3 A Hierarchy of Classes of Bounded Bitolerance Orders

In this section, we define subclasses of bounded bitolerance orders with three different kinds of restrictions in G. Isaak, K. L. Nyman and A. N. Trenk [7].

Restrictions on Intervals I_v

Definition 3.1. (Unit): P is a *unit bitolerance order* if it has a bounded bitolerance representation $\langle I, p, q \rangle$ in which $|I_x| = |I_y|$ for all $x, y \in V$.

Definition 3.2. (Proper): P is a *proper bitolerance order* if it has a bounded bitolerance representation $\langle I, p, q \rangle$ in which I_x is not properly contained in I_y for all $x, y \in V$.

Restrictions on Tolerance Points $p(v), q(v)$

Definition 3.3. (Point-core): P is a *point-core bitolerance order*(or point-core order) if it has a bounded bitolerance representation $\langle I, p, q \rangle$ in which $p(v) = q(v)$ for all $v \in V$.

Definition 3.4. (Totally bounded): P is a *totally bounded bitolerance order* if it has a bounded bitolerance representation $\langle I, p, q \rangle$ in which $p(v) \leq q(v)$ for all $v \in V$.

Restrictions on Left and Right Tolerance

Definition 3.5. (Tolerance): P is a *bounded tolerance order* if it has a bounded bitolerance representation $\langle I, p, q \rangle$ in which $t_l(v) = t_r(v)$ for all $v \in V$. In this case we write $t(v) = t_l(v) = t_r(v)$.

Lemma 3.6. (G. Isaak, K. L. Nyman and A. N. Trenk [7]) *If $P = (V, \prec)$ is a member of any of the 18 classes of bitolerance order obtained using the restrictions in Table 1, then it has a representation in which all endpoints and tolerant points are distinct. If P is a unit or proper interval order, then it has a representation in which all endpoints are distinct.*

intreval length	p and q	t_l and t_r
1. unit	a. point-core	i. tolerance
2. proper	b. totally bounded	ii. bitolerance
3. -	c. -	

Table 1: The 18 classes of bounded bitolerance orders

Proof. Fix a bounded bitolerance representation $\langle I, p, q \rangle$ of $P = (V, \prec)$. Assume that $V = \{1, 2, \cdots, |P|\}$ and let ϵ be the smallest positive distance between two distinct points in $\{L(x),\ p(x),\ q(x),\ R(x) | x \in V\}$. Form a new representation with $L^{'}(x) = L(x) - \epsilon/10 + \epsilon/10^{x+2}$, $p^{'}(x) = p(x) - \epsilon/10^2 + \epsilon/10^{x+2}$, $q^{'}(x) = q(x) + \epsilon/10^2 + \epsilon/10^{x+2}$, $R^{'}(x) = R(x) + \epsilon/10 + \epsilon/10^{x+2}$. The following are straightforward to check from the definition of $L^{'}(x)$, $p^{'}(x)$, $q^{'}(x)$, $R^{'}(x)$ and the use of terms involving ϵ: the prime representation has all endpoints and tolerant points distinct, if the original representation was proper or unit or totally bounded or satisfied the 'tolerance' property then so is the prime representation. It remains to check that the prime representatin also represents P. If $p(y) - R(x) > 0$ then by the choice of ϵ, $p(y) - R(x) > \epsilon$ and thus $p^{'}(y) - R^{'}(x) = (p(y) - \epsilon/10^2 + \epsilon/10^{y+2}) - (R(x) + \epsilon/10 + \epsilon/10^{x+2}) > 0$. If $p(y) - R(x) \le 0$ then $p^{'}(y) - R^{'}(x) = (p(y) - \epsilon/10^2 + \epsilon/10^{y+2}) - (R(x) + \epsilon/10 + \epsilon/10^{x+2}) \le 0$ since $(-\epsilon/10^2 + \epsilon/10^{y+2}) - (\epsilon/10 + \epsilon/10^{x+2}) \le 0$. So $R(x) < p(y) \Leftrightarrow R^{'}(x) < p^{'}(y)$. Similary $q(x) < L(y) \Leftrightarrow q^{'}(x) < L^{'}(y)$.

