k-1\right)x + (p+1)k(k-1) - 2pk\right),$$

and the minimum value of $\partial\left((a_2' - 2x - 1)h_2(x)\right)/\partial x$ occurs at

$x = \left(p(k-3)+3k-1\right)/6 > (k-3)/2$, and

$$\frac{\partial}{\partial x}\left((a_2' - 2x - 1)h_2(x)\right)\bigg|_{x=(k-3)/2} = (k-3)(3p - k/2 - 9/2) > 0 .$$

Hence $(a_2' - 2x - 1)h_2(x)$ is increasing in x for $0 \le x \le (k-3)/2$.

(e) $h_4(x)$ is concave in *x*, and the maximum value of $h_4(x)$ occurs at $x = k/2 - 1/b \ge (k-1)/2$.

For $0 \le x \le (k-1)/2$, by (a), (b), (c), and the fact that $k-2x-1$, $h_1'(x)$, $h_3'(x) > 0$, one can see that $(k-2x-1)h_1'(x)h_3'(x)$ is decreasing in *x*; for $0 \le x \le (k-3)/2$, by (d), (e), and the fact that $a_2' - 2x - 1$, $h_2(x)$, $h_4(x) > 0$, one can see that $(a_2' - 2x - 1)h_2(x)h_4(x)$ is increasing in *x*. Hence $f_1'(x)$ is decreasing in *x* for $0 \le x \le (k-3)/2$.

Moreover, let $h_{11}(x) = (k-2x-1)h_1'(x)h_3'(x)$,

$h_{12}(x) = (p-1)^2(a_2' - 2x - 1)h_2(x)h_4(x)$, then $f_1'(x) = h_{11}(x) - h_{12}(x)$, and

$$f_1'\left((k-2)/2\right) - f_1'\left((k-3)/2\right)$$

$$= h_{11}\left((k-2)/2\right) - h_{11}\left((k-3)/2\right) + h_{12}\left((k-3)/2\right) - h_{12}\left((k-2)/2\right).$$

Since $h_{11}(x)$ is decreasing in *x* and is nonnegative for $0 \le x \le (k-1)/2$, one can obtain that $h_{11}\left((k-3)/2\right) - h_{11}\left((k-2)/2\right) < 0$.

Note that

$$h_{12}\left((k-2)/2\right) = (p-1)^2(a_2' - k + 1)h_2\left((k-2)/2\right)h_4\left((k-2)/2\right)$$

$$= (1/4)b(p-1)^2\big(p(k-3)+1\big)(k^2-4)\big(b(k^2-4)/4+1\big),$$

$$h_{12}\big((k-3)/2\big) = (p-1)^2(a_2'-k+2)h_2\big((k-3)/2\big)h_4\big((k-3)/2\big)$$

$$= (1/4)b(p-1)^2\big(p(k-3)+2\big)(k^2-9)\big(b(k^2-9)/4+2\big),$$

one thus has

$$h_{12}\big((k-3)/2\big) - h_{12}\big((k-2)/2\big)$$

$$= (1/4)b(p-1)^2\Big(\big(p(k-3)+1\big)(k^2-4)\big(b(k^2-9)/4 - b(k^2-4)/4+1\big)$$

$$+ \big(b(k^2-9)/4+2\big)\big(k^2-5p(k-3)-14\big)\Big)$$

$$\le (1/4)b(p-1)^2\Big(\big(p(k-3)+1\big)(k^2-4)(1-5b/4)$$

$$+ \big(b(k^2-9)/4+2\big)\big(k(15-4k)/2-14\big)\Big) < 0\,.$$

Hence $f_1'\big((k-2)/2\big) - f_1'\big((k-3)/2\big) > 0$. That is, $f_1'(x)$ is decreasing in x for $0 \le x \le (k-2)/2$.

Now, since $f_1'(0) > 0$ and $f_1'\big((k-1)/2\big) < 0$, there exists t such that $f_1'(t) \le 0$, then $f_1'(x) \le 0$ for $t \le x \le [k/2]-1$, and the result follows.

(ii) Let $h_5'(x) = 2a_1' - a_2'(bx+b-1) + bx^2 + 2(b-1)x + b - 1$, then

$$g(x,b) - g(x,b-1)$$

$$= \frac{pk\big((p-1)^2 h_2(x+1) + h_1'(x+1)\big)h_5'(x)h_6(x)}{h_1'(x+1)h_2(x+1)h_5'(x)h_6(x)}$$

$$- \frac{pk\big((p-1)^2 h_6(x) + h_5'(x)\big)h_1'(x+1)h_2(x+1)}{h_1'(x+1)h_2(x+1)h_5'(x)h_6(x)}$$

$$= \frac{pk\big((p-1)^2(a_2'-2x-1)h_2(x+1)h_6(x) - (k-2x-1)h_1'(x+1)h_5'(x)\big)}{h_1'(x+1)h_2(x+1)h_5'(x)h_6(x)}$$

$$= pkf_2{}'(x)/\left(h_1'(x+1)h_2(x+1)h_5'(x)h_6(x)\right), \text{ say.}$$

Note that

$$h_1'(x+1) > 2a_1' + b(x+1)(x+1-a_2') - b(a_2' - 2x - 3)$$

$$= 2a_1' - 2b(x+1)(a_2' - x - 2)$$

$$\geq b\left((x+1)^2 + (1/2)(k-1)\left(3p(k-3) + 2p^2 - k - 1\right) + 1\right) > 0,$$

