4 Unit and Proper Bitolerance Orders

Associated with any bitolerance order on V, there is a natural extension of the order: give a representation (in which we may assume no two interval I_x and I_y have the same center), we defined the linear extension by $x \prec_c y$ if and only if the center of I_x is less than the center of I_y. We call this linear extension *central extension.* We need the following two lemmas to prove our main results in this section.

Lemma 4.1. (K. P. Bogart and G. Isaak [1]) *Let* $P = (V, \prec)$ *be a proper bitolerance order and* $N = (V', \prec)$ *an induced suborder with* $V' = \{a, b, i, j\}$ *with* $i \prec a$, $b \prec j$, $b \prec a$ *and with no other comparabilities among* $\{a, b, i, j\}$. *Then in any central extension of the ordering, either* (i) $b \prec_c i \prec_c a$ *or* (ii) $b \prec_c j \prec_c a$ *(or both).*

Proof. For a contradiction, assume there is a proper representation $\langle I, p, q\rangle$ of P whose central extension $\prec_c$ of P violates both (i) and (ii) in Figure 8. Then $i \prec_c b \prec_c a \prec_c j$, that is, the center satisfy $c(i) < c(b) < c(a) < c(j)$. Since the representation is proper, the left and right endpoint of each interval also appear in the order, thus $R(i) < R(b)$ and $L(a) < L(j)$. Now $b \prec j$ so $R(b) < p(j)$ and thus $R(i) < R(b) < p(j)$. Similary, $i \prec a$ so $q(i) < L(a) < L(j)$. Hence, $R(i) < p(j)$ and $q(i) < L(j)$ imply $i \prec j$, a contradiction. ❑

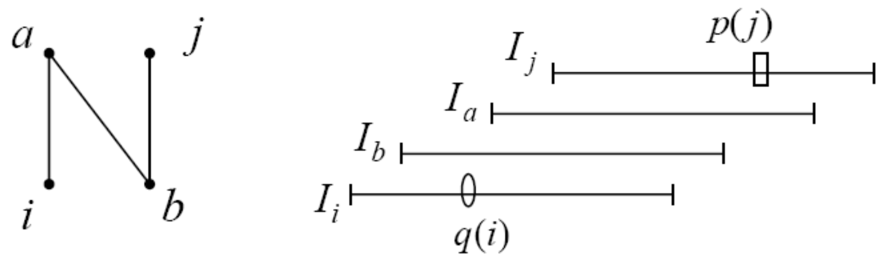

Figure 8: A proper bitolerance representation with $i \prec a$, $b \prec j$, $b \prec a$

Lemma 4.2. (K. P. Bogart and G. Isaak [1]) *Let* $P = (V, \prec)$ *be a proper bitolerance order and* $Q = (V', \prec)$ *an induced suborder* ***2+2*** *with* $V' = \{a, b, i, j\}$ *with* $i \prec a$, $b \prec j$ *and with no other comparabilities among* $\{a, b, i, j\}$. *Then in any central extension of the ordering, either* (i) $i \prec_c b \prec_c j \prec_c a$ *or* (ii) $b \prec_c i \prec_c a \prec_c j$.

Proof. Fix a proper bitolerance representation $\langle I, p, q \rangle$ of P where $I_v = [L(v), R(v)]$ and the centers of the intervals are distanct. Let $\prec_c$ be the associated central extension. Thus $i \prec_c a$ and $b \prec_c j$ and without loss of generality we assume $i \prec_c b$. It remains to show $j \prec_c a$, so for a contradiction, assume $a \prec_c j$ in Figure 9. Since the representation is proper, the left and right endpoint of each interval are also ordered by $\prec_c$, thus $R(i) < R(b)$ and $L(a) < L(j)$. Since $b \prec j$, we have $R(b) < p(j)$, thus $R(i) < R(b) < p(j)$. Similary, $i \prec a$ so $q(i) < L(a)$ and thus $q(i) < L(a) < L(j)$. Hence $R(i) < p(j)$ and $q(i) < L(j)$ imply $i \prec j$, a contradiction. ❑

