7 附錄

7.1 狀態空間

在處理時間序列模型時, 常態狀態空間 (Gaussian State Space Form) 是常用的工具, 一般常見的常態線性狀態空間形式爲:

$$y_t = c_t + Z_t\alpha_t + \varepsilon_t, \qquad \varepsilon_t \sim N(\mathbf{0}, \mathbf{G}_t), \tag{1}$$

$$\alpha_{t+1} = d_t + T_t\alpha_t + \eta_t, \qquad \eta_t \sim N(\mathbf{0}, \mathbf{H}_t), \tag{2}$$

(2) 稱爲測量方程式,(3) 稱爲遞移方程式其中 y_t 爲第期 t 的觀察值, α_t 爲第 t 期的狀態變數, 表觀測值受一不可觀察變數的影響。我們並假設,

1. 起始狀態 $\alpha_{1|0}$ 符合穩定條件, 其條件平均數爲 $a_{1|0}$, 條件共變異矩陣爲 $P_{1|0}$ 的常態分配: $\alpha_1|Y_0 \sim N(a_{1|0}, P_{1|0})$
2. $\mathbf{G}_t$ 與 $\mathbf{H}_t$ 每一期均不相關 : $\mathrm{E}(\mathbf{G}_t'\mathbf{H}_t) = 0$

對狀態空間模型的研究, 是在給定不同的訊息集合之下, 利用過濾, 預測, 平滑, 的方法, 來求取狀態變數與模型的特性。我們簡述這三種推論方法的出發點:

- 過濾: 給定 F_t , 希望能獲得 α_t , 也就是移除觀測方程式中的測量誤差。
- 預測: 給定 F_t , 預測 α_{t+h} 或 y_{t+h} 。
- 平滑: 估計 α_t 給定 F_T , 也就是在有更多新訊息之下, 再回頭估計狀態變數。

7.2 Kalman filter 推導

Kalman filter 的推導過程, 大量運用多變量常態分配下面幾個性質,

$$\mathrm{E}(x|y,z) = \mathrm{E}(x|y) + \Sigma_{xz}\Sigma_{zz}^{-1}z. \tag{3}$$

$$\mathrm{Var}(x|y,z) = \mathrm{Var}(x|y) - \Sigma_{xy}\Sigma_{zz}^{-1}\Sigma_{xz}. \tag{4}$$

令 Y_{t-1} 表到 $t-1$ 期之前觀察值的集合, 我們的目標是在給定狀態變數的起使分配之下, 在加入新觀察之之後, 求導下一期狀態的條件分配。我們知道 $p(y_t|\alpha_1,\cdots,\alpha_t,Y_{t-1}) = p(y_t|\alpha_t)$, $p(\alpha_{t+1}|\alpha_1,\cdots,\alpha_t,Y_t) = p(\alpha_{t+1}|\alpha_t)$, 同時 a_{t+1} 爲 t 期時, 對 α_{t+1} 的最佳估計,

$$a_{t+1} = \mathrm{E}(\alpha_{t+1}|Y_t) \tag{5}$$

$$P_{t+1} = \mathrm{Var}(\alpha_{t+1}|Y_t) \tag{6}$$

$$\alpha_1 \sim N(a_1,\ P_1) \tag{7}$$

P_{t+1} 表估計誤差的共變異數矩陣。在給定 a_t 與 p_t 下, 我們想要計算 a_{t+1}, 與P_{t+1},

$$\begin{aligned} a_{t+1} &= \mathrm{E}(T_t\alpha_t + R_t\eta_t|Y_t) \qquad (8)\\ &= \mathbf{T}_t E(\alpha_t|Y_t), \\ p_{t+1} &= \mathrm{Var}(T_t\alpha_t + R_t\eta_t|Y_t) \qquad (9)\\ &= T_t\mathrm{Var}(\alpha_t|Y_t)T_t' + R_tQ_tR_t' \end{aligned}$$

令 ν 表新觀測值取得時所計算的估計誤差,

$$\nu_t = y_t - \mathrm{E}(y_t|Y_{t-1}) = y_t - \mathrm{E}(Z_t\alpha_t + \varepsilon_t|Y_{t-1}) = y_t - Z_t\alpha_t$$

