4 Nonlinear problem

In this section we study the "nonlinear problem" (NP). It follows from Lemma 3.1 that $x \in X$ is a solution of (NP) if and only if it satisfies

$$x(t) = \int_0^{\sigma(T)} G(t,s)f(s,x(s))\Delta s + \sum_{k=1}^{m} G(t,t_k)I_k(x(t_k)), \quad t \in [0,\sigma(T)]_{\mathbb{T}}.$$

Introduce the operator $\Phi : X \to X$ by the formula

$$\Phi x(t) = \int_0^{\sigma(T)} G(t,s)f(s,x(s))\Delta s + \sum_{k=1}^{m} G(t,t_k)I_k(x(t_k)), \quad t \in [0,\sigma(T)]_{\mathbb{T}}.$$

Obviously, fixed points of Φ are solutions of (NP) and conversely.

Definition 4.1 *Let F be a subset of PC. We say that F is quasiequicontinuous on $[0,\sigma(T)]_{\mathbb{T}}$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $f \in F$ and $k = 0,\ldots,m$, then*

$$|f(t) - f(\tilde{t})| < \epsilon, \ \forall t, \ \tilde{t} \in J_k \text{ and } |t - \tilde{t}| < \delta.$$

In order to show that Φ is compact, we need the following compactness criteria.

Lemma 4.2 *A set $F \subset PC$ is relatively compact if the following conditions hold:*

(i) *F is bounded, that is, there exists a positive constant c such that $\|x\| \leq c$ for all $x \in F$,*

(ii) *F is quasiequicontinuous on $[0,\sigma(T)]_{\mathbb{T}}$.*

Proof. Let $\{x_n\}$ be a sequence in F. From assumptions (i) and (ii), we know that $\{x_n\}$ is uniformly bounded and equicontinuous on J_0. By Arzela's theorem, there is a convergent subsequence $\{x_n^{(1)}\}$ of $\{x_n\}$ on J_0. Since $\{x_n^{(1)}\}$ is uniformly bounded and equicontinuous on J_1, it follows from Arzela's theorem that there is a convergent subsequence $\{x_n^{(2)}\}$ of $\{x_n^{(1)}\}$ on J_1. Continuing this process, we can get a convergent subsequence $\{x_n^{(m+1)}\}$ of $\{x_n^{(m)}\}$ on J_m. It is clear that $\{x_n^{(m+1)}\}$ is a convergent subsequence of $\{x_n\}$ on $[0, \sigma(T)]_{\mathbb{T}}$. Hence F is relatively compact. $\square$

Lemma 4.3 $\Phi : X \to X$ *is compact.*

Proof. Let D be a bounded subset of X. Then there exist positive constants M and M_k, $k = 0, \ldots, m$, such that

$$|f(t, x(t))| \leq M, \ |I_k(x_{t_k})| \leq M_k, \quad \forall x \in D, \ t \in [0, \sigma(T)]_{\mathbb{T}}, \ k = 0, \ldots, m,$$

and hence

$$\begin{aligned} |\Phi x(t)| &= |\int_0^{\sigma(T)} G(t,s) f(s, x(s)) \Delta s + \sum_{k=1}^{m} G(t, t_k) I_k(x(t_k))| \\ &\leq \int_0^{\sigma(T)} |G(t,s)||f(s, x(s))| \Delta s + \sum_{k=1}^{m} |G(t, t_k)||I_k(x(t_k))| \\ &\leq AM\sigma(T) + A \sum_{k=1}^{m} M_k. \end{aligned}$$

This implies that $\Phi(D)$ is bounded.

