Chapter 2

The Model

Based on the specification in Mansoorian and Mohsin(2006), the model in this paper is composed of a small open economy with a single good. The foreign currency price of the good is P*, which is fixed and taken by the small economy as given and the domestic currency price of the good is P=EP*, where E is the exchange rate (the price of the foreign currency in terms of the domestic currency). With flexible prices, the domestic inflation rate π is equal to the rate of depreciation of the domestic currency $\dot{E}/E$. In the rest of this section we will study, sequentially, the problems of the representative household, the representative firm, the government, and finally the equilibrium dynamics of the economy.

2.1 The representative household

The discounted sum of future instantaneous utilities of the representative household is given by

$$\int_0^\infty U(c_t, l_t)e^{-\beta t}dt, \tag{1}$$

where c is consumption, l is labor supply and β is a constant rate of time preference. Here we assume both the consumption and labor are normal goods. Thus, $U_{cc}U_l - U_{cl}U_c > 0$ and $U_{ll}U_c - U_{cl}U_l < 0$. In addition, the utility function is increasing in c, decreasing in l and quasi-concave. Therefore, $U_c(c,l) > 0$, $U_l(c,l) < 0$, $U_{cc}(c,l) <$

$0,\ U_{ll}(c,l) < 0,\ U_{cc}U_{ll} - U_{cl}^2 \geq 0.$

The main difference from the discussion in Mansoorian and Mohsin(2006) is that the relationship between consumption and labor is explicitly taken into account here. We can easily divide the relationship between consumption and labor into two parts. The first is that consumption and labor are independent. The other is that consumption and labor are related. As consumption and labor are related, they are either substitutes or complements. The farther analysis will be shown later.

Money is introduced into the model through a cash in advance constraint on consumption expenditure:

$$m_t \geq c_t. \tag{2}$$

The representative household owns all the domestic firms, and receives all the dividends D_t. Besides dividends, the representative household also receives government transfers τ_t. The flow budget constraint of the representative households is

$$\dot{m}_t + \dot{b}_t = D_t + w_t l_t + r b_t + \tau_t - c_t - \pi_t m_t, \tag{3}$$

where w_t is the wage rate, b_t is foreign bonds holding, and r is real interest rate, which is fixed abroad.

Then, we define the household's total accumulable assets as $a_t\ (\equiv m_t + b_t)$. Thus, we can obtain: $\dot{a}_t = \dot{m}_t + \dot{b}_t$. Hence, Eq. (3) can be re-written as

$$\dot{a}_t = D_t + r a_t + w_t l_t + \tau_t - c_t - (r + \pi_t) m_t. \tag{4}$$

The flow budget constraint satisfies transversality condition

$$\lim_{t\to\infty} e^{-\beta t} a_t \geq 0. \tag{5}$$

and the initial condition $a_0 \geq 0$.

Money does not yield utility directly, and as the return on bonds completely dominates the return on money, it follows that Eq. (2) will always hold with strict equality. Thus, setting $m_t = c_t$ in Eq. (4). The current-value Hamiltonian for the household's optimization is given by:

$$H^h = U(c_t, l_t) + \lambda_t[D_t + ra_t + w_t l_t + \tau_t - (1 + r + \pi_t)c_t],$$

where λ is the co-state variable corresponding to assets.

The objective of the representative household is to maximize the discounted sum of future instantaneous utilities. Therefore, the optimality conditions for the representative household's problems are:

$$U_c(c_t, l_t) = \lambda_t(1 + r + \pi_t), \quad (6)$$

$$U_l(c_t, l_t) = -\lambda_t w_t, \quad (7)$$

$$\dot{\lambda}_t = \lambda_t(\beta - r), \quad (8)$$

Since β and r are both fixed, requiring $\beta = r$ from Eq. (8) for a steady state to exist. This is a standard assumption made in the literature; see, e.g., Sen and Turnovsky (1989, 1990). It implies $\dot{\lambda}_t = 0$ for all t, and λ_t is always at its steady state level. Hence, if there is any shock, whether it is temporary or permanent, λ jumps to its steady state level instantly and stays there during the adjustment process.

2. 2 The representative firm

Now we turn to the problem of the representative firm which has the standard neoclassical constant return to scale production function $Y_t = F(K_t, l_t)$, with capital K_t and labor l_t as inputs, where $F_K(K, l) > 0, F_l(K, l) > 0, F_{KK}(K, l) < 0, F_{ll}(K, l) < 0$ and $F_{KK}F_{ll} - F_{Kl}^2 = 0$.