因爲 Y_{t-1}加上ν_t 所包含的訊息與 Y_t 相同, 所以 $\mathrm{E}(\alpha_t|Y_t) = \mathrm{E}(\alpha_t|Y_{t-1}, \nu_t)$, 我們可以輕易得到下列性質

$$\begin{aligned}\mathrm{E}(\nu_t|Y_{t-1}) &= \mathrm{E}(y_t - Z_t a_t|Y_{t-1}) \\ &= \mathrm{E}(Z_t\alpha_t + \varepsilon_t - Z_t a_t|Y_{t-1}) = 0 \end{aligned} \tag{10}$$

$$\mathrm{E}(\nu_t) = \mathrm{E}[\mathrm{E}(\nu_t|Y_{t-1})] = 0 \tag{11}$$

$$\begin{aligned}\mathrm{Cov}(y_j, \nu_t) &= \mathrm{E}[\nu_t y_j] = \mathrm{E}[\mathrm{E}(\nu_t y_j|Y_{t-1})] \\ &= \mathrm{E}[y_j\mathrm{E}(\nu_t|Y_{t-1})] = 0, \qquad j = 1, \cdots, t-1 \end{aligned} \tag{12}$$

根據上述這些性質, 以及 (3) 式,

$$\begin{aligned}\mathrm{E}(\alpha_t|Y_t) &= \mathrm{E}(\alpha_t|Y_{t-1}, \nu_t) \\ &= \mathrm{E}(\alpha_t|Y_{t-1}) + \mathrm{Cov}(\alpha_t, \nu_t)[\mathrm{Var}(\nu_t)]^{-1}\nu_t \\ &= a_t + M_t F_t^{-1}\nu_t, \end{aligned} \tag{13}$$

其中 $M_t = \mathrm{Cov}(\alpha_t, \nu_t), \quad F_t = \mathrm{Var}(\nu_t)$ 爲預測均方差。

M_t 與 F_t 的計算如下

$$\begin{aligned}M_t &= \mathrm{Cov}(\alpha_t, \nu_t) = \mathrm{E}[\mathrm{E}\{\alpha_t(Z_t\alpha_t + \varepsilon_t - Z_t a_t)'|Y_{t-1}\}] \\ &= \mathrm{E}[\mathrm{E}\{\alpha_t(\alpha_t - a_t)'Z_t'|Y_{t-1}\}] = P_t Z_t', \\ F_t &= \mathrm{Var}(Z_t\alpha_t + \varepsilon_t - Z_t a_t) = Z_t P_t Z_t' + H_t \end{aligned}$$

帶回 (8) 與 (13) 式得到:

$$a_{t+1} = T_t a_t + T_t M_t F_t^{-1} \nu_t$$

$$= T_t a_t + K_t \nu_t, \qquad t = 1, \cdots, n \tag{14}$$

$$K_t = T_t M_t F_t^{-1} = T_t P_t Z_t' F_t^{-1}. \tag{15}$$

而 (9) 式中的 $\mathrm{Var}(\alpha_t | y_t)$ 可以 (4) 得到

$$\mathrm{Var}(\alpha_t | Y_t) = \mathrm{Var}(\alpha_t | Y_{t-1}, \nu_t)$$

$$= \mathrm{Var}(\alpha_t | Y_{t-1} - \mathrm{Cov}(\alpha_t, \nu_t)[\mathrm{Var}(\nu_t)]^{-1}\mathrm{Cov}(\alpha_t, \nu_t)'$$

$$= P_t - M_t F_t^{-1} M_t'$$

$$= P_t - P_t Z_t' F_t^{-1} Z_t P_t \tag{16}$$

帶回 (9) 式:

$$P_{t+1} = T_t P_t L_t' + R_t Q_t R_t', \qquad t = 1, \cdots, n.$$

$$L_t = T_t - K_t Z_t$$

Kalman Filter 藉由新樣本的進入, 對狀態的預測, 我們整理一下疊代方程式:

$$\alpha_t = d_t + T_t \alpha_t + K_t \nu_t, \qquad P_{t+1} = T_t P_t L_t' + R_t Q_t R_t'$$

$$\nu_t = y_t - Z_t \alpha_t - c_t, \qquad F_t = Z P_t Z_t' + \mathbf{H}_t$$

$$K_t = T_t P_t Z_t' F_t^{-1}, \qquad L_t = T_t - K_t Z_t$$

Shepherd 對單變量的隨機波動的設定爲

$$y_t = \exp(h_t/2)\varepsilon_t \tag{17}$$

$$\Rightarrow y^* = \log(y^2) = h_t + \log\left(\varepsilon_t^2\right) \tag{18}$$