Let $x \in D$ and t, $\tilde{t} \in J_k$, where $k = 0, \ldots, m$. Then one can easily show that

$$\begin{aligned} & |\Phi x(t) - \Phi x(\tilde{t})| \\ \leq\ & M \textstyle\int_0^{\tilde{t}} |G(t,s) - G(\tilde{t}, s)| \Delta s + M \int_{\tilde{t}}^{t} |G(t,s) - G(\tilde{t}, s)| \Delta s \\ & \quad + M \textstyle\int_t^{\sigma(T)} |G(t,s) - G(\tilde{t}, s)| \Delta s + \sum_{k=1}^{m} \left| G(t, t_k) - G(\tilde{t}, t_k) \right| M_k \\ =\ & MA \textstyle\int_0^{\tilde{t}} \left| e_p(s,t) - e_p(s, \tilde{t}) \right| \Delta s \\ & \quad + M\eta \textstyle\int_{\tilde{t}}^{t} \left| e_p(s,t) e_p(\sigma(T), 0) - e_p(s, \tilde{t}) \right| \Delta s \\ & \quad + M\eta \textstyle\int_t^{\sigma(T)} \left| e_p(s,t) - e_p(s, \tilde{t}) \right| \Delta s \\ & \quad + A \textstyle\sum_{k=1}^{j-1} \left| e_p(t_k, t) - e_p(t_k, \tilde{t}) \right| M_k \\ & \quad + \eta \textstyle\sum_{k=j}^{m} \left| e_p(t_k, t) - e_p(t_k, \tilde{t}) \right| M_k, \end{aligned}$$

where $\eta = 1/(e_p(\sigma(T),0)-1)$. It follows that

$$|\Phi x(t) - \Phi x(\tilde{t})| \to 0 \text{ uniformly in } x \in D,$$

as $|t-\tilde{t}| \to 0$, which implies that $\Phi(D)$ is quasiequicontinuous on $[0,\sigma(T)]_{\mathbb{T}}$. By Lemma 4.2, Φ is compact. This completes the proof. □

Now we are in a position to establish the existence theorem for problem (NP) by using fixed point theorems.

Theorem 4.4 *Assume that there exist positive constants l_k, $k = 1, \ldots, m$, such that*

$$|I_k(u) - I_k(v)| \le l_k |u-v| \text{ for all } u, v \in \mathbb{R},$$

and suppose also that there exists a positive constant l such that

$$|f(t,u) - f(t,v)| \le l|u-v| \text{ for all } t \in [0,\sigma(T)]_{\mathbb{T}} \text{ and } u, v \in \mathbb{R}.$$

If

$$A\left(\sigma(T)l + \sum_{k=1}^{m} l_k\right) < 1,$$

then the problem (NP) has a unique solution.

Proof. For any $u, v \in X$ and $t \in [0,\sigma(T)]_{\mathbb{T}}$, we can easily get that

$$|\Phi u(t) - \Phi v(t)| \le A\left(\sigma(T)l + \sum_{k=1}^{m} l_k\right)\|u-v\|,$$

and hence

$$\|\Phi u - \Phi v\| \le A\left(\sigma(T)l + \sum_{k=1}^{m} l_k\right)\|u-v\|.$$

This means that Φ is a contraction mapping. By Banach's fixed point theorem, Φ has a unique fixed point which is the unique solution of (NP). This completes the proof. □

Theorem 4.5 *Assume that there exist positive constants l and c_k, $k = 1, ..., m$, such that*

$$|f(t,x)| \le l|x| \text{ for all } t \in [0,\sigma(T)]_{\mathbb{T}} \text{ and } x \in \mathbb{R}, \tag{5}$$

and

$$|I_k(x)| \leq c_k \text{ for all } x \in \mathbb{R} \text{ and } k = 1, \dots, m. \tag{6}$$

If

$$lA\sigma(T) < 1, \tag{7}$$

then the problem (NP) has at least one solution.

Proof. Let $x \in X$, $t \in [0, \sigma(T)]_{\mathbb{T}}$. If x is a solution of $x = \mu\Phi x$ for some $\mu \in (0, 1)$, we have that

$$\begin{aligned} |x(t)| &= \left| \mu \int_0^{\sigma(T)} G(t,s) f(s, x(s)) \Delta s + \mu \sum_{k=1}^{m} G(t, t_k) I_k(x(t_k)) \right| \\ &\leq \mu \int_0^{\sigma(T)} |G(t,s)||f(s, x(s))| \Delta s + \mu \sum_{k=1}^{m} |G(t, t_k)||I_k(x(t_k))| \\ &\leq \mu Al\|x\|\sigma(T) + \mu A \sum_{k=1}^{m} c_k \end{aligned}$$

and hence

$$\|x\| \leq \mu Al\|x\|\sigma(T) + \mu A \sum_{k=1}^{m} c_k \leq Al\|x\|\sigma(T) + A \sum_{k=1}^{m} c_k.$$