The profits, or dividend payments, of the firm are given by

$$D_t = F(K_t, l_t) - w_t l_t - \Phi(I_t), \quad (9)$$

where $\Phi(I_t)$ represents total costs related to investment I_t. In particular,

$$\Phi(I_t) = I_t + \Psi(I_t), \quad (10)$$

where $\Psi(I_t)$ are the adjustment costs associated with I_t. The function $\Psi(I_t)$ is assumed to be a non-negative convex function, with $\Phi^{'} \geq 0$ and $\Phi^{''} > 0$. By choice of units we set $\Psi(0) = 0$ and $\Psi^{'}(0) = 0$, which implies $\Phi(0) = 0$ and $\Phi^{'}(0) = 1$.

The firm's problem is to maximize the present value of its dividend payments

$$\int_0^{\infty} D_t e^{-rt} dt = \int_0^{\infty} [F(K_t, l_t) - w_t l_t - \Phi(I_t)]\, e^{-rt} dt, \tag{11}$$

Subject to

$$\dot{K}_t = I_t, \tag{12}$$

and the initial condition K_0.

The current-value Hamiltonian for the firm's optimization is given by:

$$H^f = F(K_t, l_t) - w_t l_t - \Phi(I_t) + q_t I_t,$$

where q is the co-state variable associated with capital.

The optimality conditions for this problem are

$$F_l(K_t, l_t) = w_t, \tag{13}$$

$$\Phi^{'}(I_t) = q_t, \tag{14}$$

$$\dot{q}_t = q_t r - F_K(K_t, l_t), \tag{15}$$

and the standard transversality condition

$$\lim_{T \to \infty} e^{-rt} K_t \geq 0. \tag{16}$$

2. 3 The government

The government has no outstanding bonds or debt, and it chooses real transfer τ_t in order to satisfy its flow constraint

$$\dot{m}_t + \pi_t m_t = \tau_t, \tag{17}$$

according to which total real transfer τ_t should be equal to total government revenues from seigniorage ($\dot{m}_t + \pi_t m_t$). Thus, the central bank controls the inflation rate π, by adjusting τ continuously. The central bank does not intervene in the foreign exchange market.

2.4 The equilibrium dynamics

Then we can start to discuss the equilibrium for this economy. Combining the constraints and the optimality conditions of the representative household, the representative firm and the government, we obtain the following set of equations, which determine the equilibrium:

$$U_c(c_t, l_t) = \tilde{\lambda}(1 + r + \pi_t), \tag{18}$$

$$U_l(c_t, l_t) = -\tilde{\lambda} F_l(K_t, l_t), \tag{19}$$

$$\Phi'(I_t) = q_t, \tag{20}$$

$$\dot{q}_t = q_t r - F_K(K_t, l_t), \tag{21}$$

$$\dot{K}_t = I_t, \tag{22}$$

$$\dot{b}_t = F(K_t, l_t) + r b_t - c_t - \Phi(I_t), \tag{23}$$

where λ has been explicit setting at its steady state level during the adjustment process.

Notice, Eq. (19) is obtained from Eqs. (7) and (13). Eq. (23), which is the country's resource constraint, is obtained by combining Eqs. (4), (9), and (17). Eqs. (18), (20), (21), and (22) are, respectively, Eqs. (6), (14), (15) and (12).

To derive the perfect foresight path, we first solve Eqs. (18) and (19) for consumption and labor supply to obtain

$$c_t = c\left(K_t, \tilde{\lambda}, \pi_t\right), \tag{24}$$

$$l_t = l\left(K_t, \tilde{\lambda}, \pi_t\right), \tag{25}$$

where $c_K = \tilde{\lambda} F_{lK} U_{cl} / \Delta \gtreqless 0$,

$c_{\tilde{\lambda}} = \left[(1 + r + \pi)(U_{ll} U_c - U_l U_{cl}) + \tilde{\lambda}(1 + r + \pi) U_c F_{ll}\right] / \Delta U_c < 0,$

$c_\pi = \tilde{\lambda}(U_{ll} + \tilde{\lambda}F_{ll})/\Delta < 0,\ l_K = -\tilde{\lambda}U_{cc}F_{lK}/\Delta > 0,$

$l_{\tilde{\lambda}} = -[(1 + r + \pi)(U_{cc}U_l - U_{cl}U_c)]/\Delta U_c > 0,\ l_\pi = -\tilde{\lambda}U_{cl}/\Delta \gtreqless 0$, and $\Delta = U_{cc}(U_{ll} + \tilde{\lambda}F_{ll}) - U_{cl}^2 > 0$.[1]

