5. The rule-extraction of a 3-layer feed-forward approximation network

In this section, we explain how to back-propagate a response over a 3-layer feed-forward approximation network. That is, for each hidden node, assume the *piecewise-polynomial function g* defined in equation (9) is adopted to approximate the transfer function *tanh*. $\beta_1 > 0$; $\beta_2 < 0$; $\kappa > 0$; $\beta_1\kappa + \beta_2\kappa^2 = 1$.

$$g(net) \equiv \begin{cases} 1 & if\ \text{net} \geq \kappa \\ \beta_1 net + \beta_2 net^2 & if\ 0 \leq net < \kappa \\ \beta_1 net - \beta_2 net^2 & if\ -\kappa < net < 0 \\ -1 & if\ \text{net} \leq -\kappa \end{cases} \quad (9)$$

The feed-forward computation of a 3-layer approximation network is of the form $net_i = {}_2\theta_{i0} + \sum_{j=1}^{m} {}_2w_{ij}\, x_j$ and $y = {}_3w_0 + \sum_{i=1}^{p} {}_3w_i\, g(net_i)$. We can distinguish three sub-processes in this computation: (1) *an affine net transformation sub-process* in which $net_i = {}_2\theta_{i0} + \sum_{j=1}^{m} {}_2w_{ij}\, x_j$ and $\mathbf{net} = \Phi_{\text{affine}}(\mathbf{x})$ in which $\mathbf{net} \equiv (net_1, \ldots, net_p)^T$; (2) *an approximation function sub-process* in which $\hat{a}_i = g(net_i)$ and $\hat{\mathbf{a}} = \Phi_g(\mathbf{net})$; and (3) *a linear output transformation sub-process* in which $\hat{y} = {}_3w_0 + \sum_{i=1}^{p} {}_3w_i\, \hat{a}_i$ and $\hat{y} = \Phi_{\text{linear}}(\hat{\mathbf{a}})$. net_i can be represented as ${}_2\theta_{i0} + {}_2\mathbf{w}_i^T\, \mathbf{x}$ and ${}_3w_0 + \sum_{i=1}^{p} {}_3w_i\, \hat{a}_i$ can be represented as ${}_3w_0 + {}_3\mathbf{w}^T \hat{\mathbf{a}}$.

We will show separately how a responded output is back-propagated in these three computation sub-processes to extract the corresponding rule.

5.1 The back-propagation with respect to the linear output transformation sub-process

If we have the responded output $\hat{y}$, we can infer that ${}_3\mathbf{w}^T \hat{\mathbf{a}} = \hat{y} - {}_3w_0$. Thus, the responded output $\hat{y}$ is back-propagated to $\Phi_{\text{linear}}^{-1}(\hat{y}) \equiv \{\hat{\mathbf{a}} \mid {}_3\mathbf{w}^T \hat{\mathbf{a}} = \hat{y} - {}_3w_0\}$ which may be a hyperplane in the $\hat{\mathbf{a}}$ space.

5.2 The back-propagation with respect to the approximation function sub-process

The *piecewise-polynomial function* induces a partition of the **net** space into four areas. Thus, p independent hidden nodes result into 4^p cells in the **net** space. On each cell, the approximated activation value of each hidden node is calculated from a polynomial function. Therefore, the computation of the reciprocal image of $\hat{a}_i$ contained in a cell is a task referred to the computation of the inverse polynomial function corresponding to that cell.

For any $\hat{\mathbf{a}}$, let $\Phi_g^{-1}(\hat{\mathbf{a}})$ be the set of **net** whose images under the g function are $\hat{\mathbf{a}}$. Thus, if a given $\hat{\mathbf{a}}$ is in some cell, $\Phi_g^{-1}(\hat{\mathbf{a}})$ can be calculated through the inverse polynomial functions corresponding to that cell. When given the $\Phi_{linear}^{-1}(\hat{y})$, the cell corresponding to each $\hat{\mathbf{a}} \in \Phi_{linear}^{-1}(\hat{y})$ can be identified and thus $\Phi_g^{-1}(\Phi_{linear}^{-1}(\hat{y}))$ can be calculated through the union of $\Phi_g^{-1}(\hat{\mathbf{a}})$ with $\hat{\mathbf{a}} \in \Phi_{linear}^{-1}(\hat{y})$.

5.3 The back-propagation with respect to the affine net transformation sub-process

If we have the **net** response, we can infer that $_2\mathbf{W}\,\mathbf{x} = \mathbf{net} - {}_2\theta$. Thus, the **net** response can be back-propagated to $\Phi_{affine}^{-1}(\mathbf{net}) \equiv \{\mathbf{x} | \, _2\mathbf{W}\,\mathbf{x} = \mathbf{net} - {}_2\theta\}$ which is affine space of dimension m - $rank(_2\mathbf{W})$ in the input space.

The given region $\Phi_g^{-1}(\Phi_{linear}^{-1}(\hat{y}))$ in the **net** space can be back-propagated to $Xa(\hat{y}) \equiv \Phi_{affine}^{-1}(\Phi_g^{-1}(\Phi_{linear}^{-1}(\hat{y}))) = \{\mathbf{x} | \, _2\mathbf{W}\,\mathbf{x} = \mathbf{net} - {}_2\theta, \mathbf{net} \in \Phi_g^{-1}(\Phi_{linear}^{-1}(\hat{y}))\}$ can be calculated through the union of $\Phi_{affine}^{-1}(\mathbf{net})$s in which $\mathbf{net} \in \Phi_g^{-1}(\Phi_{linear}^{-1}(\hat{y}))$. Hence $Xa(\hat{y})$ is the set of **x**s whose images under the g function are $\hat{y}$.

