3. Capital gains tax and the stock price volatility

Our main purpose in this section is to undertake an examination of the stock price volatility by considering how the rational speculator's holding changes after the tax rate on capital gains rises under each type of unexpected shocks. From Eq.(2.11), we know that whether the rational speculators adjust their stock holdings mainly depends on the capital gains and the conditional variance of the stock price. It is reasonable to consider that raising the tax rate has direct impact on the capital gains; however, it is worth a notice that the conditional variance of the stock price determined endogenously in the model may also be affected. Therefore, we will begin this section with a focus on a higher tax rate's influence on the conditional variance of the stock price and then turn to the discussion about the changes of the stock price volatility with a higher tax rate under each type of unexpected shocks.

Now, let us compute the conditional variance of the stock price based on Eq.(2.19). Suppose that the unexpected shocks are independent of each other and all future shocks have zero expected values. At time $t+1$, the new information comes from shocks u_{t+1}, ε_{t+1}, and Δ_{t+1}; therefore, the unexpected stock price change will be:

$$p_{t+1} - E_t p_{t+1} = \frac{-(1-\lambda)}{\omega} u_{t+1} + \frac{\alpha(1-\lambda)}{\omega} \varepsilon_{t+1} - \frac{\lambda}{(1-\lambda)(1-\tau)} \Delta_{t+1} . \quad (3.1)$$

This allows us to calculate the conditional variance[1] directly:

$$Var(p_{t+1}) = E_t[p_{t+1} - E_t p_{t+1}]^2$$

$$= \frac{(1-\lambda)^2}{\omega^2} Var(u) + \frac{\alpha^2(1-\lambda)^2}{\omega^2} Var(\varepsilon) + \frac{\lambda^2}{(1-\lambda)^2(1-\tau)^2} Var(\Delta) . \quad (3.2)$$

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[1] The definition of this conditional variance is obtained from J.A. Carlson, C.L. Osler (2000).

However, it is not appropriate to view λ as a fixed parameter.[2] Let us recall that λ is the characteristic root determined by the parameters of the model, and the parameter b must be consistent with Eq.(2.12). We therefore have two simultaneous equations (2.12) and (2.17) that must be solved in order to express the equilibrium values of b and λ as functions of the parameters ω, θ, and τ. We now rewrite these two equations to obtain the following two relationships between b and λ:

$$b(1-\tau)^{-1}(1-\lambda)^2 = \lambda\omega \quad , \qquad \text{(RE)} \qquad (3.3)$$

$$b(1-\tau)^{-1}(1-\lambda)^2 = \frac{(1-\tau)^{-1}(1-\lambda)^2}{\theta Var(p_{t+1})} . \qquad \text{(SB)} \qquad (3.4)$$

The first expression is RE equation that we mentioned in Eq.(2.17). The second expression is denoted SB as a reminder that it is a transformation of the first-order condition for optimal speculative behavior.

Differentiating RE and SB with respect to b, λ, τ, and then we use the Cramer's rule to obtain:[3]

$$\frac{d\lambda}{d\tau} = \frac{f_\tau}{\omega - f_\lambda} . \qquad (3.5)$$

We are now ready to perform comparative statics to see how the capital gains tax influences the conditional variance of the stock price. Differentiating Eq.(3.2) by τ, we find that:

$$\frac{dVar(p_{t+1})}{d\tau} = \frac{-2(1-\lambda)}{\omega^2} \cdot \frac{d\lambda}{d\tau} \cdot Var(u) - \frac{2\alpha^2(1-\lambda)}{\omega^2} \cdot \frac{d\lambda}{d\tau} \cdot Var(\varepsilon) +$$

$$[\frac{2\lambda^2}{(1-\lambda)^2(1-\tau)^3} + \frac{2\lambda}{(1-\lambda)^3(1-\tau)^2} \cdot \frac{d\lambda}{d\tau}] \cdot Var(\Delta) . \qquad (3.6)$$

That is, as the tax rate rises, the conditional variance varies according to these three

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2 See McCafferty and Driskill (1980).

3 See Appendix A3.

types of shocks. In the following part, we will substitute the earlier expression for $\frac{d\lambda}{d\tau}$ into the above expression for $\frac{dVar(p_{t+1})}{d\tau}$ and then discuss the stock market response to a higher tax rate on capital gains under each type of unexpected shocks.