From Eq. (20) we can write $I_t = I(q_t)$. Substituting this into Eq. (22) and substituting from Eq. (25) into Eq. (21) and substituting Eqs.(24) and (25) into Eq.(23). We obtain

$$\dot{K}_t = I(q_t), \tag{26}$$

$$\dot{q}_t = q_t r - F_K\left(K_t, l(K_t, \tilde{\lambda}, \pi_t)\right), \tag{27}$$

$$\dot{b}_t = F\left(K_t, l(K_t, \tilde{\lambda}, \pi_t)\right) + rb_t - c(K_t, \tilde{\lambda}, \pi_t) - \Phi(I_t). \tag{28}$$

Because λ is always at its steady state level, Eqs. (26)-(28) give us a system of differential equations which can be solved for K_t, q_t and b_t. Linearizing these three equations around the steady state equilibrium results in the following differential equation system

$$\begin{bmatrix} \dot{K}_t \\ \dot{q}_t \\ \dot{b}_t \end{bmatrix} = \begin{bmatrix} 0 & 1/\Phi'' & 0 \\ -F_{KK} - F_{Kl}l_K & r & 0 \\ F_K + F_l l_K - c_K & -1/\Phi'' & r \end{bmatrix} \begin{bmatrix} K_t - \tilde{K} \\ q_t - \tilde{q} \\ b_t - \tilde{b} \end{bmatrix}. \tag{29}$$

Let μ be the characteristic root of the dynamic system. From the coefficient matrix in Eq. (29), we can then yield:

$$(r - \mu)\left(\mu^2 - r\mu + \frac{F_{KK} + F_{Kl}l_K}{\Phi''}\right) = 0. \tag{30}$$

Letting μ_1, μ_2, and μ_3 be the three characteristic roots that satisfy Eq. (30), it is clear from Eq. (30) that $\mu_1\mu_2 = \frac{F_{KK} + F_{Kl}l_K}{\Phi''} < 0$ and $\mu_3 = r > 0$, where $F_{KK} + F_{Kl}l_K = F_{KK}(U_{cc}U_{ll} - U_{cl}^2)/\Delta < 0$. Among the three characteristic roots of the system, two are positive and one is negative. This implies that the system displays a

[1] The partial derivatives of consumption demand and labor supply implicit in Eqs. (24) and (25) are found by differentiating Eqs. (18) and (19) with respect to K, λ and π respectivly. Whether the signs of c_K, c_λ, c_π, l_K, l_λ, and l_π is positive or negative depends on the characteristic of U_{cl}($U_{cl} > 0$, $U_{cl} = 0$ or $U_{cl} < 0$). The details will be shown in Appendix A.

saddle-point stability, which is common for perfect foresight models. For expository convenience, we assume $\mu_1 < 0 < \mu_2$.The general solution for K, q, and b can be described by

$$K = \tilde{K} + h_{11}A_1 e^{\mu_1 t} + h_{12}A_2 e^{\mu_2 t}, \tag{31}$$

$$q = \tilde{q} + h_{21}A_1 e^{\mu_1 t} + h_{22}A_2 e^{\mu_2 t}, \tag{32}$$

$$b = \tilde{b} + A_1 e^{\mu_1 t} + A_2 e^{\mu_2 t} + A_3 e^{rt}, \tag{33}$$

where $\mu_1 = \frac{1}{2}\left[r - \sqrt{r^2 - \frac{4}{\Phi''}(F_{KK} + F_{Kl}l_K)}\right]$, $\mu_2 = \frac{1}{2}\left[r + \sqrt{r^2 - \frac{4}{\Phi''}(F_{KK} + F_{Kl}l_K)}\right]$,

$h_{11} = \frac{\mu_1 - r}{F_K + F_l l_K - c_k - \mu_1}$, $h_{12} = \frac{\mu_2 - r}{F_K + F_l l_K - c_k - \mu_2}$, $h_{21} = \frac{\Phi''\mu_1(\mu_1 - r)}{F_K + F_l l_K - c_k - \mu_1}$, $h_{22} = \frac{\Phi''\mu_2(\mu_2 - r)}{F_K + F_l l_K - c_k - \mu_2}$,

A_1, A_2, and A_3 are as yet undetermined coefficients.