$$h_2(x+1) = b(x+1)(k-x-1) > 0,$$

$$h_5'(x) = h_1(x+1) + (a_2' - 2x - 1) > 0,$$

$$h_6(x) = h_2(x+1) - (k - 2x - 1) > (k-x-1)(bx+b-1) > 0.$$

Hence $g(x,b) \leq g(x,b-1)$ if and only if $f_2{}'(x) \leq 0$. $f_2{}'(x)$ is a fourth degree polynomial in *x*, and

$$f_2{}'(0) = b(b-1)(a_2'-1)(p-1)^2(k-1)^2$$

$$- (k-1)\left(2a_1' - b(a_2'-1)\right)\left(2a_1' - (b-1)(a_2'-1)\right)$$

$$< b(b-1)(a_2'-1)(p-1)^2(k-1)^2$$

$$- b(b-1)(k-1)(2k-1)(a_2'-1)\left(2p\left(k(k-3)+p\right) - (a_2'-1))\right)$$

$$< b(b-1)(a_2-1)(k-1)^2\left((p-1)^2 - \left(2p(k(k-3)+p\right) - (a_2'-1)\right)$$

$$= b(b-1)(a_2'-1)(k-1)^2\left(-p^2 - \left(2+(k-3)(2k-1)\right)p + k\right) < 0$$

for $p \geq k$,

$$f_2{}'\left((k-1)/2\right) = (a_2'-k)(p-1)^2\left(b(k+1)(k-1)/4\right)^2$$

$$= (1/16)p(k-3)\left((p-1)b(k^2-1)\right)^2 > 0.$$

Now through some straightforward calculation, one observe that

(f) $h_6(x)$ is concave in *x*, and the maximum value of $h_6(x)$ occurs at

$x=(k-2)/2+1/b>(k-2)/2$.

(g) $\partial\big((a_2'-2x-1)h_2(x+1)\big)/\partial x$

$=b\big(6x^2-2\big(p(k-3)+3k-5\big)x+(k-2)(a_2'-1)-2(k-1)\big)$,

and the minimum value of $\partial\big((a_2'-2x-1)h_2(x+1)\big)/\partial x$ occurs at

$x=(pk-p+3k-5)/6>(k-2)/2$, and

$$\frac{\partial}{\partial x}\big((a_2'-2x-1)h_2(x+1)\big)\Big|_{x=(k-4)/2}=2p(k-3)-(k-4)(k/2+2)>0.$$

Hence $(a_2'-2x-1)h_2(x+1)$ is increasing in *x* for $0\le x\le(k-4)/2$.

(h) $h_1'(x+1)$ is convex in *x*, and the minimum value of $h_1'(x+1)$ occurs at

$x=(a_2'-2)/2>(k-2)/2$.

(i) $h_5'(x)$ is convex in *x*, and the minimum value of $h_5'(x)$ occurs at

$x=(a_2'-2)/2+1/b>(k-2)/2$.

For $0\le x\le(k-4)/2$, by (f), (g), and the fact that $a_2'-2x-1$, $h_2(x+1)$, $h_6(x)>0$, one can see that $(a_2'-2x-1)h_2(x+1)h_6(x)$ is increasing in *x*; for $0\le x\le(k-2)/2$, by (c), (h), (i), and the fact that $k-2x-1$, $h_1'(x+1)$, $h_5'(x)>0$, one can see that $(k-2x-1)h_1'(x+1)h_5'(x)$ is decreasing in *x*. Hence $f_2'(x)$ is increasing in *x* for $0\le x\le(k-4)/2$.

Moreover, let $h_7'(x)=(p-1)^2(a_2'-2x-1)h_2(x+1)h_6(x)$, $h_8'(x)=(k-2x-1)h_1'(x+1)h_5'(x)$, then $f_2'(x)=h_7'(x)-h_8'(x)$, and

$$f_2'\big((k-3)/2\big) - f_2'\big((k-4)/2\big)$$

$$= h_7'\big((k-3)/2\big) - h_7'\big((k-4)/2\big) + h_8'\big((k-4)/2\big) - h_8'\big((k-3)/2\big),$$

$$f_2'\big((k-2)/2\big) - f_2'\big((k-3)/2\big)$$

$$= h_7'\big((k-2)/2\big) - h_7'\big((k-3)/2\big) + h_8'\big((k-3)/2\big) - h_8'\big((k-2)/2\big).$$

Since $h_8'(x)$ is decreasing in x and is nonnegative for $0 \le x \le (k-2)/2$, one can obtain that $h_8'\big((k-4)/2\big) > h_8'\big((k-3)/2\big) > h_8'\big((k-2)/2\big) > 0$.