Together with (7), we obtain

$$\|x\| \leq \frac{A\sum_{k=1}^{m} c_k}{1 - Al\sigma(T)}.$$

This implies that all solutions of $x = \mu\Phi x$ are uniformly bounded independent of $\mu \in (0, 1)$. From Lemma 2.11 and Lemma 4.3, Φ has a fixed point. This completes the proof. □

Theorem 4.6 *Assume that there exist positive constants c and* l_k, $k = 1, ..., m$, *such that*

$$|f(t, x)| \leq c \text{ for all } t \in [0, \sigma(T)]_{\mathbb{T}} \text{ and } x \in \mathbb{R}, \tag{8}$$

and

$$|I_k(x)| \leq l_k|x| \text{ for all } x \in \mathbb{R} \text{ and } k = 1, ..., m. \tag{9}$$

If

$$A \sum_{k=1}^{m} l_k < 1, \tag{10}$$

then the problem (NP) has least one solution.

Proof. Let $x \in X$, $t \in [0, \sigma(T)]_{\mathbb{T}}$. If x is a solution of $x = \mu\Phi x$, for some $\mu \in (0,1)$, we have that

$$\begin{aligned} |x(t)| &= \left|\mu \int_0^{\sigma(T)} G(t,s) f(s, x(s))\Delta s + \mu \sum_{k=1}^{m} G(t, t_k) I_k(x(t_k))\right| \\ &\leq \mu \int_0^{\sigma(T)} |G(t,s)||f(s,x(s))|\Delta s + \mu \sum_{k=1}^{m} |G(t,t_k)||I_k(x(t_k))| \\ &\leq \mu B c\sigma(T) + \mu B \sum_{k=1}^{m} l_k \|x\|, \end{aligned}$$

and hence

$$\|x\| \leq \mu A c\sigma(T) + \mu A \sum_{k=1}^{m} l_k \|x\| \leq A c\sigma(T) + A \sum_{k=1}^{m} l_k \|x\|.$$

Together with assumption (10), we can obtain that

$$\|x\| \leq \frac{A c\sigma(T)}{1 - A\sum_{k=1}^{m} l_k}.$$

This implies that all solutions of $x = \mu\Phi x$ are uniformly bounded independent of $\mu \in (0,1)$. From Lemma 2.11 and Lemma 4.3, Φ has a fixed point. This completes the proof. □

When all impulsive functions are linear, we have the following existence result.

Theorem 4.7 *For each $k = 1, \ldots, m$, let $I_k(x) = l_k x$, where l_k is a constant. Suppose that the following conditions hold:*

(1) $|f(t,x)| \leq c$ *for all* $(t,x) \in [0, \sigma(T)]_{\mathbb{T}} \times \mathbb{R}$, *where c is a positive constant,*
(2) $\prod_{k=1}^{m} b_k \neq e_p(\sigma(T), 0)$, *where* $b_k = l_k + 1$.

Then the problem (NP) has least one solution.

Proof. In this case, the problem (NP) can be rewriten as

$$x^{\triangle} + p(t)x^{\sigma} = f(t,x),\ t \in J,\ \ t \neq t_k,\ k = 1,\dots,m, \tag{11}$$

$$x(t_k+) = b_k x(t_k),\ k = 1,\dots,m, \tag{12}$$

$$x(0) = x(\sigma(T)). \tag{13}$$

To account for the problem, there are two cases to be discussed.
First, if $b_{k_0} = 0$ for some $1 \leq k_0 \leq m$ and let $y(t) = e_p(t,0)x(t)$, it is shown as follows:

$$\begin{aligned} &y^{\Delta}(t) = e_p(t,0)f(t, e_p(0,t)y(t)), \quad t \in J,\ t \neq t_k,\ k = 1,\dots,m,\\ &y(t_k+) = b_k y(t_k), \quad k \neq k_0,\\ &y(t_{k_0}+) = 0,\\ &y(0) = e_p(0,\sigma(T))y(\sigma(T)). \end{aligned} \tag{14}$$