$$h_{t+1} = \mu(1-\phi) + \phi h_t + \eta_t \tag{19}$$

我們以混和常態分配來模擬對數卡方隨機變數, 給定指示變數 s_t 下

$$\log\left(\varepsilon_t^2\right)|s_t = i \sim N\left(m_i, v_i^2\right),$$

比對 (1),(2) 式與 (18) , (19) 式可知 $c_t = m_i$, $G_t = (v_i, 0)$, $\gamma_t = h_t$, $Z_t = 1$, $d_t = \mu(1-\phi)$, $T_t = \phi$, $H_t = (0, \sigma_\eta)$,$R_t = 1$, 最後起始值條件 $h_1|Y_0 \sim N\left(\mu, \sigma_\eta^2/\left(1-\phi^2\right)\right)$。

7.3 Simulation smoother 推導

在擁有所有觀察值之下, 對干擾項的抽樣稱爲模擬平滑法, 模擬平滑法 (simulation smoother) 是由 de Jong and Shepherd (1995) 所提出。以數學推納法推導, 過程非常繁複, 且中間用到許多 disturbance smoothing 和 stae smoothing 的推導結果, 詳細請參考 Koopman (2001)。在此列出必要步驟。首先定義狀態估計誤差

$$x_t = \alpha_t - a_t, \qquad V(x_t) = P_t$$

$$\begin{aligned} \nu_t &= y_t - Z_t a_t = Z_t \alpha_t + \varepsilon_t - Z_t a_t \\ &= Z_t x_t + \varepsilon_t \end{aligned} \tag{20}$$

$$x_{t+1} = L_t x_t + R_t \nu_t - K_t \varepsilon_t \tag{21}$$

在這裡, 我們先推導測量方程式與遞移方程式誤差項的平滑估計, 給定所觀察值 $y_1, \cdots, y_n$ 下, 估計 $\hat{\varepsilon}_t = \mathrm{E}(\varepsilon_t|y)$ 與 $\hat{\eta}_t = \mathrm{E}(\eta_t|y)$ 。

因爲 $\mathrm{E}(\varepsilon_t|Y_{t-1}) = 0$ 。從 (20) 式我們得到 $\mathrm{E}\left(\varepsilon_t \nu_t'\right) = \mathrm{E}\left(\varepsilon_t x_j'\right) Z_j' + \mathrm{E}\left(\varepsilon_t \varepsilon_j'\right)$, 且 $\mathrm{E}(\varepsilon_t x_t') = 0$ for $t = 1, \cdots, n$, $j = t, \cdots, n$

$$\mathrm{E}\left(\varepsilon_t \nu_j'\right) = \begin{cases} H_t, & j = t, \\ \mathrm{E}\left(\varepsilon_t x_j'\right) Z_j', & j = t+1, \cdots, n, \end{cases} \tag{22}$$

以及:

$$\mathrm{E}\left(\varepsilon_t x'_{t+1}\right) = -H_t K'_t,$$

$$\mathrm{E}\left(\varepsilon_t x'_{t+2}\right) = -H_t K'_t L'_{t+1},$$

$$\vdots$$

$$\mathrm{E}\left(\varepsilon_t x'_n\right) = -H_t K'_t L'_{t+1} \cdots L'_{n-1},$$

這些結果我們可從 (20) , (21) 式得來帶入 (22) 式,

$$\begin{aligned}\hat{\varepsilon}_t &= H_t\Big(F_t^{-1}\nu_t - K'_t Z'_{t+1} F_{t+1}^{-1}\nu_{t+1} - K'_t L'_{t+1} Z'_{t+2} F_{t+2}^{-1}\nu_{t+2} - \cdots \\ &\quad - K'_t L'_{t+1} \cdots L'_{n-1} Z'_n F_n^{-1}\nu_n\Big) \\ &= H_t\left(F^{-1}\nu_t - K'_t r_t\right), \qquad t = n, \cdots, 1, \end{aligned} \tag{23}$$

其中 r_t 爲一遞迴式, 定義爲:

$$r_{t-1} = Z'_t F_t^{-1}\nu_t + L'_t r_t, \qquad r_n = 0, \quad t = n, \cdots, 1.$$

對於 η_t 的平滑估計, 我們首先定義 $\hat{\eta}_t = \mathrm{E}(\eta_t|y)$

$$\hat{\eta}_t = \sum_n^{j=t} \mathrm{E}(\eta_t \nu'_j) F_j^{-1}\nu_j \tag{24}$$

$$\mathrm{E}\left(\eta_t \nu'_j\right) = \begin{cases} Q_t R'_t Z'_{t+1}, & j = t+1, \\ \mathrm{E}\left(\eta_t x'_j\right) Z'_j, & j = t+2, \cdots, n, \end{cases} \tag{25}$$