Note that

$$h_7'\big((k-4)/2\big) = (p-1)^2(a_2' - k + 3)h_2\big((k-2)/2\big)h_6\big((k-4)/2\big)$$

$$= (1/4)b(p-1)^2\big(p(k-3)+3\big)(k^2-4)\big(b(k^2-4)/4-3\big),$$

$$h_7'\big((k-3)/2\big) = (p-1)^2(a_2' - k + 2)h_2\big((k-1)/2\big)h_6\big((k-3)/2\big)$$

$$= (1/4)b(p-1)^2\big(p(k-3)+2\big)(k^2-1)\big(b(k^2-1)/4-2\big),$$

$$h_7'\big((k-2)/2\big) = (p-1)^2(a_2' - k + 1)h_2(k/2)h_6\big((k-2)/2\big)$$

$$= (1/4)b(p-1)^2\big(p(k-3)+1\big)k^2(bk^2/4-1),$$

one thus has

$$h_7'\big((k-3)/2\big) - h_7'\big((k-4)/2\big)$$

$$= (1/4)b(p-1)^2\Big(\big(p(k-3)+2\big)(k^2-1)\big(b(k^2-1)/4 - b(k^2-4)/4 + 1\big)$$

$$+ \big(b(k^2-4)/4-3\big)\big(3p(k-3) - k^2 + 10\big)\Big)$$

$$\ge (1/4)b(p-1)^2\Big(\big(p(k-3)+2\big)(k^2-1)(3b/4+1)$$

$$+ \big(b(k^2-4)/4-3\big)\big(k(2k-9)+10\big)\Big) > 0$$

since $p \ge k$, and

$$h_7'\big((k-2)/2\big) - h_7'\big((k-3)/2\big)$$

$$= (1/4)b(p-1)^2\Big(\big(p(k-3)+1\big)k^2\big(bk^2/4 - b(k^2-1)/4 + 1\big)$$

$$+ \big(b(k^2-1)/4 - 2\big)\big(p(k-3) - k^2 + 2\big)\Big) > 0 \quad \text{when} \quad k \ge 5,$$

and one can see that, through straightforward calculation, for $k = 3$ and 4, $h_8'\big((k-3)/2\big) - h_8'\big((k-2)/2\big) + h_7'\big((k-2)/2\big) - h_7'\big((k-3)/2\big) > 0,$ that is, $f_2'\big((k-3)/2\big) - f_2'\big((k-4)/2\big) > 0$ and $f_2'\big((k-2)/2\big) - f_2'\big((k-3)/2\big) > 0.$ Hence $f_2'(x)$ is increasing in x for $0 \le x \le (k-2)/2$.

Now since $f_2'(0) < 0$ and $f_2'((k-1)/2) > 0$, there exists t such that $f_2'(t) \le 0$, then $f_2'(x) \le 0$ for $0 \le x \le t$, and the result follows.

The proof of Lemma 4.11 is the same as Lemma 4.4, hence is omitted

Lemma 4.11. (i) For $1 \le t \le [k/2]-1$, then $g(t,0) = \min_{(x,z)\in\Lambda} g(x,z)$ if and only if

$$g(t,0) \le g(t,1) \quad \text{and} \quad g(t,0) \le g(t-1,b-1),$$

(ii) For $t = [k/2]$, then $g([k/2],0) = \min_{(x,z)\in\Lambda} g(x,z)$ if and only if

$$g([k/2],0) \le g([k/2]-1,b-1).$$

Lemma 4.8 to 4.11 can be used to find families of A-optimal type S_0 block designs with the control line appearing in *t* crosses in each block.

Theorem 4.12. Let $1 \le t \le [k/2]-1$. A type S_0 block design $S_0(p,b,k,$ $g_0, g_1, \lambda_0, \lambda_1)$ having $s_0 = tb$, $s_1 = b(2k-t)/p$, $g_0 = tb/p$, $g_1 = 2b(k-t)/$ $p(p-1)$, $\lambda_0 = tb(2k-t)/p$, $\lambda_1 = b\big((2k-t)(2k-t-3)+2p\big)/p(p-1)$, if exists, is A-optimal when p, b, k, and t satisfy

$$b\Big((k-2t-1)\big(p(k-3)(2k-t)+2p^2-t(k-t)\big)^2-(a_2'-2t-1)\big(t(p-1)(k-t)\big)^2\Big)$$
$$\leq p(a_2'-2t-1)(k-2t-1)\big(t(p-2)(k-t)+(k-3)(2k-t)+2p\big), \qquad (4.10)$$

and

$$b\Big((a_2'-2t+1)\big(t(p-1)(k-t)\big)^2-(k-2t+1)\big(p(k-3)(2k-t)+2p^2-t(k-t)\big)^2\Big)$$
$$\leq p(a_2'-2t+1)(k-2t+1)\big(t(p-2)(k-t)+(k-3)(2k-t)+2p\big). \qquad (4.11)$$

When $t=[k/2]$, only inequality (4.11) needs to be satisfied.

Proof: Similar to Theorem 4.5, now

$$f_1'(t)=(k-2t-1)h_1'(t)h_3'(t)-(p-1)^2(a_2'-2t-1)h_2(t)h_4(t),$$

$$f_2'(t-1)=(p-1)^2(a_2'-2t+1)h_2(t)h_6(t)-(k-2t+1)h_1'(t)h_5'(t),$$

then by Lemma 4.10 and 4.11, and through some straightforward calculations, inequalities (4.10) and (4.11) are thus derived.

