2 The market framework and the model

We consider the investor in the economic environment can invest his wealth between

a domestic money market account M_d;

a foreign money market account M_f;

a domestic discount bond B_d;

a foreign discount bond B_f;

a domestic stock index S_d;

a foreign stock index S_f;

where subscript d means domestic asset, ans f means foreign asset.

We suppose that in the international financial market the trades take place during $[0, T]$. There are five risks under the environment of two countries (economy). They are represented by five independent Brownian motion $\{Z_i(t); t \in [t, T]; i = \{e, r_e, r_f, S_d, S_f\}\}$ defined on the complete probability space $(\Omega, \mathcal{F}, P)$ where Ω is the state space, $\mathcal{F}$ is a σ-field of describing the incident that can be measured and P is the probability measure. All the processes defined below are diversely affected by these sources of risk and adapted to the augmented filtration generated by the five Brownian motions. This filtration is denoted by $\mathcal{F} \equiv \{\mathcal{F}_t\}_{t \in [0,T]}$ and satisfies the usual conditions.

Following Boulier et al. (2001), we have the following diffusion process. The domestic cash asset gives an instantaneous risk-free rate $r_d(t)$ is assumed to satisfy an Orstein-Unlenbeck process

$$dr_d(t) = a_d(b_d - r_d(t))dt - \sigma_{r_d} dZ_{r_d}(t). \tag{1}$$

The domestic money market account $M_d(t)$, with an initial value of $M_d(0) = 1$, is given by following expression

$$M_d(t) = \exp\left(\int_0^t r_d(s)ds\right).$$

The price of domestic zero-coupon bond B_d maturing at date T_d satisfies

$$\begin{aligned} \frac{dB_d(t,T_d)}{B_d(t,T_d)} &= r_d(t)dt + \sigma_{B_d}(T_d - t)(dZ_{r_d}(t) + \lambda_{r_d}dt), \\ B_d(T_d,T_d) &= 1, \end{aligned} \tag{2}$$

where the premium λ_{r_d} is assumed to be constant. Consistently with the Vesicek's short interest rate model, the volatility of a domestic zero-coupon bond with a maturity τ is given by

$$\sigma_{B_d}(\tau) = \frac{1 - e^{-a_d\tau}}{a_d}\sigma_{r_d}.$$

Following Sorensen (1999), we assumed that the price of the domestic stock index S_d satisfies

$$\frac{dS_d(t)}{S_d(t)} = (\mu_{S_d}(t) + r_d(t))dt + \sigma_{S_d}(t)dZ_{S_d}(t), \tag{3}$$

where $\mu_{S_d}(t)$,$\sigma_{S_d}(t)$ are deterministic functions. The exchange rate e between the domestic and foreign market is assumed to follow a simple dynamic process of the following type:

$$\frac{de(t)}{e(t)} = \mu_e(t)dt + \sigma_e(t)dZ_e(t). \tag{4}$$

For convenient, we use the subscript f to represent foreign assets that are identical to the corresponding domestic ones. Then we have

$$dr_f(t) = a_f(b_f - r_f(t))dt - \sigma_{r_f} dZ_{r_f}(t). \tag{5}$$

The price of foreign zero-coupon bond B_f maturing at date T_f satisfies

$$\begin{aligned} \frac{dB_f(t, T_f)}{B_f(t, T_f)} &= r_f(t)dt + \sigma_{B_f}(T_f - t)(dZ_{r_f}(t) + \lambda_{r_f} dt), \\ B_f(T_f, T_f) &= 1, \end{aligned}$$

where the premium λ_{r_f} is assumed to be constant. Consistently with the Vesicek's short interest rate model, the volatility of a domestic zero-coupon bond with a maturity τ is given by

$$\sigma_{B_f}(\tau) = \frac{1 - e^{-a_f \tau}}{a_f} \sigma_{r_f},$$

and

$$\frac{dS_f(t)}{S_f(t)} = (\mu_{S_f}(t) + r_f(t))dt + \sigma_{S_f}(t) dZ_{S_f}(t).$$

According to the domestic framework, all prices of foreign assets should be converted by the real exchange rate e. All converted prices are denoted by the symbol $\widehat{\cdot}$. With Itô's lemma, the converted foreign money market $\widehat{M_f} \equiv M_f \cdot e$ satisfies

$$\frac{d\widehat{M}_f(t)}{\widehat{M}_f(t)} = (\mu_e(t) + r_f(t))dt + \sigma_e(t) dZ_e(t).$$

The converted price of foreign instantaneous stock index $\widehat{S_f} \equiv S_f \cdot e$ satisfies

$$\frac{d\widehat{S_f}(t)}{\widehat{S_f}(t)} = (\xi_f(t) + r_f(t))dt + \sigma_{S_f}(t)dZ_{S_f}(t) + \sigma_e(t)dZ_e(t), \quad (6)$$

where

$$\xi_f(t) = \mu_e(t) + \mu_{S_f}(t) + \sigma_{e,S_f}(t).$$

The converted of the zero-coupon bond price $\widehat{B_f} \equiv B_f \cdot e$ satisfies

$$\frac{d\widehat{B_f}(t,T_f)}{\widehat{B_f}(t,T_f)} = (\zeta_f(t,T_f) + r_f(t))dt + \sigma_{B_f}(T_f - t)dZ_{r_f}(t) + \sigma_e(t)dZ_e(t),$$

where

$$\zeta_f(t,T_f) = \mu_e(t) + \lambda_{r_f}\sigma_{B_f}(T_f - t) + \sigma_{e,B_f}(T_f - t).$$

We assumed that the instantaneous proportional drift, $\mu(t)$, is unknown to the investor, but is related to the predictive variable, L, by a functional relation $\mu(t) = \mu(L,t)$. A two-dimensional predictive variable $L(t) = (r_d(t), r_f(t))$ is considered in this study. The relation regarding to the drift and the predictive variable is unknown to investor. For convenient, we suppose that the investors can employ linear regression to characterize their relationship, and the regression coefficients are random and unobservable for the investors. It can be written as:

$$\mu_e(t) = \alpha + \beta^{'}L(t), \quad (7)$$

where α is an unknown scalar and β is a 2×1 vector of unknown (unobservable) predictive coefficients. The coefficients β are assumed to develop according to the following known dynamic process:

$$d\beta = (B_0(e, L, t) + B_1(e, L, t)\beta)dt + \eta(e, L, t)dZ_\beta. \tag{8}$$

Here B_0, B_1,and η are assumed to be known constants.

For simplicity, we assume that both the long-run mean of μ_e, $\overline{\mu_e}$, and the mean of the predictive variables, $\overline{L}$, are known constants such that $\alpha \equiv \overline{\mu_e} - \beta'\overline{L}$ can be calculated whenever the value of β is determined. To complete the model, we assume that the vector of predictive variables, L, follows a known joint Markov process:

$$dL = (A_0(e, L, t) + A_1(e, L, t)\beta)dt + \sigma_l(e, L, t)dW_l, \tag{9}$$

where

$$\begin{aligned}\begin{pmatrix} dr_d(t) \\ dr_f(t) \end{pmatrix} &= \left(\begin{pmatrix} a_d(b_d - r_d(t)) \\ a_f(b_f - r_f(t)) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix} \right) dt \\ &\quad + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} dZ_{\beta_1}(t) \\ dZ_{\beta_2}(t) \end{pmatrix} + \begin{pmatrix} \sigma_{r_d} & 0 \\ 0 & \sigma_{r_f} \end{pmatrix} \begin{pmatrix} dZ_d(t) \\ dZ_f(t) \end{pmatrix}.\end{aligned}$$