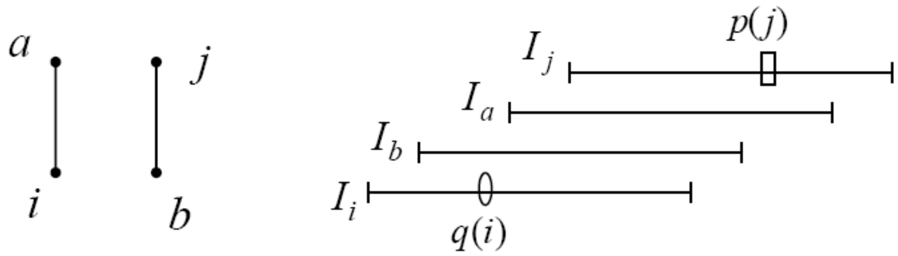

Figure 9: A proper bitolerance representation with $i \prec a$, $b \prec j$

Theorem 4.3. (K. P. Bogart and G. Isaak [1]) *For an ordered set $P = (V, \prec)$ the following are equivalent:*

(i) *P is a proper bitolerance order.*

(ii) *P has a linear extension $\prec_L$ so that*

  (a) *$N = (V', \prec)$ an induced suborder with $V' = \{a, b, i, j\}$ with $i \prec a, b \prec j, b \prec a$ and with no other comparabilities among $\{a, b, i, j\}$, then either* (i) $b \prec_L i \prec_L a$ *or* (ii) $b \prec_L j \prec_L a$*(or both).*

  (b) *$Q = (V', \prec)$ an induced suborder **2+2** with $V' = \{a, b, i, j\}$ with $i \prec a, b \prec j$ and with no other comparabilities among $\{a, b, i, j\}$, then either* (i) $i \prec_L b \prec_L j \prec_L a$ *or* (ii) $b \prec_L i \prec_L a \prec_L j$.

(iii) *P is a unit bitolerance order.*

Proof. (iii)$\Longrightarrow$ (i): This can be proved directly from the definitions of unit and proper bitolerance orders.

(i)$\Longrightarrow$ (ii): Lemma 4.1 and 4.2 have proved.

(ii)$\Longrightarrow$ (iii): Let $P = (V, \prec)$ be an ordered satisfying the condition of (ii) for linear extension $\prec_L$. According to $\prec_L$, we know that $v_1 \prec_L v_2 \prec_L \cdots \prec_L v_n \Longrightarrow L(v_1) < L(v_2) < \cdots < L(v_{n)}$. Next we let $R(v_1) > L(v_n)$ and $n = R(v_1) - L(v_1)$ for unit. Now we define the remaining right endpoint by $R(v_i) = L(v_i) + n$. Hence we construct a unit bitolerance representation $\langle I, p, q\rangle$ for P.

- Assign endpoint of I_{v_i}:
  Let $L(v_i) = i$ for $i, 1 \leq i \leq n$.
  Let $R(v_i) = n + i$ for $i, 1 \leq i \leq n$.
- Assign right tolerant point: If v_i is maximal element in P, let $q(v_i) = n + \frac{1}{2}$
  Otherwise, let a be the smallest index so that $v_i \prec v_a$ and set $q(v_i) = a - \frac{1}{2}$
- Assign left tolerant point: If v_i is minimal element in P, let $p(v_i) = n + \frac{1}{2}$
  Otherwise, let b be the largest index so that $v_b \prec v_i$ and set $p(v_i) = n + b + \frac{1}{2}$