Notice that the right hand side of the inequalities (4.10) and (4.11) are both nonnegative, the inequalities are satisfied if and only if the left hand sides are less than or equal to 0. Let the left hand side of inequality (4.10) be $bq_5(p)$, and the left hand side of inequality (4.11) be $bq_6(p)$, then

$$q_5(p)=4(k-2t-1)p^3+(k-3)\Big(4(2k-t)(k-2t-1)-\big(t(k-t)\big)^2\Big)p^2$$
$$+\Big((k-2t-1)(\big((k-3)(2k-t)\big)^2-4t(k-t)(k-2t-1)+(k+2t-5)\big(t(k-t)\big)^2\Big)p$$
$$+(k-4t+1)\big(t(k-t)\big)^2-2t(k-3)(k-t)(2k-t)(k-2t-1),$$

$$q_6(p) = 4(k-2t+1)p^3 + (k-3)\left(4(2k-t)(k-2t+1) - \left(t(k-t)\right)^2\right)p^2$$
$$+\left((k-2t+1)\left((k-3)(2k-t)\right)^2 - 4t(k-t)(k-2t+1) + (k+2t-7)\left(t(k-t)\right)^2\right)p$$
$$+(k-4t+5)\left(t(k-t)\right)^2 - 2t(k-3)(k-t)(2k-t)(k-2t+1).$$

Since both $q_5(p)$ and $q_6(p)$ are third degree polynomial in *p*, finding a range of *p* with both limits functions of *k* and *t*, such that $q_5(p) \le 0$, $q_6(p) \ge 0$ is not straightforward. However, by using a computer, one can obtain the values of *p* that satisfy $q_5(p) \le 0$, $q_6(p) \ge 0$, for given values of *k* and *t*. Table 4.1 of such $p \le 30$ is listed in the following.

Table 4.1. A Catalog of A-Optimal Designs with $p \le 30$

k	*t*	*p*
3	1	$4 \le p \le 5$
4	2	$4 \le p \le 6$
5	2	$5 \le p \le 8$
6	3	$6 \le p \le 9$
7	3	$7 \le p \le 11$
8	3	$8 \le p \le 13$
9	4	$9 \le p \le 10$
10	4	$10 \le p \le 16$
11	4	$11 \le p \le 18$
12	4	$16 \le p \le 20$
13	5	$13 \le p \le 15$
14	5	$14 \le p \le 20$
15	5	$17 \le p \le 25$
16	5	$21 \le p \le 27$

17	5	$27 \le p \le 29$
17	6	$17 \le p \le 19$
18	6	$18 \le p \le 23$
19	6	$21 \le p \le 27$
20	6	$25 \le p \le 30$
21	6	$29 \le p \le 30$

Some families of A-optimal type S_0 block designs for t = 1, together with their construction methods are given in the following.

Family 10. For $p = 4$, $b = 0 \pmod{12}$, and $k = 3$, that is, $b = 12u$, where $u \ge 1$ is an integer, a type S_0 block design $S_0(4, 12u, 3, 3u, 4u, 15u, 18u)$ is A-optimal in $D(4+1, 12u, 3)$, and can be constructed by repeating the following S_0(4, 12, 3, 3, 4, 15, 18) design u times in the row direction.

$$\begin{array}{cccc} (0,1) & (0,2) & (0,3) & (0,4) \\ (3,2) & (1,3) & (2,1) & (1,3) \\ (2,4) & (3,4) & (1,4) & (2,1) \end{array}$$

$$\begin{array}{cccc} (0,1) & (0,2) & (0,3) & (0,4) \\ (3,2) & (1,3) & (2,1) & (1,3) \\ (3,4) & (1,4) & (2,4) & (3,2) \end{array}$$

$$\begin{array}{cccc} (0,1) & (0,2) & (0,3) & (0,4) \\ (2,4) & (3,4) & (1,4) & (2,1) \\ (3,4) & (1,4) & (2,4) & (3,2) \end{array}$$

Family 11. For $p = 5$, $b = 0 \pmod 5$, and $k = 3$, that is, $b = 5u$, where $u \geq 1$ is an integer, a type S_0 block design $S_0(5, 5u, 3, u, u, 5u, 5u)$ is A-optimal in $D(5+1, 5u, 3)$, and can be constructed by repeating the following $S_0(5, 5, 3, 1, 1, 5, 5)$ design u times in the row direction.

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(5,2)	(1,3)	(2,4)	(3,5)	(4,1)
(4,3)	(5,4)	(1,5)	(2,1)	(3,2)

Some families of A-optimal type S_0 block designs for $t = 2$, together with their construction methods are given in the following.

A type S_0 block design $S_0(p, b, k, g_0, g_1, \lambda_0, \lambda_1)$ with $s_0 = 2b$, has the following values for $s_1, g_0, g_1, \lambda_0, \lambda_1$, and

$$s_1 = 2b(k-1)/p, \quad g_0 = 2b/p, \quad g_1 = 2b(k-2)/p(p-1),$$

$$\lambda_0 = 4b(k-1)/p, \quad \lambda_1 = 2b\big((k-1)(2k-5)+p\big)/p(p-1).$$

For these designs to exist, $s_1, g_0, g_1, \lambda_0, \lambda_1$ must all be integers.

Family 12. For $p=4$, $b=0\ (\mathrm{mod}\ 12)$, and $k=4$, that is, b = 12u, where $u \geq 1$ is an integer, a type S_0 block design $S_0(4, 12u, 4, 6u, 4u, 36u, 26u)$ is A-optimal in $D(4+1,12u,4)$, and can be constructed by repeating the following S_0(4, 12, 4, 6, 4, 36, 26) design u times in the row direction.