以及

$$\mathrm{E}\left(\eta_t x'_{t+2}\right) = Q_t R'_t L'_{t+1},$$

$$\mathrm{E}\left(\eta_t x'_{t+3}\right) = Q_t R'_t L'_{t+1} L'_{t+2},$$

$$\vdots$$

$$\mathrm{E}\left(\eta_t x'_n\right) = Q_t R'_t L'_{t+1} \cdots L'_{n-1}, \tag{26}$$

將 (26) 帶入 (24) 可得到

$$\begin{aligned}\hat{\eta}_t &= Q_t R_t' \Big(Z_{t+1}' F_{t+1}^{-1} \nu_{t+1} + L_{t+1}' Z_{t+2}' \nu_{t+2} + \cdots \\ &\quad + L_{t+1}' \cdots L_{t+1}' \cdots L_{n-1}' Z_n' F_n^{-1} \nu_n \Big) \\ &= Q_t R_t' r_t, \qquad t = n, \cdots, 1,\end{aligned}$$

令 $r_n = 0$, $N_n = 0$, $\bar{\varepsilon}_t$ 的遞迴式爲,

$$\bar{\varepsilon}_t = H_t \left(F_t^{-1} \nu_t - K_t' \tilde{r}_t \right), \qquad t = n, n-1, \cdots, 1, \tag{27}$$

其中 $\tilde{r}_t$ 也爲一遞迴式 令 $\tilde{r}_n = 0$, $\tilde{N}_n = 0$,

$$\tilde{r}_{t-1} = Z_t' F_t^{-1} \nu_t - \tilde{W}_t' C_t^{-1} d_t + L_t' \tilde{r}_t, \tag{28}$$

$$\tilde{W}_t = H_t \left(F_t^{-1} Z_t - K_t' \tilde{N}_t L_t \right), \tag{29}$$

$$\tilde{N}_{t-1} = Z_t' F_t^{-1} Z_t + \tilde{W}_t' C_t^{-1} \tilde{W}_t + L_t' \tilde{N}_t L_t, \tag{30}$$

我們以數學推納法求導 (27) 式 , 假設 t 在 $t+1, ..., n-1$ 都對, 當 $j > t$ 時

$$\begin{aligned}\bar{\varepsilon}_t &= \mathrm{E}(\varepsilon_t | \nu_t, \cdots, \nu_n, d_{t+1}, \cdots, d_n) \\ &= \mathrm{E}(\varepsilon_t | \nu_t, \cdots, \nu_n) + \sum_{j=t+1}^{n} \mathrm{E}\left(\varepsilon_t | d_j\right) \\ &= \hat{\varepsilon}_t + \sum_{j=t+1}^{n} \mathrm{E}\left(\varepsilon_t d_j'\right) C_j^{-1} d_j.\end{aligned} \tag{31}$$

$\hat{\varepsilon}_t$ 得自於 (23) , 當 t 以 j 替代

$$\begin{aligned}\mathrm{E}(\varepsilon_t d_j') &= -\mathrm{E}(\varepsilon_t \bar{\varepsilon}_j') \\ &= -\mathrm{E}(\varepsilon_t \nu_j') F_j^{-1} H_j + \mathrm{E}(\varepsilon_t \tilde{r}_j') K_j H_j, \\ &= H_t K_t' L_{t+1}' \cdots L_{j-1}' Z_j' F_j^{-1} H_j + \mathrm{E}(\varepsilon_t \tilde{r}_j') K_j H_j,\end{aligned} \tag{32}$$

我們利用 (22), (28), 與 (32) 式以數學歸納法可證明

$$\mathrm{E}(\varepsilon_t \tilde{r}_j') = -H_t K_t' L_{t+1}' \cdots L_j' \tilde{N}_j, \qquad j = t+1, \cdots, n, \tag{33}$$

將 (33) 式帶入 (32) 式得到

$$\begin{aligned}\mathrm{E}(\varepsilon_t d_j') &= H_t K_t' L_{t+1}' \cdots L_{j-1}' \left(Z_j' F_j^{-1} H_j - L_j' \tilde{N}_j K_j' H_j \right) \\ &= H_t K_t' L_{t+1}' \cdots L_{j-1}' \tilde{W}_j'. \end{aligned} \tag{34}$$