At the long-run equilibrium, the economy is characterized by $\dot{K} = \dot{q} = \dot{b} = 0$. From Eq. (26)-(28), we can derive the following long-run relationships with given $\tilde{\lambda}$:

$$\tilde{q} = 1, \tag{34}$$

$$\tilde{K} = \tilde{K}(\tilde{\lambda}, \pi), \tag{35}$$

$$\tilde{b} = \tilde{b}(\tilde{\lambda}, \pi), \tag{36}$$

where $\tilde{K}_{\tilde{\lambda}} = -\frac{F_{Kl}l_{\tilde{\lambda}}}{F_{KK} + F_{Kl}l_K} > 0$, $\tilde{K}_\pi = -\frac{F_{Kl}l_\pi}{F_{KK} + F_{Kl}l_K} \gtreqless 0$[2],

$\tilde{b}_{\tilde{\lambda}} = -\frac{(F_K + F_l l_K - c_K)\tilde{K}_{\tilde{\lambda}} + (F_l l_{\tilde{\lambda}} - c_{\tilde{\lambda}})}{r} < 0$, $\tilde{b}_\pi = -\frac{(F_K + F_l l_K - c_K)\tilde{K}_\pi + (F_l l_\pi - c_\pi)}{r} \gtrless 0$. Whether the signs of $\tilde{K}_\pi$ and $\tilde{b}_\pi$ is positive or negative depends on the relationship between consumption and labor. The details will be discussed later.

We have already known that the system displays a saddle-point stability, with the saddle-path given by the following equations:

$$K_t = \tilde{K} + (K_0 - \tilde{K})e^{\mu_1 t}\ , \tag{37}$$

$$q_t = \tilde{q} + \mu_1\Phi''(K_0 - \tilde{K})e^{\mu_1 t}, \tag{38}$$

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[2] If the relationship between consumption and labor are independent, the value of $\tilde{K}_\pi$ will be equal to one. If the relationship between consumption and labor are substitute, the sign of $\tilde{K}_\pi$ will be positive. If the relationship between consumption and labor are complement, the sign of $\tilde{K}_\pi$ will be negative.

Combining Eq. (37) with (38), we obtain

$$q_t = \tilde{q} + \mu_1 \Phi''(K_t - \widetilde{K}), \tag{39}$$

Eq. (39) is the negatively sloping curve XX in Figure 1. From Eqs. (26) and (27), it is clear that the $\dot{q} = 0$ locus will be downward sloping and the $\dot{K} = 0$ locus will be horizontal. Also from Eqs. (26) and (27), it is clear that $\partial\dot{q}/\partial K > 0$, and $\partial\dot{K}/\partial q > 0$. This explains the directions of the changes in K and q off the $\dot{q} = 0$ and the $\dot{K} = 0$ locus.

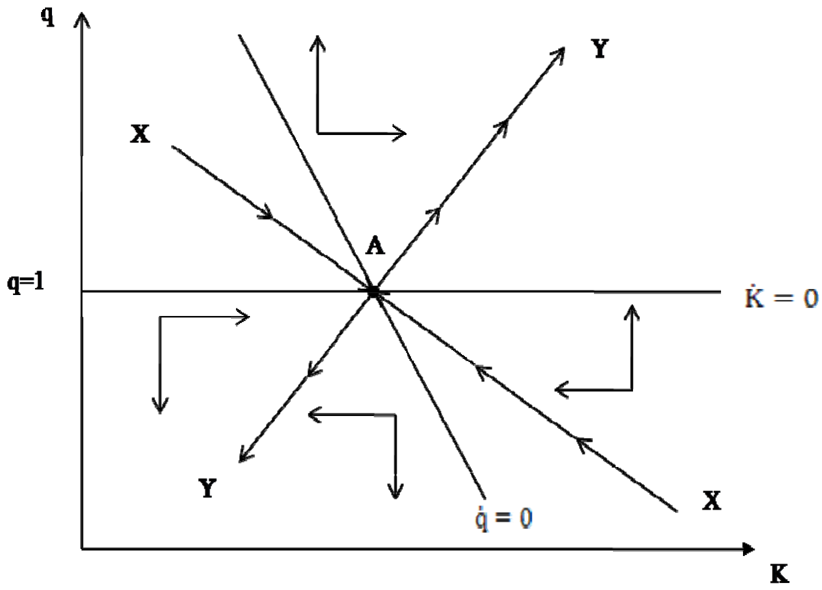

Figure 1 The phase diagram.