(0,1)	(0,2)	(0,3)	(0,4)
(0,1)	(0,2)	(0,3)	(0,4)
(3,2)	(1,3)	(2,1)	(1,3)
(2,4)	(3,4)	(1,4)	(2,1)

(0,1)	(0,2)	(0,3)	(0,4)
(0,1)	(0,2)	(0,3)	(0,4)
(3,2)	(1,3)	(2,1)	(1,3)
(3,4)	(1,4)	(2,4)	(3,2)

(0,1)	(0,2)	(0,3)	(0,4)
(0,1)	(0,2)	(0,3)	(0,4)
(2,4)	(3,4)	(1,4)	(2,1)
(3,4)	(1,4)	(2,4)	(3,2)

Family 13. For $p=5$, $b=0\ (\mathrm{mod}\ 5)$, and $k=4$, that is, b = 5u, where $u \geq 1$ is an integer, a type S_0 block design $S_0(5, 5u, 4, 2u, u, 12u, 7u)$ is A-optimal in $D(5+1,5u,4)$, and can be constructed by repeating the following S_0(5, 5, 4, 2, 1, 12, 7) design u times in the row direction.

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(5,2)	(1,3)	(2,4)	(3,5)	(4,1)
(4,3)	(5,4)	(1,5)	(2,1)	(3,2)

Family 14. For $p = 6$, $b = 0 \pmod{30}$, and $k = 4$, that is, $b = 30u$, where $u \geq 1$ is an integer, a type S_0 block design $S_0(6, 30u, 4, 10u, 4u, 60u, 30u)$ is A-optimal in $D(6+1, 30u, 4)$, and can be constructed by repeating the following $S_0(6, 30, 4, 10, 4, 60, 30)$ design u times in the row direction.

(0,6)	(0,6)	(0,6)	(0,6)	(0,6)	(0,1)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(5,2)	(1,3)	(2,4)	(3,5)	(4,1)	(3,5)
(4,3)	(5,4)	(1,5)	(2,1)	(3,2)	(2,4)

(0,3)	(0,4)	(0,5)	(0,1)	(0,2)	(0,3)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(5,2)	(1,3)	(2,4)	(3,5)	(4,1)	(5,2)
(4,6)	(5,6)	(1,6)	(2,6)	(3,6)	(4,1)

(0,5)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(3,2)	(4,3)	(5,4)	(1,5)	(2,1)	(1,3)
(4,6)	(5,6)	(1,6)	(2,6)	(3,6)	(2,4)

(0,4)	(0,5)	(0,1)	(0,2)	(0,3)	(0,4)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(3,2)	(4,3)	(5,4)	(1,5)	(2,1)	(1,3)
(5,6)	(1,6)	(2,6)	(3,6)	(4,6)	(5,2)

(0,2)	(0,3)	(0,4)	(0,5)	(0,1)	(0,2)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(4,3)	(5,4)	(1,5)	(2,1)	(3,2)	(4,1)
(5,6)	(1,6)	(2,6)	(3,6)	(4,6)	(3,5)

Family 15. For $p = 5$, $b = 0 \ (\text{mod } 10)$, and $k = 5$, that is, $b = 10u$, where $u \geq 1$ is an integer, a type S_0 block design $S_0(5, 10u, 5, 4u, 3u, 32u, 25u)$ is A-optimal in $D(5+1, 10u, 5)$, and can be constructed by repeating the following $S_0(5, 10, 5, 4, 3, 32, 25)$ design u times in the row direction.

(0,5)	(0,1)	(0,2)	(0,3)	(0,4)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(5,2)	(1,3)	(2,4)	(3,5)	(4,1)
(4,3)	(5,4)	(1,5)	(2,1)	(3,2)
(3,2)	(4,3)	(5,4)	(1,5)	(2,1)

(0,5)	(0,1)	(0,2)	(0,3)	(0,4)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(5,2)	(1,3)	(2,4)	(3,5)	(4,1)
(4,3)	(5,4)	(1,5)	(2,1)	(3,2)
(4,1)	(5,2)	(1,3)	(2,4)	(3,5)

Family 16. For $p = 6$, $b = 0 \ (\text{mod } 30)$, and $k = 5$, that is, $b = 30u$, where $u \geq 1$ is an integer, a type S_0 block design $S_0(6, 30u, 5, 10u, 6u, 80u, 52u)$ is A-optimal in $D(6+1, 30u, 5)$, and can be constructed by repeating the following $S_0(6, 30, 5, 10, 6, 80, 52)$ design u times in the row direction.