In sum, we have the following (general) rule corresponding to the 3-layer feed-forward approximation network:

If ($\mathbf{x} \in Xa(\hat{y})$), the output value equals $\hat{y}$.

5.4 An illustration of the rule-extraction

Let's take the Network III in Figure 4 to illustrate the proposed rule-extraction method. The Network III has two hidden nodes, two input nodes and one output node. net_1 and net_2 are defined in equation (10); the g function defined in equation (11) is adopted to approximate the activation function of all hidden nodes; the approximated output $\hat{y}$ is shown in equation (12). The range of non-vague $\hat{y}$ is [-3, 1].

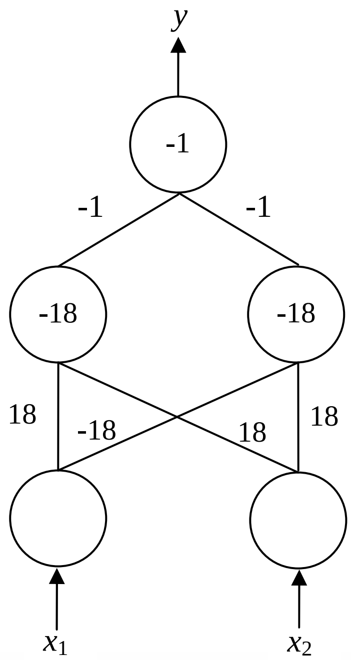

Figure 4: The Network III.

$$net_1 = 18x_1 + 18x_2 - 18$$
$$net_2 = -18x_1 + 18x_2 - 18 \quad (10)$$

$$\hat{a}_i = g(net_i) \equiv \begin{cases} 1 & if\ net_i \ge \kappa \\ \beta_1 net_i + \beta_2 net_i^2 & if\ 0 \le net_i < \kappa \\ \beta_1 net_i - \beta_2 net_i^2 & if\ -\kappa < net_i < 0 \\ -1 & if\ net_i \le -\kappa \end{cases} \quad (11)$$

$$\hat{y} = -\hat{a}_1 - \hat{a}_2 - 1 \quad (12)$$

The (piecewise-polynomial) g function induces a partition of the **net** space into four areas. Thus, two independent g functions result into 16 cells in the **net** space, and the approximate activation value of each hidden node on each cell is calculated from its corresponding polynomial approximation function. These cells and their corresponding inverse functions are as follows:

1. For any **net** $\in$ {**net**| $net_1 \ge \kappa$, $net_2 \ge \kappa$}: $\hat{a}_1 = 1$, $\hat{a}_2 = 1$, and $\hat{y} = -3$. $\Phi_{linear}^{-1}(-3) = \{(1, 1)^T\}$, $\Phi_g^{-1}((1, 1)^T)$ = {**net**| $net_1 \ge \kappa$, $net_2 \ge \kappa$}, and Φ_{affine}^{-1}(**net**) = {**x**| $18x_1 + 18x_2 - 18 = net_1$, $-18x_1 + 18x_2 - 18 = net_2$}.
2. For any **net** $\in$ {**net**| $0 \le net_1 < \kappa$, $net_2 \ge \kappa$}: $\hat{a}_1 = \beta_1 net_1 + \beta_2 net_1^2$, $0 \le \hat{a}_1 < 1$, $\hat{a}_2 = 1$, $\hat{y} = -\hat{a}_1 - 2$, and $\hat{y} \in (-3, -2]$. $\Phi_{linear}^{-1}(\hat{y})$ = {**â** | $\hat{a}_1 = -\hat{y} - 2$, $\hat{a}_2 = 1$} = $\{(-\hat{y}-2, 1)^T\}$, $\Phi_g^{-1}(\hat{\mathbf{a}})$ = {**net**| $\beta_2 net_1^2 + \beta_1 net_1 = \hat{a}_1$, $0 \le net_1 < \kappa$, $net_2 \ge \kappa$} =

{**net**| $net_1 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4\hat{a}_1\beta_2}}{2\beta_2}$, $net_2 \geq \kappa$ }, and $\Phi_{\text{affine}}^{-1}(\mathbf{net})$ = {**x**| $18x_1 + 18x_2 - 18 = net_1$, $-18x_1 + 18x_2 - 18 = net_2$}.