We then turn to the perfect foresight path between the foreign bonds holding and the capital stock. In order to get the perfect foresight path, linearize Eq. (28) around the steady state to obtain

$$\dot{b}_t = (F_K + F_l l_K - c_k)(K_t - \widetilde{K}) - \Phi'(\tilde{I})I'(\tilde{q})(q_t - \tilde{q}) + r(b_t - \tilde{b}). \tag{40}$$

In the steady state $\tilde{I} = 0$, and $\Phi'(0) = 1$. Besides, $I'(\tilde{q}) = 1/\Phi''$. Hence, Eq. (40) can re-written as:

$$\dot{b}_t = (F_K + F_l l_K - c_k)(K_t - \widetilde{K}) - \frac{1}{\Phi''}(q_t - \tilde{q}) + r(b_t - \tilde{b}). \tag{41}$$

Substituting from Eqs. (37) and (38) into Eq. (41), we obtain a differential equation for b which has the following solution

$$b_t = \tilde{b} + \frac{\Omega(K_0 - \tilde{K})}{\mu_1 - r} e^{\mu_1 t} + \left[(b_0 - \tilde{b}) - \frac{\Omega(K_0 - \tilde{K})}{\mu_1 - r} \right] e^{rt}. \tag{42}$$

For Eq. (42) to converge, the coefficient of e^{rt} must be zero: i.e., we should have

$$(b_0 - \tilde{b}) - \frac{\Omega(K_0 - \tilde{K})}{\mu_1 - r} = 0. \tag{43}$$

Combining condition (43) with Eq. (42), we obtain the solution for b_t

$$b_t - \tilde{b} = \frac{\Omega(K_0 - \tilde{K})}{\mu_1 - r} e^{\mu_1 t} = \frac{\Omega(K_t - \tilde{K})}{\mu_1 - r}, \tag{44}$$

where $\Omega = F_K + F_l l_K - c_k - \mu_1 > 0$.[3]

From Eqs. (23) and (44) we can know that the $\dot{b} = 0$ locus and ZZ curve are both downward sloping in Figure 2.

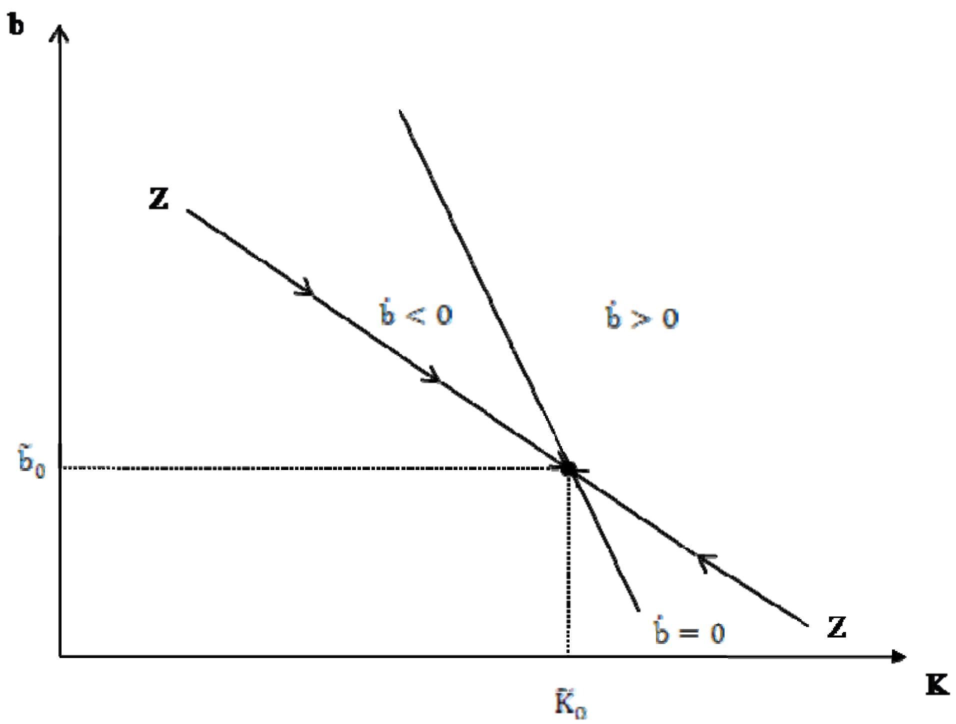

Figure 2 Negative slope of $\dot{b} = 0$ locus and saddle path ZZ.

[3] The sign of Ω will always be positive because $F_l l_K - c_k = \frac{\tilde{\lambda} F_{Kl}}{\Delta U_c} [(r + \pi) U_{cc} U_l + (U_{cc} U_l - U_{cl} U_c)] > 0$.

The negatively sloping curve ZZ in Figure 2 describes the relationship between K and b along the saddle path of the complete model. One important point to note is that, as both K and b are predetermined variables, the position of ZZ depends very much on the initial conditions$\left(\tilde{K}_0, \tilde{b}_0\right)$.