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(5,2)	(1,3)	(2,4)	(3,5)	(4,1)	(1,3)
(4,3)	(5,4)	(1,5)	(2,1)	(3,2)	(4,1)
(5,6)	(1,6)	(2,6)	(3,6)	(4,6)	(5,2)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(5,2)	(1,3)	(2,4)	(3,5)	(4,1)	(1,3)
(4,6)	(5,6)	(1,6)	(2,6)	(3,6)	(2,4)
(3,2)	(4,3)	(5,4)	(1,5)	(2,1)	(3,5)

(0,1)	(0,2)	(0,4)	(0,4)	(0,5)	(0,6)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(3,2)	(4,3)	(5,4)	(1,5)	(2,1)	(3,5)
(4,6)	(5,6)	(1,6)	(2,6)	(3,6)	(4,1)
(5,6)	(1,6)	(2,6)	(3,6)	(4,6)	(5,2)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(5,2)	(1,3)	(2,4)	(3,5)	(4,1)	(1,3)
(4,3)	(5,4)	(1,5)	(2,1)	(3,2)	(2,4)
(4,6)	(5,6)	(1,6)	(2,6)	(3,6)	(5,2)

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(4,3)	(5,4)	(1,5)	(2,1)	(3,2)	(2,4)
(3,2)	(4,3)	(5,4)	(1,5)	(2,1)	(3,5)
(5,6)	(1,6)	(2,6)	(3,6)	(4,6)	(4,1)

Family 17. For $p = 7$, $b = 0 \pmod 7$, and $k = 5$, that is, $b = 7u$, where $u \geq 1$ is an integer, a type S_0 block design $S_0(7, 7u, 5, 2u, u, 16, 9u)$ is A-optimal in $D(7+1, 7u, 5)$, and can be constructed by repeating the following S_0(7, 7, 5, 2, 1, 16, 9) design u times in the row direction.

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
(7,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,1)
(6,3)	(7,4)	(1,5)	(2,6)	(3,7)	(4,1)	(5,2)
(5,4)	(6,5)	(7,6)	(1,7)	(2,1)	(3,2)	(4,3)

Remark: For $p \geq 5$ is an odd number, $b = up$, and $k = (p+3)/2$, a type S_0 block design $S_0(p, up, (p+3)/2, 2u, u, 2u(p+1), u(p+2))$ constructed by the following method is either an A-optimal design or an efficient design. Consider the initial block $\{(C,0), (C,1), (0,2), (p-1,3), \ldots, ((p+3)/2, (p+1)/2)\}$. Cyclically developing the initial block, mod *p*, where *C* denotes the control line, 0 denotes the *p*th test lines, and control line *C* is unchanged during the cyclical procedure, will yields a type S_0 block design $S_0(p, p, (p+3)/2, 2, 1, 2(p+1), p+2)$. A type S_0 block design $S_0(p, up, (p+3)/2, 2u, u, 2u(p+1), u(p+2))$ can be constructed by repeating the $S_0(p, p, (p+3)/2, 2, 1, 2(p+1), p+2)$ design *u* times in the row direction. It is an A-optimal design for *p* = 5, 7, and is an efficient design for $9 \leq p \leq 29$ with a lower bound to the efficiency 0.8859 in $D(p+1, p, (p+3)/2)$. For example, for $p = 9$, $b = 9$, and $k = 6$, that is, $u = 1$, the following design is a $S_0(9, 9, 6, 2, 1, 20, 11)$ design, and a lower bound to the efficiency is 0.9883.

(0,9)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)
(9,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,1)
(8,3)	(9,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,1)	(7,2)
(7,4)	(8,5)	(9,6)	(1,7)	(2,8)	(3,9)	(4,1)	(5,2)	(6,3)
(6,5)	(7,6)	(8,7)	(9,8)	(1,9)	(2,1)	(3,2)	(4,3)	(5,4)

Some families of A-optimal type S_0 block designs for *t* = 3, together with their construction methods are given in the following.

A type S_0 block design $S_0(p, b, k, g_0, g_1, \lambda_0, \lambda_1)$ with $s_0 = 3b$, has the following values for $s_1, g_0, g_1, \lambda_0, \lambda_1$, and

$$s_1 = b(2k-3)/p, \quad g_0 = 3b/p, \quad g_1 = 2b(k-3)/p(p-1),$$

$$\lambda_0 = 3b(2k-3)/p, \quad \lambda_1 = 2b\big((k-3)(2k-3)+p\big)/p(p-1).$$

For these designs to exist, $s_1, g_0, g_1, \lambda_0, \lambda_1$ must all be integers.

Family 18. For $p = 7$, $b = 0 \pmod{21}$, and $k = 6$, that is, $b = 21u$, where $u \geq 1$ is an integer, a type S_0 block design $S_0(7, 21u, 6, 9u, 3u, 81u, 34u)$ is A-optimal in $D(7+1, 21u, 6)$, and can be constructed by repeating the following S_0(7, 21, 6, 9, 3, 81, 34) design u times in the row direction.

(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,1)	(0,2)
(0,4)	(0,5)	(0,6)	(0,7)	(0,1)	(0,2)	(0,3)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
(7,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,1)
(6,3)	(7,4)	(1,5)	(2,6)	(3,7)	(4,1)	(5,2)
(5,4)	(6,5)	(7,6)	(1,7)	(2,1)	(3,2)	(4,3)
(0,5)	(0,6)	(0,7)	(0,1)	(0,2)	(0,3)	(0,4)
(0,7)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
(7,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,1)
(6,3)	(7,4)	(1,5)	(2,6)	(3,7)	(4,1)	(5,2)
(5,4)	(6,5)	(7,6)	(1,7)	(2,1)	(3,2)	(4,3)
(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,1)
(0,6)	(0,7)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
(7,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,1)
(6,3)	(7,4)	(1,5)	(2,6)	(3,7)	(4,1)	(5,2)
(5,4)	(6,5)	(7,6)	(1,7)	(2,1)	(3,2)	(4,3)

Family 19. For $p = 9$, $b = 0 \pmod{36}$, and $k = 6$, that is, $b = 36u$, where $u \geq 1$ is an integer, a type S_0 block design $S_0(9, 36u, 6, 12u, 3u, 108u, 36u)$ is A-optimal in $D(9+1, 36u, 6)$, and can be constructed by repeating the following S_0(9, 36, 6, 12, 3, 108, 36) design u times in the row direction.