3. For any **net** ∈ {**net**| $-\kappa < net_1 < 0$, $net_2 \geq \kappa$}: $\hat{a}_1 = \beta_1 net_1 - \beta_2 net_1^2$, $-1 < \hat{a}_1 < 0$, $\hat{a}_2 = 1$, $\hat{y} = -\hat{a}_1 - 2$, and $\hat{y} \in (-2, -1)$. $\Phi_{\text{linear}}^{-1}(\hat{y})$ = { $\hat{\mathbf{a}}$ | $\hat{a}_1 = -\hat{y} - 2$, $\hat{a}_2 = 1$} = {$(-\hat{y}-2, 1)^T$}, $\Phi_g^{-1}(\hat{\mathbf{a}})$ = {**net**| $-\beta_2 net_1^2 + \beta_1 net_1 = \hat{a}_1$, $-\kappa < net_1 < 0$, $net_2 \geq \kappa$} = {**net**| $net_1 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4\hat{a}_1\beta_2}}{2\beta_2}$, $net_2 \geq \kappa$ }, and $\Phi_{\text{affine}}^{-1}(\mathbf{net})$ = {**x**| $18x_1 + 18x_2 - 18 = net_1$, $-18x_1 + 18x_2 - 18 = net_2$}.
4. For any **net** ∈ {**net**| $net_1 \leq -\kappa$, $net_2 \geq \kappa$}: $\hat{a}_1 = -1$, $\hat{a}_2 = 1$, and $\hat{y} = -1$. $\Phi_{\text{linear}}^{-1}(-1)$ = {$(-1, 1)^T$}, $\Phi_g^{-1}((-1, 1)^T)$ = {**net**| $net_1 \leq \kappa$, $net_2 \geq \kappa$}, and $\Phi_{\text{affine}}^{-1}(\mathbf{net})$ = {**x**| $18x_1 + 18x_2 - 18 = net_1$, $-18x_1 + 18x_2 - 18 = net_2$}.
5. For any **net** ∈ {**net**| $net_1 \geq \kappa$, $0 \leq net_2 < \kappa$}: $\hat{a}_1 = 1$, $\hat{a}_2 = \beta_1 net_2 + \beta_2 net_2^2$, $0 \leq \hat{a}_2 < 1$, $\hat{y} = -\hat{a}_2 - 2$, and $\hat{y} \in (-3, -2]$. $\Phi_{\text{linear}}^{-1}(\hat{y})$ = { $\hat{\mathbf{a}}$ | $\hat{a}_1 = 1$, $\hat{a}_2 = -\hat{y} - 2$} = {$(1, -\hat{y}-2)^T$}, $\Phi_g^{-1}(\hat{\mathbf{a}})$ = {**net**| $net_1 \geq \kappa$, $\beta_2 net_2^2 + \beta_1 net_2 = \hat{a}_2$, $0 \leq net_2 < \kappa$} = {**net**| $net_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4\hat{a}_2\beta_2}}{2\beta_2}$, $net_1 \geq \kappa$}, and $\Phi_{\text{affine}}^{-1}(\mathbf{net})$ = {**x**| $18x_1 + 18x_2 - 18 = net_1$, $-18x_1 + 18x_2 - 18 = net_2$}.
6. For any **net** ∈ {**net**| $0 \leq net_1 < \kappa$, $0 \leq net_2 < \kappa$}: $\hat{a}_1 = \beta_1 net_1 + \beta_2 net_1^2$, $0 \leq \hat{a}_1 < 1$, $\hat{a}_2 = \beta_1 net_2 + \beta_2 net_2^2$, $0 \leq \hat{a}_2 < 1$, $\hat{y} = -\hat{a}_1 - \hat{a}_2 - 1$ and $\hat{y} \in (-3, -1]$; $\Phi_{\text{linear}}^{-1}(\hat{y})$ = { $\hat{\mathbf{a}}$ | $\hat{a}_1 + \hat{a}_2 = -\hat{y} - 1$}, $\Phi_g^{-1}(\hat{\mathbf{a}})$ = {**net**| $\beta_2 net_1^2 + \beta_1 net_1 = \hat{a}_1$, $0 \leq net_1 < \kappa$, $\beta_2 net_2^2 + \beta_1 net_2 = \hat{a}_2$, $0 \leq net_2 < \kappa$} = {**net**| $net_1 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4\hat{a}_1\beta_2}}{2\beta_2}$, $net_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4\hat{a}_2\beta_2}}{2\beta_2}$ }, and $\Phi_{\text{affine}}^{-1}(\mathbf{net})$ = {**x**| $18x_1 + 18x_2 - 18 = net_1$, $-18x_1 + 18x_2 - 18 = net_2$}.
7. For any **net** ∈ {**net**| $-\kappa < net_1 < 0$, $0 \leq net_2 < \kappa$}: $\hat{a}_1 = \beta_1 net_1 - \beta_2 net_1^2$, $-1 < \hat{a}_1 < 0$, $\hat{a}_2 = \beta_1 net_2 + \beta_2 net_2^2$, $0 \leq \hat{a}_2 < 1$, $\hat{y} = -\hat{a}_1 - \hat{a}_2 - 1$ and $\hat{y} \in (-2, 0)$; $\Phi_{\text{linear}}^{-1}(\hat{y})$ = { $\hat{\mathbf{a}}$ | $\hat{a}_1 + \hat{a}_2 = -\hat{y} - 1$}, $\Phi_g^{-1}(\hat{\mathbf{a}})$ = {**net**| $-\beta_2 net_1^2 + \beta_1 net_1 = \hat{a}_1$, $-\kappa < net_1 < 0$, $\beta_2 net_2^2 + \beta_1 net_2 = \hat{a}_2$, $0 \leq net_2 < \kappa$} = {**net**| $net_1 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4\hat{a}_1\beta_2}}{2\beta_2}$, $net_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4\hat{a}_2\beta_2}}{2\beta_2}$ }, and $\Phi_{\text{affine}}^{-1}(\mathbf{net})$ = {**x**| $18x_1 + 18x_2 - 18 = net_1$, $-18x_1 + 18x_2 - 18 = net_2$}.
8. For any **net** ∈ {**net**| $net_1 \leq -\kappa$, $0 \leq net_2 < \kappa$}: $\hat{a}_1 = -1$, $\hat{a}_2 = \beta_1 net_2 + \beta_2 net_2^2$, $0 \leq \hat{a}_2 < 1$, $\hat{y} = -\hat{a}_2$ and $\hat{y} \in (-1, 0]$; $\Phi_{\text{linear}}^{-1}(\hat{y})$ = { $\hat{\mathbf{a}}$ | $\hat{a}_1 = -1$, $\hat{a}_2 = -\hat{y}$ } = {$(-1, -\hat{y})^T$}, $\Phi_g^{-1}(\hat{\mathbf{a}})$ = {**net**| $net_1 \leq -\kappa$, $\beta_2 net_2^2 + \beta_1 net_2 = \hat{a}_2$, $0 \leq net_2 < \kappa$} =

