Chapter 2

Valuation Methods

In this chapter, we review several American put option pricing techniques compared in this thesis. These methods covered all approaches, including lattice approach, finite difference method, Monte Carlo simulation and analytic approximations, for pricing American put option. The abbreviations of these methods are shown in table 2.1. The CRR, ABT, FD, GJ and Quad methods are widely cited in many literatures, as shown in table 2.1. The LTB, BIN and BBSR methods are simply modified from the CRR method. Boyle (1988) shows that if the probabilities of the trinomial method are identical, the numerical results of trinomial method will be better. The TEIN1 is modified from BOYLE method and set the probabilities to be equal. The LSM method is the critical improvement on Monte Carlo simulation for pricing American option. The EFB and IBZ methods are new methods. The IBZ method, from Ibanez (2003), is modified from RM and E'HSY methods and is more accurate. We can find the efficiency of modification from these methods.

It is convenient to employ the following notations:

S = asset value of put option underlying,

K = exercise price of put option,

r = the risk-free interest rate,

T = time to option expiration (in years),

s = the standard deviation of the rate of return on the underlying asset,

N = the number of time steps into which the time interval of length T is divided,

? t = T/N the time interval.

We assume that the risk-free interest rate and the standard deviation are both constant during the life of the option.

Table 2.1 the option pricing methods in related literatures

Method	Literature
CRR: The binomial method of Cox, Ross, and Rubinstein(1979)	Hull and White (1988), Huang, Subrahmanyam, and Yu (1996), Trigeorgis and Lenos (1991), Breen (1991), Tian (1993), Tain (1999), Broadie and Detemple (1996), Amin (1991), Ju and Zhong (1999), Bunch and Johnson (2000)
CV: The control-variate method of Hull and White (1988)	Hull and White (1988)
GJ: The analytic approximation method of Geske and Johnson (1984)	Geske and Johnson (1984), Hull and White (1988), Bunch and Johnson (1992), Barone-Adesi and Whaley (1987), Huang, Subrahmanyam, and Yu (1996), Broadie and Detemple (1996), Bunch and Johnson (2000), Breen (1991), Trigeorgis and Lenos (1991), Ju and Zhong (1999)
MGJ2: The modified two-point Geske-Johnson method of Bunch and Johnson(1992)	Bunch and Johnson (1992), Broadie and Detemple (1996), Ju and Zhong (1999)
CR: The method used in Cox and Rubinstein (1985)	Geske and Johnson (1984) Bunch and Johnson (1992)
ABT: The accelerate binomial method of Breen (1991)	Huang, Subrahmanyam, and Yu (1996), Breen (1991), Barone-Adesi and Whaley (1987), Broadie and Detemple (1996), Ju and Zhong (1999), Bunch and Johnson (2000)
FD: The finite difference method of Schwartz (1977)	Barone-Adesi and Whaley (1987), Trigeorgis and Lenos (1991), Huang, Subrahmanyam, and Yu (1996), Longstaff and Schwartz (2001), Bunch and Johnson (2000)
RM: The Recursive Integration Method of Huang, Subrahmanyam and Yu(1996)	Trigeorgis and Lenos (1991), Huang, Subrahmanyam, and Yu (1996), Broadie and Detemple (1996), Bunch and Johnson (2000)
IBZ: the analytic approximation method of Ibanez (2003)	Ibanez (2003)
LTB: The log-transformed binomial method of Trigeorgis (1991)	Trigeorgis and Lenos (1991)

The abbreviations of pricing methods are shown in the left side.

Table 2.1 the option pricing methods in related literatures (cont.)