For point-core representation do as above omitting terms $\epsilon/10^2$ to form $p^{'}(x)$ and $q^{'}(x)$ so that $p(x) = q(x) \Rightarrow p^{'}(x) = q^{'}(x)$. Finally, suppose P is a unit or proper interval order with a representation in which $v \in V$ is assigned the interval $I_v = [L(v), R(v)]$. Define $L^{'}(x)$ and $R^{'}(x)$ as above. The prime representation has all endpoints distinct and it is straightforward to check that the unit or proper property is maintained. In the manner similar to above we can check that $R(x) < L(y) \Leftrightarrow R^{'}(x) < L^{'}(y)$ and hence the prime intervals also represent P. ❑

We choose an interval $I_x = [L(x), R(x)]$ and the center c_x of I_x for each $x \in V$. We define $x \prec y$ if and only if $R(x) < c_y$ and $L(y) > c_x$. Thus an order has a tolerance representation in which the sum of the tolerances $t_l(x)$ and $t_r(x)$ is the interval length

$|I_x|$ for each elements x. We call such ordered sets point-core bitolerance orders. The representation of point-core bitolerance orders are called *Fishburn representations* by K. P. Bogart and G. Isaak [1]. Later, we will show that the classes of unit bitolerance orders and point-core bitolerance orders are the same.

Proposition 3.7. (G. Isaak, K. L. Nyman and A. N. Trenk [6])
The classes of unit point-core bitolerance orders (*1, a, ii*) *and proper point-core bitolerance orders* (*2, a, ii*) *are equivalent. The classes of unit totally-bounded bitolerance orders* (*1, b, ii*) *and proper totally-bounded bitolerance orders* (*2, b, ii*) *are equivalent.*

Proof. (i) By definition, we know that unit point-core bitolerance orders are contained in proper point-core bitolerance orders and unit totally-bounded bitolerance orders are contained in proper totally-bounded bitolerance orders immediately. So we need only show the reverse inclusions.

(ii) Observe that two bitolerance representation for which the relative order of the interval endpoints and tolerant points is the same orders. Using this observations, we next show that a proper bitolerance representation can be transformated into a unit bitolerance representation of the same order. Afterwards we note that the transformation preserves the "point-core" and "totally bounded" properties.

By induction, we assume that any proper bitolerance representation $\langle I, p, q\rangle$ of an order $P = (V, \prec)$ with $|V| < n$ can be transformed into a unit bitolerance representation of P. Furthermore, assume this can be accomplished so that the relative order of the set of endpoints and tolerant points is unchanged.

Let $P = (V, \prec)$ be a proper bitolerance order with $|V| = n$ and using Lemma 3.6. Fix a proper bitolerance representation $\langle I, p, q\rangle$ of P in which all endpoints and tolerant points are distinct. Let x be the element with smallest left endpoint. Since the representation is proper, $R(x)$ is also the smallest right endpoint. By induction, fix a unit bitolerance representation $\langle I', p', q'\rangle$ of $P - x$ in which the points in $\{L'(v), p'(v), q'(v), R'(v) : v \in V - x\}$ appear in the same order as the corresponding points in $\{L(v), p(v), q(v), R(v) : v \in V - x\}$.