{**net**| $net_1 \le -\kappa$, $net_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4\hat{a}_2\beta_2}}{2\beta_2}$ }, and Φ_{affine}^{-1}(**net**) = {**x**| $18x_1 + 18x_2 - 18 = net_1$, $-18x_1 + 18x_2 - 18 = net_2$}.

9. For any **net** ∈ {**net**| $net_1 \ge \kappa$, $-\kappa < net_2 < 0$}: $\hat{a}_1 = 1$, $\hat{a}_2 = \beta_1 net_2 - \beta_2 net_2^2$, $-1 < \hat{a}_2 < 0$, $\hat{y} = -\hat{a}_2 - 2$ and $\hat{y} \in (-2, -1)$; $\Phi_{linear}^{-1}(\hat{y})$ = {$\hat{\mathbf{a}}$| $\hat{a}_1 = 1$, $\hat{a}_2 = -\hat{y}-2$} = {$(1, -\hat{y}-2)^T$}, $\Phi_g^{-1}(\hat{\mathbf{a}})$ = {**net**| $net_1 \ge \kappa$, $-\beta_2 net_2^2 + \beta_1 net_2 = \hat{a}_2$, $-\kappa < net_2 < 0$} = {**net**| $net_2 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4\hat{a}_2\beta_2}}{2\beta_2}$, $net_1 \ge \kappa$}, and Φ_{affine}^{-1}(**net**) = {**x**| $18x_1 + 18x_2 - 18 = net_1$, $-18x_1 + 18x_2 - 18 = net_2$}.

10. For any **net** ∈ {**net**| $0 \le net_1 < \kappa$, $-\kappa < net_2 < 0$}: $\hat{a}_1 = \beta_1 net_1 + \beta_2 net_1^2$, $0 \le \hat{a}_1 < 1$, $\hat{a}_2 = \beta_1 net_2 - \beta_2 net_2^2$, $-1 < \hat{a}_2 < 0$, $\hat{y} = -\hat{a}_1 - \hat{a}_2 - 1$ and $\hat{y} \in (-2, 0)$; $\Phi_{linear}^{-1}(\hat{y})$ = {$\hat{\mathbf{a}}$| $\hat{a}_1 + \hat{a}_2 = -\hat{y}-1$}, $\Phi_g^{-1}(\hat{\mathbf{a}})$ = {**net**| $\beta_2 net_1^2 + \beta_1 net_1 = \hat{a}_1$, $0 \le net_1 < \kappa$, $-\beta_2 net_2^2 + \beta_1 net_2 = \hat{a}_2$, $-\kappa < net_2 < 0$} = {**net**| $net_1 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4\hat{a}_1\beta_2}}{2\beta_2}$, $net_2 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4\hat{a}_2\beta_2}}{2\beta_2}$ }, and Φ_{affine}^{-1}(**net**) = {**x**| $18x_1 + 18x_2 - 18 = net_1$, $-18x_1 + 18x_2 - 18 = net_2$}.

11. For any **net** ∈ {**net**| $-\kappa < net_1 < 0$, $-\kappa < net_2 < 0$}: $\hat{a}_1 = \beta_1 net_1 - \beta_2 net_1^2$, $-1 < \hat{a}_1 < 0$, $\hat{a}_2 = \beta_1 net_2 - \beta_2 net_2^2$, $-1 < \hat{a}_2 < 0$, $\hat{y} = -\hat{a}_1 - \hat{a}_2 - 1$ and $\hat{y} \in (-1, 1)$; $\Phi_{linear}^{-1}(\hat{y})$ = {$\hat{\mathbf{a}}$| $\hat{a}_1 + \hat{a}_2 = -\hat{y}-1$}, $\Phi_g^{-1}(\hat{\mathbf{a}})$ = {**net**| $-\beta_2 net_1^2 + \beta_1 net_1 = \hat{a}_1$, $-\kappa < net_1 < 0$, $-\beta_2 net_2^2 + \beta_1 net_2 = \hat{a}_2$, $-\kappa < net_2 < 0$} = {**net**| $net_1 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4\hat{a}_1\beta_2}}{2\beta_2}$, $net_2 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4\hat{a}_2\beta_2}}{2\beta_2}$ }, and Φ_{affine}^{-1}(**net**) = {**x**| $18x_1 + 18x_2 - 18 = net_1$, $-18x_1 + 18x_2 - 18 = net_2$}.