LBA: The lower bound approximation of Broadie and Detemple (1996)	Broadie and Detemple (1996)
LUBA: The lower and upper bound approximation of Broadie and Detemple(1996)	Broadie and Detemple (1996) Ju and Zhong (1999)
Quad: The quadratic analytical approximation of MacMillan(1986) and Barone-Adesi and Whaley (1987)	Barone-Adesi and Whaley (1987), Trigeorgis and Lenos (1991), Broadie and Detemple (1996), Ju and Zhong (1999)
BOYLE: The trinomial method of Boyle(1988)	Tian (1993), Amin (1991)
TRIN: The trinomial method of Kamrad and Ritchken (1991)	Tain (1999), Broadie and Detemple (1996), Amin (1991)
BIN: The modified binomial method of Tian (1993)	Amin (1991), Tian (1993)
TRI1: The modified trinomial method of Tian(1993)	Tian (1993)
TRI2: The modified trinomial method of Tian(1993)	Tian (1993)
EFB: The extrapolated Flexible Binomial Model of Tian (1999)	Tain (1999)
LSM: The least-squares approach of Longstaff and Schwartz(2001)	Longstaff and Schwartz (2001)
MC: The Monte Carlo valuation of Rogers (2002)	Rogers (2002)
ML: The Method of Lines from Carr and Faguet (1994)	Pantazopoulos, Houstis and Kortesis (1998), Broadie and Detemple (1996)
FT: The Front-Tracking Finite Difference Method of Pantazopoulos, Houstis and Kortesis(1998)	Pantazopoulos, Houstis and Kortesis (1998)
BBS: The binomial Black and Scholes Method of Broadie and Detemple(1996)	Broadie and Detemple (1996)
LC: The method of Pantazopoulos, Houstis and Kortesis(1998)	Pantazopoulos, Houstis and Kortesis (1998)
PN: The method of Parkinson (1977)	Geske and Johnson (1984)
E'HSY	Huang et al. (1996)

The abbreviations of pricing methods are shown in the left side.

2.1 Lattice Approach

The binomial and trinomial option pricing methods are the key members of the lattice approach for pricing American put option. In the lattice approach, the backward procedures are the same when we evaluate the payoff of the American put option as in a European case. We describe the backward procedure in binomial method clearly. We begin with the binomial method.

2.1.1 The Binomial Option Pricing Method (CRR)

Cox, Ross, and Rubinstein (1979) (CRR) developed the binomial option pricing formula. Assume that the stock prices follow a multiplicative binomial process over discrete periods. The rate of return on the stock over each period can have two possible values: $u - 1$ with probability q, or $d - 1$ with probability $1 - q$, where $u > 1$ and $d < 1$. The stock price at the end of the period will be either uS or dS. The parameters u, d, and p satisfy the relationship below

$$u = e^{s\sqrt{\Delta t}}, \qquad d = e^{-s\sqrt{\Delta t}}, \qquad p = \frac{e^{r\Delta t} - d}{u - d}$$

Each node of a binomial tree is shown in figure 2.1. Therefore, the price of the underlying stock at define j node (i , j) is shown below:

$$S_i^j = Sd^{i-j}u^j$$

At the expiration date, i=N, the terminal prices of stock on the nodes are as follows:

$$S_N^j = Sd^{N-j}u^j$$

Let V_i^j be the value of an American put option at node (i , j) . At the expiration date, the value of the put option on all the terminal nodes is known as:

$$V_N^j = \max\{0, K - Sd^{N-j}u^j\}$$

To evaluate the value of an American put option at time 0, we start at time T and work backward through the binomial tree recursively. The value of an American put option at node (i , j) can be evaluated by the following procedure. In the first step, we calculate the expected value of put option at time step i? t from the values at time-step (i+1)? t and then discount this value with the risk-free interest rate r to obtain the present value, that is,

