匯率連動平均利率選擇權：對數常態市場利率模型
Quanto Average Interest Rate Options in a Lognormal Interest Rate Market Model

吳庭斌 Ting-Pin Wu *
Department of Statistics, National Taipei University, Taiwan.
國立台北大學統計系助理教授

陳松男 Son-Nan Chen**
Department of Banking and Finance, National Chengchi University, Taiwan.
國立政治大學金融系教授

(Received: Apr. 15, 2007; Revision: May. 17, 2007; Accepted: Dec. 24, 2007)

摘要

本文根據 Amin 和 Jarrow (1991)，將單一貨幣的 LIBOR 市場模型擴充成跨貨幣的 LIBOR 市場模型，並在此模型架構下，分別以 Vorst (1992) 和 Levy (1992) 所提出的近似方法，求得匯率連動平均利率選擇的近似定價公式。此兩種近似公式經由蒙地卡羅模擬驗證，準確性非常高，因此對實務應用上，有相當的幫助。

關鍵字：LIBOR 市場模型；平均利率選擇權;匯率連動選擇權

Abstract

This paper extends the single-currency LMM to the cross-currency LMM based on the Amin and Jarrow (1991) framework, and the resulting model is applied to deriving the approximate pricing formula of the quanto average interest options via two different approximation approaches, presented by Vorst (1992) and Levy (1992). These two approximation formulas have been examined to be very accurate as compared with Monte Carlo simulation. The model calibration procedure is also presented in detail for practical implementation.

Keywords: LIBOR Market Model; Average Interest Rate Options; Quanto Options

* Address: 151, University Rd., San Shia, Taipei, 237 Taiwan. Tel.: 886-2-86746785. Fax: 886-2-86715907, E-mail: tpwu@mail.ntpu.edu.tw

** Address: NO.64,Sec.2,ZhiNan Rd.,Wenshan District,Taipei City 11605,Taiwan (R.O.C). Tel: 02-29393091 ext81016. Fax: 886-2-29398004. E-mail: slchen@nccu.edu.tw

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1. Introduction

With the rapid growth of global financial markets in recent years, firms may raise funds from foreign countries and involve foreign interest rate risks. Controlling interest rate risks is thus one of the most primary and vital tasks of a financial manager and has become more complicated and difficult. To help the managers hedge against foreign interest rate risks, many typical financial instruments such as quanto caps, quanto floors, quanto collars and others have been developed. One of the most widely used hedging products is the quanto cap which is composed of a series of individual quanto interest rate call options with maturities coinciding with the end-user's foreign interest payment dates. Via quanto caps, financial managers can ceil their foreign interest cost for each individual payment lower than or equal to the cap rate. However, some firms may be satisfied with their hedging need by hedging their average costs of overall foreign interest payments rather than individual ones. Moreover, it is well known that an option on an average price of an underlying asset is cheaper than a portfolio of options. Due to the cheaper hedging costs[1], quanto average interest rate options (hereafter, QAIROs) have recently been introduced and become popular.

QAIROs have several uses. First, QAIROs can be used to hedge against foreign interest rate risk with lower hedging cost. Second, if a company have a large and complicated position that is exposed to foreign interest rate risks, it is usually inefficient to hedge the risks individually. QAIROs can be employed to hedge efficiently against the average costs of overall foreign interest payments. For example, consider a financial institution having issued many kinds of foreign interest rate-related products, such as foreign interest rate-linked structure notes, and thus resulting in complicated positions exposed to foreign interest rate risks. Via QAIROs, the financial institution can hedge effectively against foreign interest rate risks at a lower cost. Moreover, QAIROs are less liable to unanticipated events or the market manipulation by the options' counterparties since their final payoffs depend on the average interest rate during their lives.