12. For any **net** ∈ {**net**| $net_1 \le -\kappa$, $-\kappa < net_2 < 0$}: $\hat{a}_1 = -1$, $\hat{a}_2 = \beta_1 net_2 - \beta_2 net_2^2$, $-1 < \hat{a}_2 < 0$, $\hat{y} = -\hat{a}_2$ and $\hat{y} \in (0, 1)$; $\Phi_{linear}^{-1}(\hat{y})$ = {$\hat{\mathbf{a}}$| $\hat{a}_1 = -1$, $\hat{a}_2 = -\hat{y}$} = {$(-1, -\hat{y})^T$}, $\Phi_g^{-1}(\hat{\mathbf{a}})$ = {**net**| $net_1 \le -\kappa$, $-\beta_2 net_2^2 + \beta_1 net_2 = \hat{a}_2$, $-\kappa < net_2 < 0$} = {**net**| $net_1 \le -\kappa$, $net_2 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4\hat{a}_2\beta_2}}{2\beta_2}$ }, and Φ_{affine}^{-1}(**net**) = {**x**| $18x_1 + 18x_2 - 18 = net_1$, $-18x_1 + 18x_2 - 18 = net_2$}.

13. For any **net** ∈ {**net**| $net_1 \ge \kappa$, $net_2 \le -\kappa$}: $\hat{a}_1 = 1$, $\hat{a}_2 = -1$, and $\hat{y} = -1$. $\Phi_{linear}^{-1}(-1)$ = {$(1, -1)^T$}, $\Phi_g^{-1}((-1, 1)^T)$ = {**net**| $net_1 \ge \kappa$, $net_2 \le -\kappa$}, and Φ_{affine}^{-1}(**net**) = {**x**| $18x_1 + 18x_2 - 18 = net_1$, $-18x_1 + 18x_2 - 18 = net_2$}.

14. For any **net** ∈ {**net**| $0 \le net_1 < \kappa$, $net_2 \le -\kappa$}: $\hat{a}_1 = \beta_1 net_1 + \beta_2 net_1^2$, $0 \le \hat{a}_1 < 1$, $\hat{a}_2 = -1$, $\hat{y} = -\hat{a}_1$, and $\hat{y} \in (-1, 0]$. $\Phi_{linear}^{-1}(\hat{y})$ = {$\hat{\mathbf{a}}$| $\hat{a}_1 = -\hat{y}$, $\hat{a}_2 = -1$} = {$(-\hat{y}, -1)^T$}, $\Phi_g^{-1}(\hat{\mathbf{a}})$ = {**net**| $\beta_2 net_1^2 + \beta_1 net_1 = \hat{a}_1$, $0 \le net_1 < \kappa$, $net_2 \le -\kappa$} =

{**net**| $net_1 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4\hat{a}_1\beta_2}}{2\beta_2}$, $net_2 \le -\kappa$}, and Φ_{affine}^{-1}(**net**) = {**x**| $18x_1 + 18x_2 - 18 = net_1$, $-18x_1 + 18x_2 - 18 = net_2$}.

15. For any **net** ∈ {**net**| $-\kappa < net_1 < 0$, $net_2 \le -\kappa$}: $\hat{a}_1 = \beta_1 net_1 - \beta_2 net_1^2$, $-1 < \hat{a}_1 < 0$, $\hat{a}_2 = -1$, $\hat{y} = -\hat{a}_1$, and $\hat{y} \in (0, 1)$. $\Phi_{linear}^{-1}(\hat{y})$ = {$\hat{\mathbf{a}}$| $\hat{a}_1 = -\hat{y}$, $\hat{a}_2 = -1$} = {$(-\hat{y}, -1)^T$}, $\Phi_g^{-1}(\hat{\mathbf{a}})$ = {**net**| $-\beta_2 net_1^2 + \beta_1 net_1 = \hat{a}_1$, $-\kappa < net_1 < 0$, $net_2 \le -\kappa$} = {**net**| $net_1 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4\hat{a}_1\beta_2}}{2\beta_2}$, $net_2 \le -\kappa$}, and Φ_{affine}^{-1}(**net**) = {**x**| $18x_1 + 18x_2 - 18 = net_1$, $-18x_1 + 18x_2 - 18 = net_2$}.
16. For any **net** ∈ {**net**| $net_1 \le -\kappa$, $net_2 \le -\kappa$}: $\hat{a}_1 = -1$, $\hat{a}_2 = -1$, and $\hat{y} = 1$. $\Phi_{linear}^{-1}(1) = \{(-1, -1)^T\}$, $\Phi_g^{-1}((-1, 1)^T)$ = {**net**| $net_1 \le -\kappa$, $net_2 \le -\kappa$}, and Φ_{affine}^{-1}(**net**) = {**x**| $18x_1 + 18x_2 - 18 = net_1$, $-18x_1 + 18x_2 - 18 = net_2$}.