$$e^{-r\Delta t}\left(pV_{i+1}^{j+1} + (1-p)V_{i+1}^{j}\right).$$

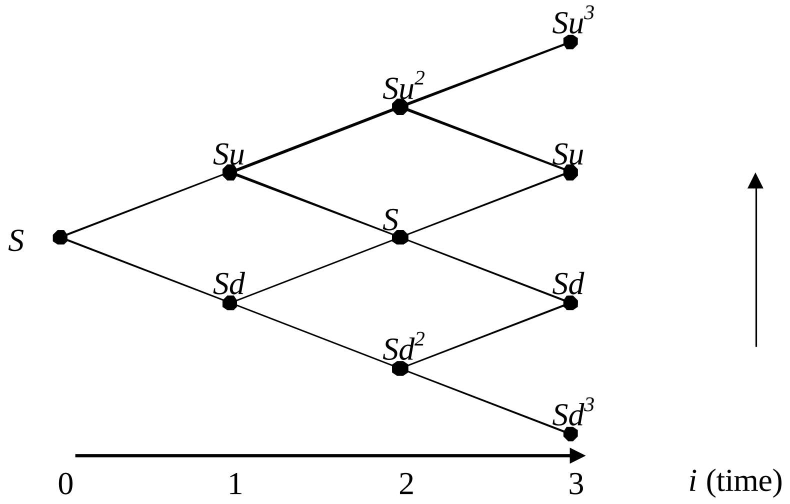

Figure 2.1 the asset value of binomial tree for N = 3

The exercise value at node (*i* , *j*) is given by $\max\left(\mathrm{K} - S_i^j, 0\right)$ In the second step, we compare the discount value with exercise value. And obtain the value of the option on node (*i* , *j*), which is the maximum of these two, that is

$$V_N^j = \max\left\{\mathrm{K} - S_i^j, e^{-r\Delta t}\left(pV_{i+1}^{j+1} + (1-p)V_{i+1}^{j}\right)\right\}.$$

Finally, V_0^0 is the value of the American put option at time 0.

2.1.2 The Trinomial Option Pricing Method (BOYLE)

Trinomial methods had been proposed in Boyle (1988), Kamrad and Ritchken (1991), Omberg (1988) and Parkinson (1977). We test the Boyle(1988) version. For the three-jump process, the asset price *S* can change to *Su* with probability P_1, or *S* with probability P_2, or *Sd* with probability P_3. And the parameters of the trinomial tree are show as follow:

$$P_1 = \frac{(V + M^2 - M)u - (M-1)}{(u-1)(u^2-1)}$$

$$P_3 = \frac{u^2(V + M^2 - M) - u^3(M-1)}{(u-1)(u^2-1)}$$

$$p_2 = 1 - p_1 - p_3$$

$$u = \exp[\, l\, s\, \sqrt{\Delta t}\,]$$

where $M = \exp(r\Delta t)$, $V = M^2[\exp(s^2 \Delta t) - 1]$, ud = 1, ? is greater than 1. Boyle(1988) shows that the best results were obtained when the probabilities were roughly equal. We set ? = $\sqrt{3/2}$ and the probabilities were roughly equal.

Each node (i,j) of a trinomial tree is difined in figure 3.2. The backward procedure is the same as the binomial method. The expected payoff of the option on node (i , j) should be valued by (i+1 , j) with probability P_3, (i+1 , j+1) with probability P_2, and (i+1 , j+2) with probability P_1, that is

$$e^{-r\Delta t}\left(p_1 V_{i+1}^{j+2} + p_2 V_{i+1}^{j+1} + p_3 V_{i+1}^{j}\right).$$

And the value of the option on node (i , j) will become

$$V_N^j = \max\left\{\mathrm{K} - S_i^j ,\, e^{-r\Delta t}\left(p_1 V_{i+1}^{j+2} + p_2 V_{i+1}^{j+1} + p_3 V_{i+1}^{j}\right)\right\}.$$

V_0^0 is the value of American put option.