Now, we consider the following initial value problem

$$\begin{aligned} &y^{\Delta}(t) = e_p(t,0)f(t, e_p(0,t)y(t)), \quad t \in J_{k_0},\\ &y(t_{k_0}+) = 0. \end{aligned} \tag{15}$$

Define an operator $L_{k_0} : C(J_{k_0}) \to C(J_{k_0})$ by

$$(L_{k_0}y)(t) = \int_{t_{k_0}}^{t} e_p(s,0)f(s, e_p(0,s)y(s))\Delta s, \quad t \in J_{k_0}.$$

Obviously, the fixed points of L_{k_0} are solutions to (15).
Firstly, we shall prove that L_{k_0} is compact operator. The proof is divided into two steps.
Step 1: To show that $L_{k_0}(D)$ is uniformly bounded.
Let $D \subseteq C(J_{k_0}) = \{y : y\ \textit{is continuous on}\ J_{k_0}\}$ be a bounded set.

Let $y \in D$, $t \in J_{k_0}$,

$$\begin{aligned}
|(L_{k_0}y)(t)| &= |\int_{k_0}^{t} e_p(s,0) f(s, e_p(0,s)y(s))\Delta s| \\
&\le \int_{t_{k_0}}^{t} |e_p(s,0)||f(s, e_p(0,s)y(s))|\Delta s \\
&\le c\int_{t_{k_0}}^{t} |e_p(s,0)|\Delta s \\
&\le c e_p(\sigma(T),0)(t_{k_0+1} - t_{k_0}).
\end{aligned}$$

This implies that $L_{k_0}(D)$ is bounded.

Step 2: For any $t, \tilde{t} \in J_{k_0}$, $y \in C(J_{k_0})$, we have

$$\begin{aligned}
&|(L_{k_0}y)(t) - (L_{k_0}y)(\tilde{t})| \\
&= \left|\int_{t_{k_0}}^{t} e_p(s,0) f(s, e_p(0,s)y(s))\Delta s - \int_{t_{k_0}}^{\tilde{t}} e_p(s,0) f(s, e_p(0,s)y(s))\Delta s\right| \\
&= \left|\int_{\tilde{t}}^{t} e_p(s,0) f(s, e_p(0,s)y(s))\Delta s\right|
\end{aligned}$$

It follows that

$$|(L_{k_0}y(t) - (L_{k_0}y(\tilde{t})| \to 0 \text{ uniformly in } y \in D,$$

as $|t - \tilde{t}| \to 0$, which implies that L_{k_0} is quasiequicontinuous on $[0, \sigma(T)]_{\mathbb{T}}$. Consequently, Step 1 and Step 2 together with Lemma 4.2 show that L_{k_0} is a compact operator. Consider the operator equation

$$y = \mu L_{k_0} y, \quad \mu \in (0,1) \tag{16}$$

If y is a solution of Eq(16), we have that

$$\begin{aligned}
|y(t)| &= \left|\mu \int_{t_{k_0}}^{t} e_p(s,0) f(s, e_p(0,t)y(t))\Delta s\right| \\
&\le \int_{t_{k_0}}^{t} |e_p(s,0)||f(s, e_p(0,t)y(t))|\Delta s \\
&\le c|e_p(\sigma(T),0)|\sigma(T),
\end{aligned}$$

and hence

$$\|y\| \le c|e_p(\sigma(T),0)|\sigma(T).$$

This implies that all solutions of $y = \mu L_{k_0} y$ are uniformly bounded independent of $\mu \in (0,1)$. From Lemma 2.11, L_{k_0} has a fixed point. Hence (15) has at least one solution and let y_{k_0} be a soluton.
It determines a value $y_{k_0}(t_{k_0+1})$ that we use as the initial value for the following problem

$$\begin{aligned} &y^{\Delta}(t) = e_p(t,0) f(t, e_p(0,t) y(t)), \quad t \in J_{k_0+1}, \\ &y(t_{k_0+1}+) = b_{k_0+1} y_{k_0}(t_{k_0+1}). \end{aligned} \tag{17}$$