將 (34) 式帶到 (31) 式

$$\bar{\varepsilon}_t = \hat{\varepsilon}_t + H_t K_t' \sum_{j=t+1}^{n} L_{t+1}' \cdots L_{j-1}' \tilde{W}_j' C_j^{-1} d_j,$$

從 (23) 式我們得到

$$\hat{\varepsilon}_t = H_t F_t^{-1} \nu_t - H_t K_t' \sum_{j=t+1}^{n} L_{t+1}' \cdots L_{j-1}' Z_j' F_j^{-1} \nu_j.$$

將最後兩式相加, 並利用 (28) 式的結果

$$\tilde{r}_t = \sum_{j=t+1}^{n} L_{t+1}' \cdots L_{j-1}' (Z_j F_j^{-1} \nu_j - \tilde{W}_j' C_j^{-1} d_j).$$

我們就可得到 (27) 式

$$\begin{aligned}\bar{\varepsilon}_t &= H_t F_t^{-1} \nu_t - H_t K_t' \sum_{j=t+1}^{n} L_{t+1}' \cdots L_{j-1}' \left(Z_j' F_j^{-1} \nu_j - \tilde{W}_j' C_j^{-1} d_j \right) \\ &= H_t \left(F_t^{-1} \nu_t - K_t' \tilde{r}_t \right)\end{aligned}$$

模擬殘差, 我們利用 $\varepsilon_t = \bar{\varepsilon}_t + d_t$, 其中 $d_t \sim \mathbf{N}(0, C_t)$, C_t 推導如下

$$
\begin{aligned}
C_t &= \mathrm{Var}(\varepsilon_t | y, \varepsilon_{t+1}, \cdots, \varepsilon_n) \\
&= \mathrm{Var}(\varepsilon_t | \nu_t, \nu_{t+1}, \cdots, \nu_n, d_{t+1}, \cdots, d_n) \\
&= H_t - H_t D_t H_t - \sum_{j=t+1}^{n} \mathrm{E}(\varepsilon_t d_j') C_j^{-1} \mathrm{E}(\varepsilon_t d_j')' \\
&= H_t - H_t \Bigg[F_t^{-1} + K_t' \sum_{j=t+1}^{n} L_{t+1}' \cdots L_{j-1}' \Big(Z_j' F_j^{-1} Z_j \\
&\quad + \tilde{W}_j' C_j^{-1} \tilde{W}_j' \Big) L_{j-1}' \cdots L_{t+1} K_t \Bigg] H_t。
\end{aligned}
$$

其中

$$
\tilde{N}_t = \sum_{j=t+1}^{n} L_{t+1}' \cdots L_{j-1}' \left(Z_{jj}' F_j^{-1} Z_j + \tilde{W}_j' C_j^{-1} \tilde{W}_j \right) L_{j-1} \cdots L_{t+1}。
$$

則

$$
C_t = H_t - H_t \tilde{D}_t H_t, \qquad t = n, \cdots, 1, \tag{35}
$$

$$
\tilde{D}_t = F_t^{-1} + K_t' \tilde{N}_t K_t, \tag{36}
$$

我們將模擬平滑的遞迴式重新整理一遍

$$
C_t = H_t - H_t \tilde{D}_t H_t, \qquad \tilde{D}_t = F_t^{-1} + K_t' \tilde{N}_t K_t, \tag{37}
$$

$$
d_t \sim N(0, C_t), \qquad \tilde{W}_t = H_t \left(F_t^{-1} Z_t - K_t' \tilde{N}_t L_t \right) \tag{38}
$$

$$
\tilde{r}_{t-1} = Z_t' F_t^{-1} \nu_t + L_t' \tilde{r}_t - \tilde{W}_t' C_t^{-1} d_t \tag{39}
$$

$$
\tilde{N}_{t-1} = Z_t' F_t^{-1} Z_t + L_t' \tilde{N}_t L_t + \tilde{W}_t' C^{-1} \tilde{W}_t。 \tag{40}
$$

則我們可以從條件分配 $c_t + Z_t \gamma_t | y, \theta, c_{t+1} + Z_{t+1}\gamma_{t+1}, \cdots, c_n + Z_n \gamma_n$ 中模擬出殘差項, $y_t - \Sigma_t \eta_t - \kappa_t$, 藉由對殘差的模擬, 進而形成對狀態變數的模擬。