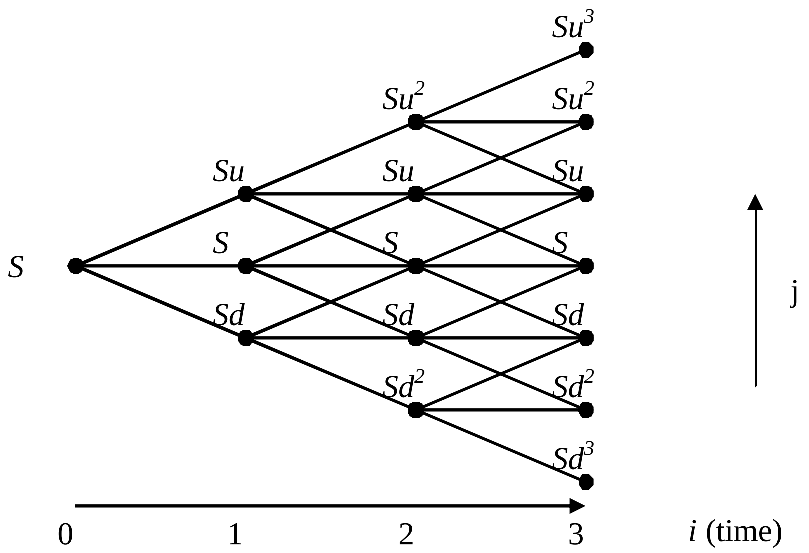

Figure 2.2 the asset value of trinomial tree for N = 3

2.1.3 The Log-Transformed Binomial Option Pricing Method (LTB)

Trigeorgis (1991) developed Log-Transformed Binomial Method. Let X = log S follows an arithmetic Brownian motion. Under risk neutrality,

$$dX = d\log S = (r - \tfrac{1}{2}s^{2})dt + s\,dz \quad \text{..(2.1)}$$

with r being risk-free rate. Within each discrete subinterval ? t, X follows a Markov random walk moving up by an amount $?X = H$ with probability P, and down by the same amount ($?X = -H$) with probability 1 – P. The key variables:

$$K \equiv s^{2}T/N\,; \qquad m \equiv r/s^{2} - \tfrac{1}{2}\,;$$

$$H \equiv \sqrt{K + (mK)^{2}}\,;\ \ P \equiv \tfrac{1}{2}(1 + mK/H)\,;$$

Let j be the integer number of time-steps, i be the integer index of the state variable X corresponding to the net number of ups less downs (i.e., $X(i) = X_0 + iH$). For each state i, the underlying asset $V(i) = e^{X_0 + iH}$, and $R(i)$ denote the total investment opportunity value.

The log-transformed binomial algorithm consists of four main steps: First, parameters affecting option values (i.e., V, r, s^2, and exercise price, EX) must be specified as well. Second, calculate the preliminary parameters needed for the rest of the algorithm (i.e., K, μ, H, and P). Third, determine the terminal values (at $j = N$), asset value: $V(i) = e^{X_0 + iH}$ and opportunity value: $R(i) = \max(V(i) - EX, 0)$. The fourth step follows a backward iterative process, with adjustments for cash flows (dividends) and real options at appropriate times. The opportunity value $R(i)$ at each node is

$$R(i) = \max\left\{\max(V(i) - EX, 0),\, e^{-(r-d)(k/s^{2})}[pR(i+1) + (1-p)R(i-1)]\right\}$$

2.1.4 The Modified Binomial Option Pricing Method (BIN)

Tian (1993) developed a modified approach to the selection of lattice parameters. We call the modified binomial method of Tian (1993) as "BIN". Suppose the stock price follows Eq. (2.1). A BIN option pricing method can be developed as follows. During each time period, the stock price is showed in figure 2.1. The parameters of the BIN option pricing method are as follow:

$$p = \frac{M - d}{u - d}, \qquad q = 1 - p$$

$$u = \frac{MV}{2}\left[(V + 1) + \sqrt{V^2 + 2V - 3}\right]$$

$$d = \frac{MV}{2}\left[(V + 1) - \sqrt{V^2 + 2V - 3}\right]$$

where

$$M = \exp(r\Delta t), \quad V = \exp(s^2 \Delta t)$$

The American put option value can be valued by using a backward recursive procedure as binomial method.

2.1.5 The Modified Trinomial Option Pricing Method (TRIN1)

Tian (1993) developed two modified trinomial method. We test the First Modified Trinomial Method (TRIN1). Because Tian (1993) indicated that it is more accuracy than the other one. The stock price of TRIN1 is showed in figure 2.2. The parameters are as follow:

$$p_1 = p_2 = p_3 = \frac{1}{3}$$

$$u = K + \sqrt{K^2 - m^2}, \qquad d = K - \sqrt{K^2 - m^2}$$

$$m = \frac{M(3 - V)}{2}$$

where

$$M = \exp(r\Delta t), \quad V = \exp(s^{2}\Delta t)$$

$$K = M(V+3)/4\,.$$

The American put option value can be valued by using a backward recursive procedure as trinomial method.