Similarly, we can get a solution $y_{t_{k_0+1}}$ for the problem (17).Continuing the process, we know that the following initial value problem

$$\begin{aligned} &y^{\Delta}(t) = e_p(t,0) f(t, e_p(0,t) y(t)), \quad t \in J_m, \\ &y(t_m+) = b_m y_{m-1}(t_m). \end{aligned}$$

has a solution y_m.
Similarly, we can consider the following initial value problem

$$\begin{aligned} &y^{\Delta}(t) = e_p(t,0) f(t, e_p(0,t) y(t)), \quad t \in J_0, \\ &y(0) = y_m(\sigma(T)). \end{aligned} \tag{18}$$

It is easy to see that the problem (18) has a solution y_0. Continuing the before process, we construct a solution for (14). So, there exist a solution of (NP).

Second, if $b_k \neq 0$ for k=1,...,m, we let x be any solution of (13) and set

$$y(t) = x(t) \prod_{k \in \mathcal{F}} b_k^{-1} \text{ for } t > 0,$$

where

$$\mathcal{F} = \{i \in \{1, \dots, m\} : 0 \leq t_i < t\},$$

when $k \geq 1$, we have

$$y(t_k+) = b_k x(t_k) \prod_{0 \leq t_i \leq t_k} b_k^{-1} = x(t_k) \prod_{0 \leq t_i < t_k} b_k^{-1} = y(t_k),$$
$$y(y_k-) = x(t_k) \prod_{0 \leq t_i < t_k} b_k^{-1} = y(t_k).$$

Hence, $y(t)$ is continuous on J.

It is obvious that $\{k : 0 \leq t_k\} = \{k : 0 \leq \sigma(t_k)\}$. Therefore, we see that $y(t)$ satisfies

$$\begin{aligned} &y^{\Delta}(t) + p(t)y(\sigma(t)) = f(t, y(t) \prod_{0 \leq t_k < t} b_k) \prod_{0 \leq t_k < t} b_k^{-1}, \\ &y(0) = y(\sigma(T)) \prod_{k=1}^{m} b_k. \end{aligned} \tag{19}$$

So, if $y(t)$ is a solution to (19), $x(t) = y(t) \prod_{k=1}^{m} b_k$ satisfies (13). Besides, let

$$F(t, y(y)) = f(t, y(y) \prod_{0 \leq t_k < t} b_k) \prod_{0 \leq t_k < t} b_k^{-1},$$

(19) can be simplified as follows:

$$\begin{aligned} &y^{\Delta}(t) + p(t)y(\sigma(t)) = F(t, y(t)), \quad t \in J, \\ &y(0) = y(\sigma(T)) \prod_{k=1}^{m} b_k. \end{aligned} \tag{20}$$

It follows that (20) has a solution if and only if the integral equation

$$y(t) = \int_0^{\sigma(T)} G(t, s) F(s, y(s)) \Delta s$$

is solvable. Here,

$$G(t, s) = \frac{1}{e_p(\sigma(T), 0) - \prod_{k=1}^{m} b_k} \begin{cases} e_p(\sigma(T), 0) e_p(s, t), & 0 \leq s \leq t \leq \sigma(T), \\ \prod_{k=1}^{m} b_k e_p(s, t), & 0 \leq t < s \leq \sigma(T). \end{cases}$$