Thus, we have the following induced approximation rules:

1. When $\hat{y} = -3$: $\Phi_{linear}^{-1}(-3) = \{(1, 1)^T\}$; $\Phi_g^{-1}(\Phi_{linear}^{-1}(-3)) = \Phi_g^{-1}((1, -1)^T)$ = {**net**| $net_1 \ge \kappa$, $net_2 \ge \kappa$}; $\Phi_{affine}^{-1}(\Phi_g^{-1}(\Phi_{linear}^{-1}(-3)))$ = {**x**| $18x_1+18x_2 \ge \kappa+18$, $-18x_1+18x_2 \ge \kappa+18$}. That is, Xa(-3) = {**x**| $18x_1+18x_2 \ge \kappa+18$, $-18x_1+18x_2 \ge \kappa+18$}, which is an open polyhedron in the input space.
2. When $\hat{y} \in (-3, -2]$: $\Phi_{linear}^{-1}(\hat{y}) = \{(-\hat{y}-2, 1)^T\} \cup \{(1, -\hat{y}-2)^T\}$ ∪ {**a**| $\hat{a}_1 + \hat{a}_2 = -\hat{y}-1$, $0 \le \hat{a}_1 < 1$, $0 \le \hat{a}_2 < 1$}; $\Phi_g^{-1}(\Phi_{linear}^{-1}(\hat{y}))$ = {**net**| $net_1 = \frac{-\beta_1 + \sqrt{\beta_1^2 - 4(\hat{y}+2)\beta_2}}{2\beta_2}$, $net_2 \ge \kappa$} ∪ {**net**| $net_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 - 4(\hat{y}+2)\beta_2}}{2\beta_2}$, $net_1 \ge \kappa$} ∪ {**net**| $net_1 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4\hat{a}_1\beta_2}}{2\beta_2}$, $net_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4(-\hat{y}-1-\hat{a}_1)\beta_2}}{2\beta_2}$, $0 \le \hat{a}_1 < 1$}; $\Phi_{affine}^{-1}(\Phi_g^{-1}(\Phi_{linear}^{-1}(\hat{y})))$ = {**x**| $18x_1 + 18x_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 - 4(\hat{y}+2)\beta_2}}{2\beta_2} + 18$, $-18x_1+18x_2-18 \ge \kappa$} ∪ {**x**| $-18x_1 + 18x_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 - 4(\hat{y}+2)\beta_2}}{2\beta_2} + 18$, $18x_1+18x_2-18 \ge \kappa$} ∪ {**x**| $18x_1 + 18x_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4\hat{a}_1\beta_2}}{2\beta_2} + 18$, $-18x_1 + 18x_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4(-\hat{y}-1-\hat{a}_1)\beta_2}}{2\beta_2} + 18$, $0 \le \hat{a}_1 < 1$}. Hence Xa($\hat{y}$) consists of two half-lines and one curve segment.
3. When $\hat{y} \in (-2, -1)$. $\Phi_{linear}^{-1}(\hat{y}) \equiv \{(-\hat{y}-2, 1)^T\}$ ∪ {$\hat{\mathbf{a}}$| $\hat{a}_1 + \hat{a}_2 = -\hat{y}-1$, $0 \le \hat{a}_1 < 1$, $0 \le \hat{a}_2 < 1$} ∪ {$\hat{\mathbf{a}}$| $\hat{a}_1 + \hat{a}_2 = -\hat{y}-1$, $-1 < \hat{a}_1 < 0$, $0 \le \hat{a}_2 < 1$} ∪ {$(1, -\hat{y}-2)^T$} ∪ {$\hat{\mathbf{a}}$| $\hat{a}_1 + \hat{a}_2 = -\hat{y}-1$, $0 \le \hat{a}_1 < 1$, $-1 < \hat{a}_2 < 0$}; $\Phi_{tanh}^{-1}(\Phi_{linear}^{-1}(\hat{y}))$ =

{**net**| $net_1 = \frac{\beta_1 - \sqrt{\beta_1^2 + 4(\hat{y}+2)\beta_2}}{2\beta_2}$, $net_2 \geq \kappa$} ∪ {**net**| $net_1 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4\hat{a}_1\beta_2}}{2\beta_2}$, $net_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4(-\hat{y}-1-\hat{a}_1)\beta_2}}{2\beta_2}$, $0 \leq \hat{a}_1 < 1$} ∪ {**net**| $net_1 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4\hat{a}_1\beta_2}}{2\beta_2}$, $net_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4(-\hat{y}-1-\hat{a}_1)\beta_2}}{2\beta_2}$, $-1 < \hat{a}_1 < 0$} ∪ {**net**| $net_2 = \frac{\beta_1 - \sqrt{\beta_1^2 + 4(\hat{y}+2)\beta_2}}{2\beta_2}$, $net_1 \geq \kappa$} ∪ {**net**| $net_1 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4\hat{a}_1\beta_2}}{2\beta_2}$, $net_2 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4(-\hat{y}-1-\hat{a}_1)\beta_2}}{2\beta_2}$, $0 \leq \hat{a}_1 < 1$}; $\Phi_{affine}^{-1}(\Phi_{tanh}^{-1}(\Phi_{linear}^{-1}(\hat{y})))$ = {**x**| $18x_1 + 18x_2 = \frac{\beta_1 - \sqrt{\beta_1^2 + 4(\hat{y}+2)\beta_2}}{2\beta_2} + 18$, $-18x_1+18x_2-18 \geq \kappa$} ∪ {**x**| $18x_1 + 18x_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4\hat{a}_1\beta_2}}{2\beta_2} + 18$, $-18x_1 + 18x_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4(-\hat{y}-1-\hat{a}_1)\beta_2}}{2\beta_2} + 18$, $0 \leq \hat{a}_1 < 1$} ∪ {**x**| $18x_1 + 18x_2 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4\hat{a}_1\beta_2}}{2\beta_2} + 18$, $-18x_1 + 18x_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4(-\hat{y}-1-\hat{a}_1)\beta_2}}{2\beta_2} + 18$, $-1 < \hat{a}_1 < 0$} ∪ {**x**| $-18x_1 + 18x_2 = \frac{\beta_1 - \sqrt{\beta_1^2 + 4(\hat{y}+2)\beta_2}}{2\beta_2} + 18$, $18x_1+18x_2-18 \geq \kappa$} ∪ {**x**| $18x_1 + 18x_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4\hat{a}_1\beta_2}}{2\beta_2} + 18$, $-18x_1 + 18x_2 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4(-\hat{y}-1-\hat{a}_1)\beta_2}}{2\beta_2} + 18$, $0 \leq \hat{a}_1 < 1$}. Hence Xa($\hat{y}$) consists of three curve segments and two half-lines.