2.1.6 The Extensible Flexible Binomial Option Pricing Method (EFB)

Tian(1999) developed a flexible binomial method (FB) with a “tilt” parameter that alters the shape and span of the binomial tree. In flexible binomial method, the tree can tilt upwards or downwards if the tilt parameter is positive or negative, respectively. And standard and barrier options can be valued more accurately. Specifically, smooth convergence is possible for standard European and American options, and extrapolation methods are used to enhance the rate of convergence.

The stock price is showed in figure 2.1. The parameters u, d, and p satisfy the below relationship

$$u = e^{s\sqrt{\Delta t} + l s^{2}\Delta t}, \qquad d = e^{-s\sqrt{\Delta t} + l s^{2}\Delta t}, \qquad p = \frac{e^{r\Delta t} - d}{u - d}$$

where λ is an arbitrary constant, called the “tilt parameter,” that can positive, zero of negative.

By selecting a tilt parameter, a node in the tree coincides exactly with the strike price at the maturity of the option. Let the initial value of λ be: λ_0 = 0, which represents the CRR method. The node closet to the strike price K, (N , j_0), can be determined by solving the following equation:

$$S_0\, u_0^{j_0}\, d_0^{N-j_0} = K$$

The closest node to K is found by defining j_0 as:

$$j_0 = \left[\frac{\log(K/S_0) - N\log(d_0)}{\log(u_0/d_0)} \right]$$

where [•] denotes the closest integer argument. To ensure that node (N , j_0) coincides exactly with the strike price *X*, a new λ is select such that:

$$S_0 u^{j_0} d^{N-j_0} = K$$

we can get

$$\lambda = \frac{\log(K / S_0) - (2 j_0 - N)\sigma \sqrt{\Delta t}}{N\sigma^2 \Delta t}$$

with this particular choice of λ, the strike price is always locate on node (N , j_0) at the maturity of the option.

With λ, FB method can converge smoothly, as shown in figure 2 of Tian(1999). An extrapolation technique can be used to greatly enhance the rate of convergence for the FB method:

$$\hat{P}(2N) = \frac{\rho P(2N) - P(N)}{\rho - 1}$$

where ρ is the limit of the error ratio series. When we use FB Method to pricing American put options, Tian(1999) proved that ρ converges to 2. The value of American put option can be valued backward as binomial method.

2.1.7 The Binomial Black and Scholes Option Pricing Method with Richardson extrapolation (BBSR)

The binomial Black and Scholes option pricing method (BBS) was suggested by Broadie and Detemple (1996). Broadie and Detemple (1996) used the Black and Scholes formula to replace the value of the time step just before option maturity. Broadie and Detemple (1996) showed that BBS can converge smoothly and the Richardson extrapolation can be used to improve the accuracy. We use two-point Richardson extrapolation, $P = 2P(2) - P(1)$, to improve the accuracy, where P(2) is the BBS method with *N* and P(1) with *N*/2 time steps.

2.1.8 The Accelerated binomial method (ABT)

Breen (1991) suggested the accelerated binomial method. Breen (1991) used the Geske-Johnson analytic formula in the binomial option pricing method. That is

$$P = P(3) + 3.5 * [P(3) - P(2)] - 0.5 * [P(2) - P(1)]$$

where $P(1)$ = the binomial option pricing value can only exercise at maturity T, $P(2)$ = the binomial option pricing value can only exercise at T and $T/2$, $P(3)$ = the binomial option pricing value can only exercise at T, $2T/3$, and $T/3$. We use the CRR method as the binomial option pricing method. It is the same as Breen (1991).

2.2 Finite Difference Method

There are many kinds of finite difference methods. We test the Explicit and Implicit finite difference method of Hull and White (1990).