(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,1)	(0,2)
(0,8)	(0,9)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)
(9,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,1)
(7,4)	(8,5)	(9,6)	(1,7)	(2,8)	(3,9)	(4,1)	(4,1)	(6,3)
(6,5)	(7,6)	(8,7)	(9,8)	(1,9)	(2,1)	(3,2)	(4,3)	(5,4)

(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,1)	(0,2)	(0,3)	(0,4)
(0,6)	(0,7)	(0,8)	(0,9)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)
(9,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,1)
(8,3)	(9,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,1)	(7,2)
(7,4)	(8,5)	(9,6)	(1,7)	(2,8)	(3,9)	(4,1)	(5,2)	(6,3)

(0,7)	(0,8)	(0,9)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,4)	(0,5)	(0,6)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)
(9,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,1)
(8,3)	(9,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,1)	(7,2)
(6,5)	(7,6)	(8,7)	(9,8)	(1,9)	(2,1)	(3,2)	(4,3)	(5,4)

(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,1)
(0,9)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)
(6,5)	(7,6)	(8,7)	(9,8)	(1,9)	(2,1)	(3,2)	(4,3)	(5,4)
(8,3)	(9,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,1)	(7,2)
(7,4)	(8,5)	(9,6)	(1,7)	(2,8)	(3,9)	(4,1)	(5,2)	(6,3)

Family 20. For $p = 7$, $b = 0 \pmod{21}$, and $k = 7$, that is, $b = 21u$, where $u \geq 1$ is an integer, a type S_0 block design $S_0(7, 21u, 7, 9u, 4u, 99u, 51u)$ is A-optimal in $D(7+1, 21u, 7)$, and can be constructed by repeating the following $S_0(7, 21, 7, 9, 4, 99, 51)$ design u times in the row direction.

(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,1)	(0,2)
(0,4)	(0,5)	(0,6)	(0,7)	(0,1)	(0,2)	(0,3)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
(7,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,1)
(6,3)	(7,4)	(1,5)	(2,6)	(3,7)	(4,1)	(5,2)
(5,4)	(6,5)	(7,6)	(1,7)	(2,1)	(3,2)	(4,3)
(7,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,1)

(0,5)	(0,6)	(0,7)	(0,1)	(0,2)	(0,3)	(0,4)
(0,7)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
(7,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,1)
(6,3)	(7,4)	(1,5)	(2,6)	(3,7)	(4,1)	(5,2)
(5,4)	(6,5)	(7,6)	(1,7)	(2,1)	(3,2)	(4,3)
(6,3)	(7,4)	(1,5)	(2,6)	(3,7)	(4,1)	(5,2)

(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,1)
(0,6)	(0,7)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
(7,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,1)
(6,3)	(7,4)	(1,5)	(2,6)	(3,7)	(4,1)	(5,2)
(5,4)	(6,5)	(7,6)	(1,7)	(2,1)	(3,2)	(4,3)
(5,4)	(6,5)	(7,6)	(1,7)	(2,1)	(3,2)	(4,3)

Family 21. For $p = 9$, $b = 0 \pmod{36}$, and $k = 7$, that is, $b = 36u$, where $u \geq 1$ is an integer, a type S_0 block design $S_0(9, 36u, 7, 12u, 4u, 132u, 53u)$ is A-optimal in $D(9+1, 36u, 7)$, and can be constructed by repeating the following $S_0(9, 36, 7, 12, 4, 132, 53)$ design u times in the row direction.

(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,2)
(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,3)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)
(9,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,1)
(8,3)	(9,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,1)	(7,2)
(7,4)	(8,5)	(9,6)	(1,7)	(2,8)	(3,9)	(4,1)	(5,2)	(6,3)
(6,5)	(7,6)	(8,7)	(9,8)	(1,9)	(2,1)	(3,2)	(4,3)	(5,4)

(0,2)	(0,2)	(0,2)	(0,2)	(0,2)	(0,2)	(0,3)	(0,3)	(0,3)
(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,4)	(0,5)	(0,6)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)
(9,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,1)
(8,3)	(9,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,1)	(7,2)
(7,4)	(8,5)	(9,6)	(1,7)	(2,8)	(3,9)	(4,1)	(5,2)	(6,3)
(6,5)	(7,6)	(8,7)	(9,8)	(1,9)	(2,1)	(3,2)	(4,3)	(5,4)

(0,3)	(0,3)	(0,3)	(0,4)	(0,4)	(0,4)	(0,4)	(0,4)	(0,5)
(0,7)	(0,8)	(0,9)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,6)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)
(9,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,1)
(8,3)	(9,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,1)	(7,2)
(7,4)	(8,5)	(9,6)	(1,7)	(2,8)	(3,9)	(4,1)	(5,2)	(6,3)
(6,5)	(7,6)	(8,7)	(9,8)	(1,9)	(2,1)	(3,2)	(4,3)	(5,4)