3.1 The issuing shock

Suppose there only exists the issuing shock; that is, $\varepsilon_t = \Delta_t = 0$. To simply our analysis, let $\bar{r}$ be zero. In this case, the speculator's stock holding given a tax rate becomes:

$$h_t = \frac{1}{\theta Var(p_{t+1})(1-\tau)} \cdot (E_t p_{t+1} - p_t)$$

$$= \frac{(1-\lambda)}{\theta Var(p_{t+1})(1-\tau)} \cdot (\bar{p} - p_t) . \qquad (3.7)$$

In addition, for intuitive convenience, assume we enter the period with the stock price at its equilibrium level. Therefore, in Eq.(2.19), p_{t-1} can be regarded as $\bar{p}$. Now let us substitute $\varepsilon_t = \Delta_t = 0$ into Eq.(2.19), we have:

$$p_t = \bar{p} - \frac{1-\lambda}{\omega} u_t . \qquad (3.8)$$

It is clear that $p_t < \bar{p}$ with $u_t > 0$, so that $h_t > 0$. That is, the issuing shock causes a lowering-price effect so that the rational speculators would increase their stock holdings with an expectation of a rising stock price.

Now let us return to the point about how the rational speculators react to a higher tax rate on capital gains. First, we have to find out the relationship between the tax rate and the conditional variance of the stock price. Combining Eq.(3.5) with

$Var(\varepsilon) = Var(\Delta) = 0$, we have $\frac{d\lambda}{d\tau} > 0$ [4], and thus Eq.(3.6) becomes:

$$\frac{dVar(p_{t+1})}{d\tau} = \frac{-2(1-\lambda)}{\omega^2} \cdot Var(u) \cdot \frac{d\lambda}{d\tau} < 0 . \quad (3.9)$$

Eq.(3.9) means that the conditional variance of the stock price is monotonically decreasing with the tax rate.

Differentiating Eq.(3.7) with respect to τ and substituting the result of Eq.(3.9) into Eq.(3.7), we find that:[5]

$$\frac{dh_t}{d\tau} = \frac{\omega \cdot u}{(1-\tau)^2 Var(u)} > 0 . \quad (3.10)$$

Judging from the above, we can see that as the tax rate rises, the rational speculators would want to increase their stock holdings more. When the issuing shock lowers the stock price below the equilibrium level, this buying pressure raises the stock price and therefore accelerates the current price to converge toward the equilibrium price. In this case, we see that a higher tax rate on capital gains leads the rational speculators to buy more stocks to offset the lowering-price effect of the issuing shock. Therefore, it is reasonable to conclude that if only the issuing shock occurs, raising the tax rate on capital gains can be a way to stabilize the stock market.

3.2 The dividend shock

Now let us consider how the stock price volatility changes as the tax rate on capital gains rises if only the dividend shock exists. The speculator's holding given a tax rate can be expressed as:

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[4] See appendix A4.

[5] See appendix A5.

$$h_t = \frac{1}{\theta Var(p_{t+1})(1-\tau)} \cdot (E_t p_{t+1} - p_t)$$

$$= \frac{(1-\lambda)}{\theta Var(p_{t+1})(1-\tau)} \cdot (\bar{p} - p_t)\,. \qquad (3.11)$$

In addition, re-writing Eq.(2.19) with $u_t = \Delta_t = 0$ and $p_{t-1} = \bar{p}$ yields:

$$p_t = \bar{p} + \frac{\alpha(1-\lambda)}{\omega}\varepsilon_t\,. \qquad (3.12)$$

This represents that $p_t > \bar{p}$ with $\varepsilon_t > 0$, and this results in $h_t < 0$. Thus, this raising-price effect of the dividend shock leads the rational speculators to reduce their stock holdings in order to take advantage of this profit-making opportunity.

Similarly, we now try to find out a higher tax rate's influence on the conditional variance of the stock price. From Eq.(3.5), we have $\frac{d\lambda}{d\tau} > 0$ [6] with $Var(u) = Var(\Delta) = 0$. Then, Eq.(3.6) becomes:

$$\frac{dVar(p_{t+1})}{d\tau} = \frac{-2\alpha^2(1-\lambda)}{\omega^2} \cdot Var(\varepsilon) \cdot \frac{d\lambda}{d\tau} < 0\,. \qquad (3.13)$$

It shows that the conditional variance of the stock price is monotonically decreasing with the tax rate.

Now differentiating Eq.(3.11) with respect to τ and substituting the result of Eq.(3.13) into Eq.(3.11), we find that:[7]

$$\frac{dh_t}{d\tau} = \frac{-\omega\varepsilon}{\alpha(1-\tau)^2 Var(\varepsilon)} < 0\,. \qquad (3.14)$$

It is clear that the higher the tax rate becomes, the fewer stocks the rational speculators want to hold. That is, the speculators would chose to sell more stocks as

[6] See appendix A4.
[7] See appendix A5.

the tax rate rises. When the dividend shock raises the current price above the equilibrium level, more of those sales would put more downward pressure on the stock price so that p_t deviates less from $\bar{p}$. Therefore, a higher tax rate on the capital gains reduces the stock price volatility when the dividend shock occurs.