2.2.1 The Explicit Finite Difference Method (EFD)

The finite difference approach was suggested by Schwartz (1977) and Brennan and Schwartz (1978). Hull and White (1990) use finite difference methods with ln S rather than S as the underlying variable.

The explicit finite difference method to the Black-Scholes PDE is

$$rf = \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}s^{2}S^{2}\frac{\partial^{2} f}{\partial S^{2}} \quad \text{.......} (2.2)$$

In terms of Z = ln S, it will become

$$rf = \frac{\partial f}{\partial t} + (r - s^{2}/2)\frac{\partial f}{\partial Z} + \frac{1}{2}s^{2}\frac{\partial^{2} f}{\partial Z^{2}}$$

We divide the life of the option in to N-period (N = 1) with equal interval of length ? $t = T/N$ and divide the stock price in to $2N$+1 points. To implement the explicit finite difference method, a small time interval, $\triangle t$, and a small change in Z, $\triangle Z$, are chosen. It is

$$Z_0, Z_0 + \Delta Z, Z_0 + 2\Delta Z, \ldots, Z_{\max} ,$$

and time is equal to

$$0, \Delta t, 2\Delta t, \ldots, T .$$

To obtain the explicit finite difference methods we approximate the Black-Scholes equation using a forward difference for $\partial f / \partial t$ and central difference for $\partial^2 f / \partial^2 Z^2$ and $\partial f / \partial Z$. We can obtain

$$rf_{i,j} = \frac{f_{i+1} - f_{i,j}}{\Delta t} + (r - s^2/2)\frac{f_{i+1,j+1} - f_{i+1,j-1}}{2\Delta Z} + \frac{1}{2}s^2 \frac{f_{i+1,j+1} - 2f_{i+1,j} + f_{i+1,j-1}}{\Delta Z^2}$$

which can be rewritten as

$$f_{i,j} = \frac{1}{1 + r\Delta t}\left(a_j f_{i+1,j-1} + b_j f_{i+1,j} + c_j f_{i+1,j+1}\right)$$

$$a_j = -\frac{\Delta t}{2\Delta Z}(r - s^2/2) + \frac{\Delta t}{2\Delta Z^2} s^2$$

$$b_j = 1 - \frac{\Delta t}{2\Delta Z^2} s^2$$

$$c_j = \frac{\Delta t}{2\Delta Z}(r - s^2/2) + \frac{\Delta t}{2\Delta Z^2} s^2$$

We set $\Delta Z = s\sqrt{3\Delta t}$ and it is numerically efficient, shown in Hull (2003).

2.2.2 The Implicit Finite Difference Method (IFD)

Consider now approximating the transformed Black-Sholes PDE, Eq. (2.2), by replacing the space derivatives with central difference at time step *i* rather than at *i*+1. We can obtain

$$rf_{i,j} = \frac{f_{i+1} - f_{i,j}}{\Delta t} + (r - s^2/2)\frac{f_{i,j+1} - f_{i,j-1}}{2\Delta Z} + \frac{1}{2}s^2 \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{\Delta Z^2}$$

which can be rewritten as

$$f_{i+1,j} = P_d f_{i+1,j-1} + P_m f_{i+1,j} + P_u f_{i+1,j+1}$$

$$P_d = -\frac{1}{2}\Delta t\left(\frac{s^2}{\Delta Z^2} - \frac{(r - s^2/2)}{\Delta Z}\right)$$

$$P_m = 1 + \frac{\Delta t}{2\Delta Z^2} s^2 + r\Delta t$$

$$P_u = -\frac{1}{2}\Delta t\left(\frac{s^2}{\Delta Z^2} + \frac{(r - s^2/2)}{\Delta Z}\right)$$

Each equation for $j = -N_j+1,\ldots, -N_j-1$ can be solved for the option values at time step i.

2.3 Simulation Approach

The simulation approach is first introduced in Boyle(1977). Longstaff and Schwartz(2001) suggested a simple approach, named LSM, to deal with the early exercise problem. The key point of this approach is estimating the conditional expected payoff by least squares to decide whether to exercise the option or not.