For concreteness, translate and scale the new representations of $P - x$ so that the smallest left endpoints is $L^{'}(y) = 0$ and $|I_v^{'}| = 1$ for all v. Now place $R^{'}(x)$ so that its position with respect to the points in $\{L^{'}(v), p^{'}(v), q^{'}(v), R^{'}(v) : v \in V - x\}$ matches the position of $R(x)$ with respect to the points in $\{L(v), p(v), q(v), R(v) : v \in V - x\}$. We know that $R^{'}(x)$ will be the smallest right endpoint in $\langle I^{'}, p^{'}, q^{'}\rangle$, thus $R^{'}(x) < R^{'}(y) = 1$. Set $L^{'}(x) = R^{'}(x) - 1 < 0$, thus $L^{'}(x)$ will be the smallest left endpoin in $\langle I^{'}, p^{'}, q^{'}\rangle$, as desired. Finally, place $p^{'}(x)$(resp. $q^{'}(x)$) so that its position relative to points in $\{L^{'}(v), p^{'}(v), q^{'}(v), R^{'}(v) : v \in V - x\}$ matches the position of $p(x)$(resp. $q(x)$) with respect to the corresponding points in $\{L(v), p(v), q(v), R(v) : v \in V - x\}$.

Hence the new representation is unit. Moreover, it has the same relative ordering of the interval endpoints and tolerant points as the original. So by our observation above, it represent the same order. If the original representation was point-core ($p(v) = q(v)$ for all v) then the new representation is point-core since $p^{'}(v) = q^{'}(v)$ for all v. Likewise, if the original representation was totally bounded ($p(v) \leqslant q(v)$ for all v) then the new representation is totally bounded since $p^{'}(v) \leqslant q^{'}(v)$ for all v. This completes the proof. ❑

Theorem 3.8. (L. Langley [8]) *The following are equivalent statements about an ordered set P.*

(i) *P is a unit bitolerance order.*

(ii) *P is a point-core bitolerance order.*

(iii) *P has a bitolerance representation with constant cores.*

Proof. (i)$\Longrightarrow$(ii): Suppose $P = (V, \prec)$ is a unit bitolerance order and fix a representation $\langle I, p, q\rangle$ of it where $I_v = [L(v), R(v)]$. Let $T = \{T_v \mid v \in V\}$ be the associated trapezoid representation of P. Note that since all intervals are unit length, the representation has the property that the left diagonal D_v has the same slope for each $v \in V$. Create a new set of trapezoids $T^{'} = \{T_v \mid v \in V\}$ by sliding all points on the line L_1 to the right by a constant k until the diagonals become vertical. Note that $T_x \ll T_y$ if and only if $T_x^{'} \ll T_y^{'}$ so $T^{'}$ is also a trapezoid representation of P.

The representation $T^{'}$ is a right-leaning trapezoid representation of P and we may transform it to a left-leaning representation $T^{''}$ of P by reflecting it around the line L_2 as discussed in Remark 2.9. The bounded representation associated with $T^{''}$ is a point-core bitolerance representation. Figure 7 shows the transfomation.

(ii)$\Longrightarrow$(iii): By definition, the point-core is a special case of constant cores.

(iii)$\Longrightarrow$(i): Let $\langle I, p, q \rangle$ be a constant core bounded bitolerance representation of P where $I_v = [L(v), R(v)]$. Thus there is a constant c for which $|I_v| - (t_r(v) + t_l(v)) = (R(v) - L(v)) - (t_r(v) + t_l(v)) = (R(v) - t_r(v)) - (L(v) + t_l(v)) = q(v) - p(v) = c$ for all $v \in V$. Let $T = \{T_v \mid v \in V\}$ be the associated trapezoid representation of P and noted that constant core condition translates into the condition that the right diagonals of each trapezoid each have the same slope. As in Remark 2.9, reflect each trapezoid around the line L_2 to get a new trapezoid representation $T^{'}$ of P which has the property that left diagonals of each trapezoid has the same slope. Finally, slide all points on the line L_1 to the left until the non-horizontal sides of each trapezoid have negative slope. The result is a left-leaning trapezoid representation $T^{''}$ of P in which all the left diagonals have the same slope and hence the same length. So the bounded bitolerance representation associated with $T^{''}$ is a unit bitolerance representation. ❑