4. When $\hat{y} = -1$: $\Phi_{linear}^{-1}(-1)$ = {$(-1, 1)^T$} ∪ {$\hat{\mathbf{a}}$| $\hat{a}_1 + \hat{a}_2 = -\hat{y}-1$, $0 \leq \hat{a}_1 < 1$, $0 \leq \hat{a}_2 < 1$} ∪ {$\hat{\mathbf{a}}$| $\hat{a}_1 + \hat{a}_2 = -\hat{y}-1$, $-1 < \hat{a}_1 < 0$, $0 \leq \hat{a}_2 < 1$} ∪ {$\hat{\mathbf{a}}$| $\hat{a}_1 + \hat{a}_2 = -\hat{y}-1$, $0 \leq \hat{a}_1 < 1$, $-1 < \hat{a}_2 < 0$} ∪ {$(1, -1)^T$}; $\Phi_{tanh}^{-1}(\Phi_{linear}^{-1}(-1))$ = {**net**| $net_1 \leq -\kappa$, $net_2 \geq \kappa$} ∪ {**net**| $net_1 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4\hat{a}_1\beta_2}}{2\beta_2}$, $net_2 = -net_1$, $0 \leq \hat{a}_1 < 1$} ∪ {**net**| $net_1 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4\hat{a}_1\beta_2}}{2\beta_2}$, $net_2 = -net_1$, $-1 < \hat{a}_1 < 0$} ∪ {**net**| $net_1 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4\hat{a}_1\beta_2}}{2\beta_2}$, $net_2 = -net_1$, $0 \leq \hat{a}_1 < 1$} ∪ {**net**| $net_1 \geq \kappa$, $net_2 \leq -\kappa$}; $\Phi_{affine}^{-1}(\Phi_{tanh}^{-1}(\Phi_{linear}^{-1}(-1)))$ = {**x**| $18x_1+18x_2 \leq -\kappa+18$, $-18x_1+18x_2 \geq \kappa+18$} ∪ {**x**| $18x_1 + 18x_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4\hat{a}_1\beta_2}}{2\beta_2} + 18$, $x_2 = 1$, $0 \leq \hat{a}_1 < 1$} ∪ {**x**| $18x_1 + 18x_2 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4\hat{a}_1\beta_2}}{2\beta_2} + 18$, $x_2 = 1$, $-1 < \hat{a}_1 < 0$} ∪ {**x**| $18x_1 + 18x_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4\hat{a}_1\beta_2}}{2\beta_2} + 18$, $x_2 = 1$, $0 \leq \hat{a}_1 < 1$} ∪ {**x**| $18x_1+18x_2 \geq \kappa+18$, $-18x_1+18x_2 \leq -\kappa+18$}. Hence Xa(-1) consists of three line segments and two open polyhedra in the input space.