2.3.1 Least-Squares Simulation (LSM)

We repeat the numerical example on page 115-120 of Longstaff and Schwartz (2001) in pricing American put options. The initial stock matrix is shown in Longstaff and Schwartz (2001). The option values at time 3 are tabulated below.

Path	t = 1	t = 2	t = 3
1	—	—	0.00
2	—	—	0.00
3	—	—	0.07
4	—	—	0.18
5	—	—	0.00
6	—	—	0.20
7	—	—	0.09
8	—	—	0.00

Consider the five paths which are in the money at time 2. Let X denote the stock prices at time 2 for these five paths and Y denote the corresponding discounted option value received at time 3. The arrays X and Y are tabulated below.

Path	Y	X
1	0.00 ×0.94176	1.08
2	—	—
3	0.07 ×0.94176	1.07
4	0.18 ×0.94176	0.97
5	—	—
6	0.20 ×0.94176	0.77
7	0.09 ×0.94176	0.84
8	—	—

By regressing Y on a constant, X, X^2, the resulting conditional expectation function at time 2 is $E[Y \mid X] = -1.070 + 2.983X - 1.813X^2$. The exercise values and expected values of continuation at time 2 are as follows.

Path	Exercise	Continue
1	0.02	0.0369
2	—	—
3	0.03	0.0461
4	0.13	0.1176
5	—	—
6	0.33	0.1520
7	0.26	0.1565
8	—	—

If the exercise value is greater than the expected value of continuation, the option value is set to the exercise value; otherwise, the option value is set to the discounted option value from time 3. The option values at time 2 are as follows.

Path	t = 1	t = 2	t = 3
1	—	0.00	0.00
2	—	0.00	0.00
3	—	0.00	0.07
4	—	0.13	0.00
5	—	0.00	0.00
6	—	0.33	0.00
7	—	0.26	0.00
8	—	0.00	0.00

Repeat the procedure until time 0, the option value can be determined by the average value of the eight paths.

We summarize the valuing procedure of the LSM algorithm as follows:

1. Generate the stock price matrix.
2. Determine the basis function form of $E[Y \mid X]$ and the order of the basis function.
3. For in-the-money paths, regress corresponding discounted option values.
4. For in-the-money paths, compare the exercise values and the expected values of continuation to determine the option values at that time.
5. Go to step 3 and repeat the procedure until time 0.

2.4 Analytic Approximation Methods

There are many analytic approximation methods. We test Barone-Adesi and Whaley(1987) method, Geske and Johnson(1984) and Ibanez (2003) methods.

2.4.1 Barone-Adesi and Whaley method (Quad)

Barone-Adesi and Whaley(1987) suggested an analytic approximation method for pricing American options. The key insight is that the price of American options is the price of European options adding the early exercise premium. First, we define the early exercise premium $e(S,t)$ is

$$e(S,t) = P_A(S,t) - P_E(S,t)$$

where $P_A(S,t)$ is the price of American put options, $P_E(S,t)$ is the price of European put options. The partial differential equation is showed in Eq. (2.2). We can get

$$S^2 f_{SS} + NSf_S - (M / K) f = 0$$

where $M = 2r/\sigma^2$ and $N = 2b/\sigma^2$, b is the cost-of-carry rate. The general solution to above equation is

$$f(S) = a_1 S^{q_1} + a_2 S^{q_2}$$

where $q_1 = [-(N-1) - \sqrt{(N-1)^2 + 4M/K}]/2$,

$$q_2 = [-(N-1) + \sqrt{(N-1)^2 + 4M/K}]/2.$$

In above equations, the term $a_1 S^{q_1}$ must approach 0 as S approaches positive infinity. The term $a_2 S^{q_2}$, violates the boundary condition, so a_2 is set to zero. The approximate value of the American put option becomes