Define the operator $B : C(J) \to C(J)$ by

$$By = \int_0^{\sigma(T)} G(t, s) F(s, y(s)) \Delta s$$

Hence (20) is equivalent to $y = By$. It is easy to show that B is compact. We consider the equation

$$y = \mu By, \quad \mu \in (0,1) \tag{21}$$

If y is a solution to Eq(21) for $t \in [0, \sigma(T)]_{\mathbb{T}}$, then

$$|y(t)| \leq \int_0^{\sigma(T)} |G(t,s)||F(s,y(s))|\Delta s \leq c_1 c_2,$$

where

$$c_1 = \sup\{c \prod_{0 \leq t_k < t} |b_k|^{-1} : t \in [0, \sigma(T)]_{\mathbb{T}}\}$$
$$c_2 = \sup\{\int_0^{\sigma(T)} |G(t,s)|\Delta s : t \in [0, \sigma(T)]_{\mathbb{T}}\}$$

and hence

$$\|y\| \leq c_1 c_2.$$

This implies that all solutions of $y = \mu By$ are uniformly bounded independent of $\mu \in (0,1)$. From Lemma 2.11, B has a fixed point. Furthermore, (13) has at least one solution. This completes the proof. □

Theorem 4.8 *Suppose that the following conditions hold:*

(a) $\lim_{|x| \to \infty} \frac{f(t,x)}{x} = 0$ *uniformly for* $t \in [0, \sigma(T)]_{\mathbb{T}}$,

(b) $\lim_{|x| \to \infty} \frac{I_k(x)}{x} = 0$ *for all* $k = 1, \ldots, m$.
Then the problem (NP) has least one solution.

Proof. Let

$$H\Phi = \{x \in X : x = \mu \Phi x, \ for \ some \ \mu \in (0,1)\}. \tag{22}$$

Then $H\Phi$ is bounded. Indeed, suppose that $H\Phi$ is unbounded, then there exist sequences $\{x_n\}_{n=1}^{\infty}$ in X and $\{u_n\}_{n=1}^{\infty}$ in $(0,1)$ such that $\|x_n\| \geq n$ and

$$\begin{aligned}
&x_n^{\triangle}(t) + p(t)x_n(\sigma(t)) = \mu_n f(t, x_n(t)), \quad t \in J,\ t \neq t_k,\ k = 1, \ldots, m,\\
&x_n(t_k+) - x_n(t_k-) = \mu_n I_k(x_n(t_k)), \quad k = 1, \ldots, m,\\
&x_n(0) = x_n(\sigma(T)).
\end{aligned}$$

Now we let $v_n = x_n/\|x_n\|$. Then $\|v_n\| = 1$ and v_n satisfies

$$\begin{aligned}
&v_n^{\triangle}(t) + p(t)v_n(\sigma(t)) = g_n(t),, \quad t \in J,\ t \neq t_k,\ k = 1, \ldots, m,\\
&v_n(t_k+) - v_n(t_k-) = \theta_{n,k}, \quad k = 1, \ldots, m,\\
&v_n(0) = v_n(\sigma(T)),
\end{aligned}$$

where

$$g_n(t) = \frac{\mu_n f(t, x_n(t))}{\|x_n\|} \text{ and } \theta_{n,k} = \frac{\mu_n I_k(x_n(t_k))}{\|x\|}.$$

By Lemma 3.1, we get

$$v_n(t) = \int_0^{\sigma(T)} G(t,s) g_n(s) \Delta s + \sum_{k=1}^{m} G(t, t_k)\theta_{n,k}, \quad t \in [0, \sigma(T)]_{\mathbb{T}},$$

From assumptions (a) and (b), we have

$$|g_n(t)| \leq \frac{|f(t, x_n(t))|}{\|x_n\|} \to 0,$$

and

$$|\theta_{n,k}| \leq \frac{|\mu_n I_k(x_n(t_k))|}{\|x\|} \to 0\ ,\ \ k = 1, \ldots, m,$$

uniformly for $t \in [0, \sigma(T)]_{\mathbb{T}}$ as $n \to \infty$, so that

$$|v_n(t)| \leq \int_0^{\sigma(T)} |G(t,s)||g_n(s)|\Delta s + \sum_{k=1}^{m} |G(t, t_k)||\theta_{n,k}| \to 0,$$

uniformly for $t \in [0, \sigma(T)]_{\mathbb{T}}$ as $n \to \infty$. Hence $\|v_n(t)\| \to 0$ as $n \to \infty$, which contradicts the fact that $\|v_n\|$=1. From Lemma 2.11, the problem (NP) has at least one solution. Therefore the proof is complete. □

The following corollaries can be obtained from Theorem 4.8.