5. When $\hat{y} \in (-1, 0]$: $\Phi_{linear}^{-1}(\hat{y}) = \{\hat{\mathbf{a}} \mid \hat{a}_1 + \hat{a}_2 = -\hat{y}-1, -1 < \hat{a}_1 < 0, 0 \le \hat{a}_2 < 1\} \cup \{(-1, -\hat{y})^T\} \cup \{\hat{\mathbf{a}} \mid \hat{a}_1 + \hat{a}_2 = -\hat{y}-1, 0 \le \hat{a}_1 < 1, -1 < \hat{a}_2 < 0\} \cup \{\hat{\mathbf{a}} \mid \hat{a}_1 + \hat{a}_2 = -\hat{y}-1, -1 < \hat{a}_1 < 0, -1 < \hat{a}_2 < 0\} \cup \{(-\hat{y}, -1)^T\}$; $\Phi_{tanh}^{-1}(\Phi_{linear}^{-1}(\hat{y})) = \{\mathbf{net} \mid net_1 = \frac{\beta_1 - \sqrt{\beta_1^2 + 4\hat{y}\beta_2}}{2\beta_2}, net_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4(-\hat{y}-1-\hat{a}_1)\beta_2}}{2\beta_2}, -1 < \hat{a}_1 < 0\} \cup \{\mathbf{net} \mid net_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 - 4\hat{y}\beta_2}}{2\beta_2}, net_1 \le -\kappa\} \cup \{\mathbf{net} \mid net_1 = \frac{-\beta_1 + \sqrt{\beta_1^2 - 4\hat{y}\beta_2}}{2\beta_2}, net_2 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4(-\hat{y}-1-\hat{a}_1)\beta_2}}{2\beta_2}, 0 \le \hat{a}_1 < 1\} \cup \{\mathbf{net} \mid net_1 = \frac{\beta_1 - \sqrt{\beta_1^2 + 4\hat{y}\beta_2}}{2\beta_2}, net_2 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4(-\hat{y}-1-\hat{a}_1)\beta_2}}{2\beta_2}, -1 < \hat{a}_1 < 0\} \cup \{\mathbf{net} \mid net_1 = \frac{-\beta_1 + \sqrt{\beta_1^2 - 4\hat{y}\beta_2}}{2\beta_2}, net_2 \le -\kappa\}$; $\Phi_{affine}^{-1}(\Phi_{tanh}^{-1}(\Phi_{linear}^{-1}(\hat{y}))) = \{\mathbf{x} \mid 18x_1 + 18x_2 = \frac{\beta_1 - \sqrt{\beta_1^2 + 4\hat{y}\beta_2}}{2\beta_2} + 18, -18x_1 + 18x_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 + 4(-\hat{y}-1-\hat{a}_1)\beta_2}}{2\beta_2} + 18, -1 < \hat{a}_1 < 0\} \cup \{\mathbf{x} \mid -18x_1 + 18x_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 - 4\hat{y}\beta_2}}{2\beta_2} + 18, 18x_1+18x_2-18 < -\kappa\} \cup \{\mathbf{x} \mid 18x_1 + 18x_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 - 4\hat{y}\beta_2}}{2\beta_2} + 18, -18x_1 + 18x_2 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4(-\hat{y}-1-\hat{a}_1)\beta_2}}{2\beta_2} + 18, 0 \le \hat{a}_1 < 1\} \cup \{\mathbf{x} \mid 18x_1 + 18x_2 = \frac{\beta_1 - \sqrt{\beta_1^2 + 4\hat{y}\beta_2}}{2\beta_2} + 18, net_2 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4(-\hat{y}-1-\hat{a}_1)\beta_2}}{2\beta_2}, -1 < \hat{a}_1 < 0\} \cup \{\mathbf{x} \mid 18x_1 + 18x_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 - 4\hat{y}\beta_2}}{2\beta_2} + 18, -18x_1+18x_2 \le -\kappa + 18\}$. Hence $Xa(\hat{y})$ consists of three curve segments and two half-lines.

6. When $\hat{y} \in (0, 1)$: $\Phi_{linear}^{-1}(\hat{y}) = \{\hat{\mathbf{a}} \mid \hat{a}_1 + \hat{a}_2 = -\hat{y}-1, -1 < \hat{a}_1 < 0, -1 < \hat{a}_2 < 0\} \cup \{(-1, -\hat{y})^T\} \cup \{(-\hat{y}, -1)^T\}$; $\Phi_{tanh}^{-1}(\Phi_{linear}^{-1}(\hat{y})) = \{\mathbf{net} \mid net_1 = \frac{\beta_1 - \sqrt{\beta_1^2 + 4\hat{y}\beta_2}}{2\beta_2}, net_2 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4(-\hat{y}-1-\hat{a}_1)\beta_2}}{2\beta_2}, -1 < \hat{a}_1 < 0\} \cup \{\mathbf{net} \mid net_2 = \frac{\beta_1 - \sqrt{\beta_1^2 + 4\hat{y}\beta_2}}{2\beta_2}, net_1 \le -\kappa\} \cup \{\mathbf{net} \mid net_1 = \frac{\beta_1 - \sqrt{\beta_1^2 + 4\hat{y}\beta_2}}{2\beta_2}, net_2 \le -\kappa\}$; $\Phi_{affine}^{-1}(\Phi_{tanh}^{-1}(\Phi_{linear}^{-1}(\hat{y}))) = \{\mathbf{x} \mid 18x_1 + 18x_2 = \frac{\beta_1 - \sqrt{\beta_1^2 + 4\hat{y}\beta_2}}{2\beta_2} + 18, net_2 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4(-\hat{y}-1-\hat{a}_1)\beta_2}}{2\beta_2}, -1 < \hat{a}_1 < 0\} \cup \{\mathbf{x} \mid -18x_1 + 18x_2 = \frac{\beta_1 - \sqrt{\beta_1^2 + 4\hat{y}\beta_2}}{2\beta_2} + 18, 18x_1+18x_2-18 \le -\kappa\} \cup \{\mathbf{x} \mid 18x_1 + 18x_2 = \frac{\beta_1 - \sqrt{\beta_1^2 + 4\hat{y}\beta_2}}{2\beta_2} + 18, -18x_1+18x_2-18 \le -\kappa\})$. Hence $Xa(\hat{y})$ consists of one curve segment and two half-lines.

7. When $\hat{y} = 1$: $\Phi_{linear}^{-1}(1) \equiv \{(-1, -1)^T\}$; $\Phi_{tanh}^{-1}(\Phi_{linear}^{-1}(1)) = \Phi_{tanh}^{-1}((-1, -1)^T) = \{\mathbf{net} | \ net_1 \leq -\kappa, net_2 \leq -\kappa\}$; $\Phi_{affine}^{-1}(\Phi_{tanh}^{-1}(\Phi_{linear}^{-1}(1))) = \{\mathbf{x} | \ 18x_1+18x_2 \leq -\kappa+18, -18x_1+18x_2 \leq -\kappa+18\}$. That is, Xa(1) = $\{\mathbf{x} | \ 18x_1+18x_2 \leq -\kappa+18, -18x_1+18x_2 \leq -\kappa+18\}$, which is one open polyhedron.