$$P(S,T) = p(S,T) + \mathrm{A}_1(\mathrm{S}/\mathrm{S}^{**})^{q_1} \quad when \quad S > S^{**} \quad and$$
$$P(S,T) = X - S \quad when \quad S \le S^{**}$$

where $p(S,t)$ is the Black-Scholes put option price,

$$A_1 = -(S^{**}/q_1)\{1 - e^{(b-r)T} N[-d_1(S^{**})]\},$$

S^{**} is critical commodity price, that is determined by solving

$$X - S^{**} = p(S^{**},T) - \{1 - e^{(b-r)T} N[-d_1(S^{**})]\}S^{**}/q_1$$

2.4.2 Geske and Johnson's method (GJ2)

Geske and Johnson (1984) proposed the American put option valuation. They point out that the option holder would exercise the put if the payoff from exercising the put exceeds the value of the put if it is not exercised. This indicates that the critical stock price (S^*) is independent of the current stock and determine from the condition of P(S,T) ≧ max(K-S). For some $S = S^*$ and any T, the exercise proceeds equaled the American put value or K-S = $P(S^*,T)$. They price the American put as discounted expected value of all future cash flows. The cash flow arises because the put can be exercised at the next instant, dt, or the following instant, $2dt$, if not previously exercised. In addition, the put will be exercised when the stock price was below the critical stock price and we obtain the following equation for the value of an American put option：

$$P = Kw_2 - Sw_1$$

where

$$w_1 = \left\{ N_1\left(-d_1(\overline{S}_{dt}, dt)\right) + N_2\left(d_1(\overline{S}_{dt}, dt), -d_1(\overline{S}_{2dt}, 2dt)\right); -r_{12} \right\}$$

$$w_2 = \left\{ e^{-rdt} N_1\left(-d_2(\overline{S}_{dt}, dt)\right) + e^{-2rdt} N_2\left(d_2(\overline{S}_{dt}, dt), -d_2(\overline{S}_{2dt}, 2dt)\right); -r_{12} \right\}$$

and

$$d_1(q,T) = \frac{\ln\left(\frac{s}{q}\right) + \left(r + \frac{1}{2}s^2\right)T}{s\sqrt{T}}, \quad d_2(q,T) = d_1 - s\sqrt{T}, \quad r_{i,j} = \sqrt{\frac{i}{j}}$$

ρ is the correlation coefficients. $\overline{S}_{dt}$, $\overline{S}_{2dt}$ are the first and second critical stock price; N_1, N_2 are the standard cumulative univariate, bivariate normals.

2.4.3 The analytic approximation method of Ibanez (2003) (IBZ)

Ibanez (2003) introduced an analytic approximation method for pricing American put options. It is based on the argument that the American put option can be decomposed into an equivalent European option plus an early exercise premium and Richardson extrapolation. The American put option can be approximated by following equation:

$$P_t^N(S_t) = p_t(S_t) + e_t^N(S_t)$$

$$e_{t_n}^N = (1 - e^{-r\Delta t})K \sum_{m=n+1}^{N-1} e^{-rt_m} N(-d_2(S_{t_n}, B_{t_n}^N, t_m)) +$$
$$c(1 - e^{-r\Delta t})Ke^{-rt_N} N(-d_2(S_{t_n}, B_{t_N}^N, t_N))$$

where c = 1/2, $t_m = t_m - t_n$, $B_{t_n}^N$ is the optimal exercise frontier, determined by solving

$$K - B_{t_n}^N = p_{t_n}(B_{t_n}^N) + e_{t_n}^N(B_{t_n}^N)$$

Ibanez (2003) suggests a conservative criterion rule for extrapolation:

- Never use the first three terms $\Delta t = \{t, t/2, t/3\}$.
- Only use terms such that $t/N_j < 1$.
- Use just three terms for extrapolation, $J = 3$.
- For short-term options, with less than 1 year to maturity ($t < 1$), use $N_1 = 4$, $N_2 = 5$, $N_3 = 6$.
- For long-term options ($t = 1$), if $t = 1.0$ ($N = 8$), if $t = 2.0$ ($N = 15$), if $t = 3.0$ ($N = 22$)