Corollary 4.9 *(Bounded case) Assume that the nonlinearity f is bounded and that the impulsive functions I_k, $k = 1, \ldots, m$, are bounded. Then the nonlinear problem (NP) has at least one solution.*

Proof. For each $k = 1, \ldots, m$, let c and c_k be positive constants such that

$$|f(t,x)| \leq c, \quad |I_k(x)| \leq c_k, \text{ for all } x \in \mathbb{R},\ t \in [0, \sigma(T)]_{\mathbb{T}}.$$

Then it is easy to see that the assumptions (a) and (b) of Theorem 4.8 hold. Hence by Theorem 4.8, the problem (NP) has at least one solution. □

Corollary 4.10 *(Sublinear growth) Suppose that there exist $a \in X$, $b \in \mathbb{R}$ and $\alpha \in [0,1)$ such that*

$$|f(t,x)| \leq a(t) + b|x|^{\alpha} \text{ for all } t \in [0, \sigma(T)]_{\mathbb{T}} \text{ and } x \in \mathbb{R},$$

and suppose also that there exist positive constants $a_k, b_k \in \mathbb{R}$, $\alpha_k \in [0,1)$, k=1,…,m, with

$$|I_k(x)| \leq a_k + b_k|x|^{\alpha_k} \text{ for all } x \in \mathbb{R}.$$

Then the problem (NP) has at least one solution.

Proof. Let $x \in X$, $t \in [0, \sigma(T)]_{\mathbb{T}}$. Since

$$\left|\frac{f(t,x)}{x}\right| \leq \frac{a(t)}{|x|} + \frac{b|x|^{\alpha}}{|x|} \text{ and } \left|\frac{I_j(x)}{x}\right| \leq \frac{a_k}{|x|} + \frac{b_k|x|^{\alpha_k}}{|x|},$$

one can easily see that the assumptions (a) and (b) of Theorem 4.8 hold. Hence by Theorem 4.8, the problem (NP) has at least one solution. □

Corollary 4.11 *Assume that the following conditions hold:*

(a) $\lim_{|x|\to\infty} \frac{f(t,x)}{x} = q(t)$ *uniformly in $[0, \sigma(T)]_{\mathbb{T}}$, where $q \leq p$ and $|1 + \mu q| \geq \eta$ in $[0, \sigma(T)]_{\mathbb{T}}$ for some positive constant η,*

(b) $\lim_{|x|\to\infty} \frac{I_k(x)}{x} = 0$ *for all $k = 1, \ldots, m$.*

Then the problem (NP) is solvable.

Proof. We rewrite (1) as

$$x^{\Delta} + \tilde{p}(t)x^{\sigma} = \tilde{f}(t,x),$$

where

$$\tilde{p}(t) = \frac{p(t) - q(t)}{1 + \mu(t)q(t)} \text{ and } \tilde{f}(t,x) = \frac{f(t,x) - q(t)x}{1 + \mu(t)q(t)}.$$

From assumption (a), we see that

$$\frac{\tilde{f}(t,x)}{x} \to 0,$$

as $x \to \infty$,uniformly for $t \in [0, \sigma(T)]_{\mathbb{T}}$. Hence the conclusion follows from Theorem 4.8. □

Remark We consider the more general first-order nonlinear dynamic system

$$\begin{aligned} &x^{\Delta} = g(t,x), \quad t \in J := [0,T]_{\mathbb{T}}, \ t \neq t_k, \ k = 1, \dots, m, \\ &x(t_k+) - x(t_k-) = I_k(x(t_k-)), \quad k = 1, \dots, m, \\ &x(0) = x(\sigma(T)). \end{aligned} \tag{23}$$

Select $\lambda \in (0, 1/\sigma(T))$. Then one can easily verify that system (23) is equivalent to (NP) with

$$p(t) = \frac{\lambda}{1 - \mu\lambda} \geq 0, \text{ and } f(t,x) = \frac{g(t,x) + \lambda x}{1 - \mu\lambda}.$$

Hence the existence of solution for system (23) can be obtained from Theorem 3.3-4.8 and Corollary 4.9-4.11.