3 Linear problem

In this section we consider the "linear problem"

$$
\begin{aligned}
&x^{\Delta} + p(t)x^{\sigma} = h(t), \quad t \in J := [0,T]_{\mathbb{T}}, t \neq t_k, k = 1, \ldots, m, \\
&x(t_k+) - x(t_k-) = I_k(x(t_k-)), \quad k = 1, \ldots, m, \\
&x(0) = x(\sigma(T)).
\end{aligned}
$$

For convenience, we shall refer to this problem as (LP). Note that (LP) is not really a linear problem since the impulsive functions I_k, $k = 1, \ldots, m$, may or may not be linear.

The following basic lemmas will be used later.

Lemma 3.1 *Suppose that $h : [0, \sigma(T)]_{\mathbb{T}} \to \mathbb{R}$ is rd-continuous. Then x is a solution of (LP) if and only if x is a solution of*

$$
x(t) = \int_0^{\sigma(T)} G(t,s)h(s)\Delta s + \sum_{k=1}^{m} G(t,t_k) I_k(x(t_k)), \quad t \in [0, \sigma(T)]_{\mathbb{T}}, \tag{4}
$$

where

$$
G(t,s) = \begin{cases} \dfrac{e_p(s,t)e_p(\sigma(T),0)}{e_p(\sigma(T),0) - 1}, & 0 \leq s \leq t \leq \sigma(T), \\[2ex] \dfrac{e_p(s,t)}{e_p(\sigma(T),0) - 1}, & 0 \leq t < s \leq \sigma(T). \end{cases}
$$

Proof. The proof can be found in [9] and therefore we omit it here.

Lemma 3.2 *Let $G(t,s)$ be defined as Lemma 3.1. Then*

$$
0 \leqslant G(t,s) \leqslant \frac{e_p(\sigma(T),0)}{e_p(\sigma(T),0) - 1} \triangleq A \quad \textit{for all } t, s \in [0, \sigma(T)]_{\mathbb{T}}.
$$

Theorem 3.3 *Suppose that there exist positive constants l_k, $k = 1, \ldots, m$,* *such that*

$$|I_k(x) - I_k(y)| \leq l_k|x - y| \text{ for all } x, y \in \mathbb{R}.$$

If

$$A\sum_{k=1}^{m} l_k < 1,$$

then the problem (LP) has a unique solution for any $h \in PC$.

Proof. First, we define the operator $\Psi : X \to X$ by

$$\Psi x(t) = \int_0^{\sigma(T)} G(t,s)h(s)\Delta s + \sum_{k=1}^{m} G(t,t_k)I_k(x(t_k)), \quad t \in [0, \sigma(T)]_{\mathbb{T}},$$

so that fixed points of Ψ are solutions of (LP) and vice versa. Next, we consider $u, v \in X$ and $t \in [0, \sigma(T)]_{\mathbb{T}}$. Then

$$\begin{aligned}
|(\Psi u)(t) - (\Psi v)(t)| &= \left|\sum_{k=1}^{m} G(t,t_k)I_k(u(t_k)) - \sum_{k=1}^{m} G(t,t_k)I_k(v(t_k))\right| \\
&\leq \sum_{k=1}^{m} |G(t,t_k)||I_k(u(t_k)) - I_k(v(t_k))| \\
&\leq \sum_{k=1}^{m} Al_k|u(t_k) - v(t_k)| \\
&\leq \sum_{k=1}^{m} Al_k\|u - v\|,
\end{aligned}$$

and hence

$$\|\Psi u - \Psi v\| \leq A\sum_{k=1}^{m} l_k\|u - v\|.$$

This means that Ψ is a contraction mapping. By Banach's fixed point theorem, Ψ has a unique fixed point $x \in X$ so that (LP) has exactly one solution. □