4 The Martingale Method

We assume that the international financial market is free of frictions and arbitrage opportunities, so there exists a probability measure which is equivalent to the historical probability measure P with respect to a given numéraire such that the prices expressed in terms of this numéraire are martingales.

We select the numéraire as the riskless asset yielding r_d and the corresponding probability measure Q is the so-called risk neutral probability. The Radon-Nikodym derivative dQ/dP is given by

$$\frac{dQ}{dP} = \delta(t) = \exp\left\{ -\int_0^t \Phi(\tau)^\intercal dZ(\tau) - \frac{1}{2}\int_0^t \Phi(\tau)^\intercal \Phi(\tau) d\tau \right\},$$

and $\Phi(\tau)$ is defined by means of $\Theta(t)$, which is

$$\Theta(t) = \begin{bmatrix} \sigma_e(t) & 0 & 0 & 0 & 0 \\ 0 & \sigma_{B_d}(t,T_d) & 0 & 0 & 0 \\ \sigma_e(t) & 0 & \sigma_{B_f}(t,T_f) & 0 & 0 \\ 0 & 0 & 0 & \sigma_{S_d}(t) & 0 \\ \sigma_e(t) & 0 & 0 & 0 & \sigma_{S_f}(t) \end{bmatrix},$$

and

$$
\begin{aligned}
\Phi(t) &= \Theta(t)^{-1} \begin{bmatrix} \overline{\mu}_e(t) + b_1(t)(r_d(t) - b_d) + b_2(t)(r_f(t) - b_f) + r_f(t) - r_e(t) \\ \lambda_{r_d} \sigma_{B_d}(T_d - t) \\ \zeta_f(t, T_f) + r_f(t) - r_d(t) \\ \mu_{S_d}(t) \\ \xi_f(t) + r_f(t) - r_d(t) \end{bmatrix} \\
&= \Theta(t)^{-1} \begin{bmatrix} \overline{\mu}_e(t) + b_1(t)(r_d(t) - b_d) + b_2(t)(r_f(t) - b_f) \\ \lambda_{r_d} \sigma_{B_d}(T_d - t) \\ \zeta_f(t, T_f) \\ \mu_{S_d}(t) \\ \xi_f(t) \end{bmatrix} + \Theta(t)^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \left(r_f(t) - r_d(t)\right), \\
&= \Phi_1(t) + \Phi_2(t)(r_f(t) - r_d(t)),
\end{aligned}
$$

where $\Phi_1(t)$, $\Phi_2(t)$ are 5×1 deterministic functions.

In a complete market, all the risks brought about by the economic factors (the state variables) must be embedded in the stochastic discount factor (the pricing kernel), so that the market price of risk sums up all the relevant information available on the market.