(0,5)	(0,5)	(0,5)	(0,6)	(0,6)	(0,6)	(0,7)	(0,7)	(0,8)
(0,7)	(0,8)	(0,9)	(0,7)	(0,8)	(0,9)	(0,8)	(0,9)	(0,9)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)
(9,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,1)
(8,3)	(9,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,1)	(7,2)
(7,4)	(8,5)	(9,6)	(1,7)	(2,8)	(3,9)	(4,1)	(5,2)	(6,3)
(6,5)	(7,6)	(8,7)	(9,8)	(1,9)	(2,1)	(3,2)	(4,3)	(5,4)

Family 22. For $p = 11$, $b = 0 \pmod{55}$, and $k = 8$, that is, $b = 55u$, where $u \geq 1$ is an integer, a type S_0 block design $S_0(11, 55u, 8, 15u, 5u, 195u, 76u)$ is A-optimal in $D(11+1, 55u, 8)$, and can be constructed by repeating the following $S_0(11, 55, 8, 15, 5, 195, 76)$ design u times in the row direction.

(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,2)
(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)	(0,3)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)
(11,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,10)	(9,11)	(10,1)
(10,3)	(11,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,10)	(7,11)	(8,1)	(9,2)
(9,4)	(10,5)	(11,6)	(1,7)	(2,8)	(3,9)	(4,10)	(5,11)	(6,1)	(7,2)	(8,3)
(8,5)	(9,6)	(10,7)	(11,8)	(1,9)	(2,10)	(3,11)	(4,1)	(5,2)	(6,3)	(7,4)
(7,6)	(8,7)	(9,8)	(10,9)	(11,10)	(1,11)	(2,1)	(3,2)	(4,3)	(5,4)	(6,5)

(0,2)	(0,2)	(0,2)	(0,2)	(0,2)	(0,2)	(0,2)	(0,2)	(0,3)	(0,3)	(0,3)
(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)	(0,4)	(0,5)	(0,6)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)
(11,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,10)	(9,11)	(10,1)
(10,3)	(11,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,10)	(7,11)	(8,1)	(9,2)
(9,4)	(10,5)	(11,6)	(1,7)	(2,8)	(3,9)	(4,10)	(5,11)	(6,1)	(7,2)	(8,3)
(8,5)	(9,6)	(10,7)	(11,8)	(1,9)	(2,10)	(3,11)	(4,1)	(5,2)	(6,3)	(7,4)
(7,6)	(8,7)	(9,8)	(10,9)	(11,10)	(1,11)	(2,1)	(3,2)	(4,3)	(5,4)	(6,5)

(0,3)	(0,3)	(0,3)	(0,3)	(0,3)	(0,4)	(0,4)	(0,4)	(0,4)	(0,4)	(0,4)
(0,7)	(0,8)	(0,9)	(0,10)	(0,11)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)
(11,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,10)	(9,11)	(10,1)
(10,3)	(11,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,10)	(7,11)	(8,1)	(9,2)
(9,4)	(10,5)	(11,6)	(1,7)	(2,8)	(3,9)	(4,10)	(5,11)	(6,1)	(7,2)	(8,3)
(8,5)	(9,6)	(10,7)	(11,8)	(1,9)	(2,10)	(3,11)	(4,1)	(5,2)	(6,3)	(7,4)
(7,6)	(8,7)	(9,8)	(10,9)	(11,10)	(1,11)	(2,1)	(3,2)	(4,3)	(5,4)	(6,5)

(0,4)	(0,5)	(0,5)	(0,5)	(0,5)	(0,5)	(0,5)	(0,6)	(0,6)	(0,6)	(0,6)
(0,11)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)	(0,7)	(0,8)	(0,9)	(0,10)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)
(11,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,10)	(9,11)	(10,1)
(10,3)	(11,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,10)	(7,11)	(8,1)	(9,2)
(9,4)	(10,5)	(11,6)	(1,7)	(2,8)	(3,9)	(4,10)	(5,11)	(6,1)	(7,2)	(8,3)
(8,5)	(9,6)	(10,7)	(11,8)	(1,9)	(2,10)	(3,11)	(4,1)	(5,2)	(6,3)	(7,4)
(7,6)	(8,7)	(9,8)	(10,9)	(11,10)	(1,11)	(2,1)	(3,2)	(4,3)	(5,4)	(6,5)

(0,6)	(0,7)	(0,7)	(0,7)	(0,7)	(0,8)	(0,8)	(0,8)	(0,9)	(0,9)	(0,10)
(0,11)	(0,8)	(0,9)	(0,10)	(0,11)	(0,9)	(0,10)	(0,11)	(0,10)	(0,11)	(0,11)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)
(11,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,10)	(9,11)	(10,1)
(10,3)	(11,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,10)	(7,11)	(8,1)	(9,2)
(9,4)	(10,5)	(11,6)	(1,7)	(2,8)	(3,9)	(4,10)	(5,11)	(6,1)	(7,2)	(8,3)
(8,5)	(9,6)	(10,7)	(11,8)	(1,9)	(2,10)	(3,11)	(4,1)	(5,2)	(6,3)	(7,4)
(7,6)	(8,7)	(9,8)	(10,9)	(11,10)	(1,11)	(2,1)	(3,2)	(4,3)	(5,4)	(6,5)