First, we are sure that $q(v_i) \in I_{v_i}$ for all $v_i \in V$. If $q(v_i) = n + \frac{1}{2}$, then $L(v_i) \leq n < n + 1 \leq R(v_i)$. Otherwise, $q(v_i) = a - \frac{1}{2}$ where $i < a$. Since i and a are integers, $i \leq a - 1$ and $L(v_i) = i \leq a - 1 < q(v_i) < a \leq n < R(v_i)$. Also $p(v_i) \in I_{v_i}$ for all $v_i \in V$.
So the intervals $I_{v_i} = [L(v_i), R(v_i)]$ and the tolerant points $p(v_i)$, $q(v_i)$ give a unit bitolerance representation of an order $Q = (V, \prec')$. Now, we want to show that $P = Q$, that is, $v_i \prec v_j \Longleftrightarrow v_i \prec' v_j$. Without loss of generality, we let $i < j$. We have the following two cases.

Case 1: $v_i \prec v_j$ in P

We know $v_i \prec v_j$. So v_i is not maximal, by definition we have $q(v_i) = a - \frac{1}{2}$, and we know that $a \leq j$. Then $q(v_i) = a - \frac{1}{2} < a \leq j = L(v_j)$.
By the same way, v_j is not minimal, we have $p(v_j) = n + b + \frac{1}{2}$, and we know that $i \leq b$. Then $p(v_j) = n + b + \frac{1}{2} > n + b \geq n + i = R(v_i)$. So we can obtain $v_i \prec' v_j$ in Q.

Case 2: $v_i \parallel v_j$ in P.

Since $p(v_i) < R(v_i) < R(v_j)$, we know $v_j \not\prec' v_i$. Hence we must show $v_i \not\prec' v_j$.

For a contradiction, suppose $v_i \prec' v_j \Rightarrow q(v_i) < L(v_j)$ and $R(v_i) < p(v_j)$.

Since $q(v_i) < L(v_j) \leq n$, we know $q(v_i) \neq n + \frac{1}{2}$. So v_i is not maximal in P, $q(v_i) = a - \frac{1}{2}$. Hence, a is the smallest index such that $v_i \prec v_a$.

Similarly, since $n + 1 \leq R(v_i) < p(v_j)$, we know $p(v_j) \neq n + \frac{1}{2}$. So v_j is not minimal in P, $p(v_j) = n + b + \frac{1}{2}$ where b is the largest index such that $v_b \prec v_j$. We have the following two claims.

(1) claim $a < j$:

Because $q(v_i) = a - \frac{1}{2} < L(v_j) = j$, we know $a \leq j$. However, $v_i \prec v_a$ and $v_i \parallel v_j \Rightarrow a \neq j$.

(2) claim $b > i$:

Because $n + i = R(v_i) < p(v_j) = n + b + \frac{1}{2}$, we know $b \geq i$. However $v_b \prec v_j$ and $v_i \parallel v_j \Rightarrow b \neq i$.

In addition, if $a = b$, we obtain $v_i \prec v_a = v_b \prec v_j$. Since $v_i \parallel v_j$, we know that $a \neq b$. Therefore, we have shown that $v_i \prec v_a$, $v_b \prec v_j$, and $v_i \parallel v_j$ in P already. We must consider the relation of v_a and v_b in $V' = \{v_i, v_j, v_a, v_b\}$ in the following two cases.

(a) If $v_b \prec v_a$ in P then the induced order $(V', \prec)$ is the order N described in (ii)(a). We have $v_b \prec_L v_i \prec_L v_a$ or $v_b \prec_L v_j \prec_L v_a$. However, it means $b < i < a$ or $b < j < a$, which contradicts that fact $b > i$ and $a < j$.

(b) If $v_b \parallel v_a$ in P then the induced order $(V', \prec)$ is the order $2 + 2$ described in (ii)(b). We have $v_i \prec_L v_b \prec_L v_j \prec_L v_a$ or $v_b \prec_L v_i \prec_L v_a \prec_L v_j$. However, it means $i < b < j < a$ or $b < i < a < j$, which contradicts that fact $a < j$ and $b > i$.

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