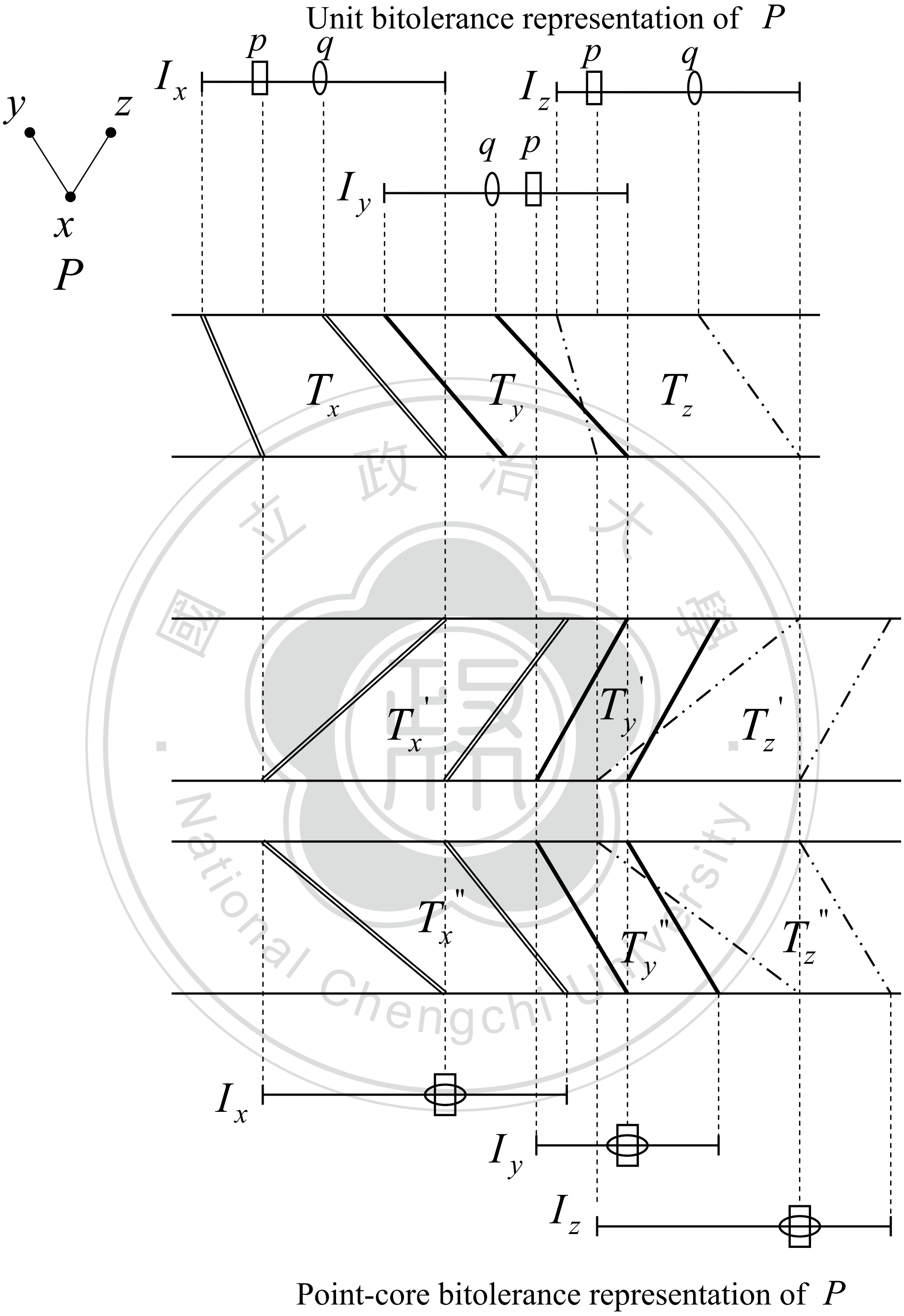

Figure 7: The correspondence between unit bitolerance and point-core representation of an order P