3.3 The margin-rate shock

Now suppose there only exists the margin-rate shock. Therefore, the speculator's stock holding given a tax rate is:

$$h_t = \frac{1}{\theta Var(p_{t+1})} \cdot [\frac{1}{1-\tau} \cdot (E_t p_{t+1} - p_t) - \frac{1}{(1-\tau)^2} \cdot \Delta_t]$$

$$= \frac{1}{\theta Var(p_{t+1})} \cdot [\frac{1-\lambda}{1-\tau} \cdot (\bar{p} - p_t) - \frac{1}{(1-\tau)^2} \cdot \Delta_t]. \tag{3.15}$$

Now re-writing Eq.(2.19) with $u_t = \varepsilon_t = 0$ and $p_{t-1} = \bar{p}$, we obtain:

$$p_t = \bar{p} - \frac{\lambda}{(1-\lambda)(1-\tau)} \Delta_t . \tag{3.16}$$

From the above expression, we see that $p_t < \bar{p}$ with $\Delta_t > 0$. Thus, the speculator would increase their stock holdings because of this lowering-price effect of Δ_t. However, $\Delta_t > 0$ also implies a rising margin-rate effect that may cause the speculator to reduce their holdings. Therefore, from Eq.(3.15), it is apparent that h_t has a negative effect of Δ_t which is absent in the earlier two cases. Since h_t in this case is affected by two different effects, we need a further discussion to see which effect dominates the change in the stock holdings.

According to Eq.(3.15), the holding position can be divided into two parts. Considering the lowering-price effect, the speculator wants to hold:

$$h_t^l = \frac{1}{\theta Var(p_{t+1})} \cdot [\frac{1-\lambda}{1-\tau} \cdot (\bar{p} - p_t)] > 0 \,. \tag{3.17}$$

In addition, the rising margin-rate effect leads the stock holding to be:

$$h_t^m = \frac{1}{\theta Var(p_{t+1})} \cdot [-\frac{1}{(1-\tau)^2} \cdot \Delta_t] \,. \tag{3.18}$$

However, from Eq.(3.16), Δ_t can be re-expressed as:

$$\Delta_t = \frac{(1-\lambda)(1-\tau)}{\lambda} \cdot (\bar{p} - p_t) \,. \tag{3.19}$$

Thus, re-writing Eq.(3.18) with Eq.(3.19) yields:

$$h_t^m = \frac{1}{\theta Var(p_{t+1})} \cdot [-\frac{1-\lambda}{\lambda(1-\tau)} \cdot (\bar{p} - p_t)] < 0 \,. \tag{3.20}$$

Now combining Eq.(3.17) and Eq.(3.20), we obtain:

$$h_t = h_t^l + h_t^m = \frac{-(1-\lambda)^2}{\theta Var(p_{t+1})\lambda(1-\tau)} \cdot (\bar{p} - p_t) < 0 \,. \tag{3.21}$$

It shows that the total effect of Δ_t on h_t is negative. That is, the rising margin-rate effect has greater impact on the stock holding and then leads the speculator to sell the stock.

Next, we turn to our main purpose of the market reaction to a rising tax rate on capital gains. First, let us take a look at the changes of the conditional variance of the stock price when the tax rate rises. Combining $\frac{d\lambda}{d\tau}$ with $Var(u) = Var(\varepsilon) = 0$, we obtain $\frac{d\lambda}{d\tau} < 0$ [8]. Then, we substitute it into Eq.(3.6):

$$\frac{dVar(p_{t+1})}{d\tau} = [\frac{2\lambda^2}{(1-\lambda)^2(1-\tau)^3} + \frac{2\lambda}{(1-\lambda)^3(1-\tau)^2} \cdot \frac{d\lambda}{d\tau}] \cdot Var(\Delta)$$

[8] See appendix A4.

$$= \frac{2\lambda^2(1-\tau)^2\omega + \dfrac{2(1-\lambda)^3(1-\tau)^3(2\lambda+1)}{\theta\lambda Var(\Delta)}}{(1-\lambda)^2(1-\tau)^5\{\omega + \dfrac{2(1-\tau)(1-\lambda)^3(1+\lambda)}{\theta\lambda^3 Var(\Delta)}} > 0. \qquad (3.22)$$

Eq.(3.22) shows that the conditional variance of the stock price is monotonically increasing with the tax rate.

Now, we take differentiation of Eq.(3.15) with respect to τ and substitute Eq.(3.22) into Eq.(3.15):[9]

$$\frac{dh_t}{d\tau} = \frac{(1-\lambda)^2(2+\lambda)\Delta_t}{\lambda^2 Var(\Delta)} \cdot \frac{d\lambda}{d\tau} < 0. \qquad (3.23)$$

The above equation implies that a rise in the tax rate causes the rational speculators to sell more stocks. Thus, when the margin-rate shock lowers the current price below the equilibrium price, those sales put a downward pressure on the stock price so that p_t deviates more from $\bar{p}$. Therefore, raising the tax rate on capital gains does not play a stabilizing role when the margin-rate shock occurs.

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[9] See appendix A5.