Even though few articles were written on the QAIROs, some were conducted on the pricing of average interest rate options (AIROs). Under the Vasicek (1977) model, Longstaff (1995) derived the pricing formula of AIROs. However, there are some problems making the Longstaff model not easy for practical utilization. First of all, the instantaneous short rate modeled in the Vasicek model is abstract and market-unobservable, thereby leading to the difficulty in the parameter calibration. Second, the parameters involved in the Vasicek model are assumed to be constant, which is not flexible enough to provide a perfect fit to the current term structure of interest rates and volatilities. Third, the setting of the underlying average rate in Longstaff is inconsistent with market practice which the average rate is calculated on the basis of the LIBOR rates rather than the continuouslycompounded short rates. In addition, the short rate characterized by the model is normally distributed, and thus it may become negative with positive probability which may lead to some pricing error in many cases. [2]

Under the Hull and White (1990, HW) interest rate model, Cheuk and Vorst (1999, CV) modified the underlying average rate in Longstaff (1995) and employed the arithmetic average of market-observable rates, namely LIBOR rates. However, the LIBOR rates used in CV are still indirectly derived from the abstract short rate modeled in HW, which may make the derivation process of pricing formulas more complicated. Similar to Longstaff (1995), the HW model is expressed in terms of instantaneous short rates which are not directly observable in the market and may cause some difficulties in calibrating the model to the market prices of actively traded products. In addition, the short rate characterized in HW is also normally distributed and therefore some pricing error may occur due to the negative rate.

The main purpose of this article is to extends the (single-currency) LIBOR market model (hereafter, LMM) to a cross-currency LMM (hereafter, CLMM) based on Amin and Jarrow (1991, AJ) and then apply the results to develop a pricing formula of QAIROs. The pricing model of QAIROs is more general, and if the model setting degenerates to the single-currency case, the pricing model of QAIROs will become the AIRO pricing model in the LMM framework. In addition, pricing QAIROs under CLMM can avoid the problems as mentioned earlier and is more feasible for practice.

The LIBOR market model is developed by Musiela and Rutkowski (1997), Miltersen et al. (1997) and Brace, Gatarek and Musiela (1997). The reasons for employing the CLMM rather than the instantaneous short rate model, such as Vasicek and HW, are mentioned as follows. The rate modeled in the CLMM is the LIBOR rate which is commonly used in the financial industry, and thus the model is more suitable for practical implementation. The LMM is widely used by practitioners due to the advantage that the cap pricing formula in the LMM framework is the Black's formula which is consistent with market practice and makes the calibration procedure easier. The underlying interest rates in QAIROs are usually the foreign LIBOR rates, and hence pricing QAIROs under the CLMM structure is more straightforward. Furthermore, the distribution of LIBOR rate under CLMM is lognormal distribution, which can avoid pricing error due to the negative rate. Thus, pricing QAIROs based on the CLMM is more natural, more direct and simpler than based on the instantaneous short rate model.

This paper is organized as follows. Section 2 briefly specifies the results in AJ (1991) and employs them to derive the CLMM. In addition, some useful techniques in CLMM, such as the approximate lognormalization and changing-measure, are also presented. Section 3 outlines the general contract of an QAIRO and presents two different approaches to deriving the closed-form solutions of the QAIRO. In Section 4, we provide the calibration procedure and examine the accuracy of the approximate formula via Monte Carlo simulation. The conclusion is made in the last section.

2. The Cross-Currency LIBOR Market Model

We assume that trading takes place continuously in time over an interval [0, T], 0< T < ∞. The uncertainty is described by the filtered spot martingale probability space $(\Omega, F, Q, \{F_t\}_{t Î [0,T]})$ and an m-dimentional independent standard Brownain motion $W(t) = (W_1(t), W_2(t), \ldots, W_m(t))$ is defined on it. The flow of information accruing to all the agents in the economy is represented by the filtration $\{F_t\}_{t Î [0,T]}$ which satisfies the usual hypotheses.[3] Note that Q denotes the domestic spot martingale probability measure.

We briefly specify the results of AJ (1991) in the first subsection and then apply them to extend the ordinary LMM to the CLMM.

2.1 The Results in AJ (1991)

We list the notations with 'd' for domestic and 'f' for foreign as follows :

$f_k(t,T)$ = the kth country's forward interest rate contracted at time t for instantaneous borrowing and lending at time T with $0 < t < T < T$ where k Î $\{d, f\}$.

$P_k(t,T)$ = the time t price of the kth country's zero coupon bond (ZCB) paying one dollar at time T, where k Î $\{d, f\}$.

$r_k(t)$ = the kth country's risk-free short rate at time t.

$X(t)$ = the spot exchange rate at time T Î $[0, T]$ for one unit of the foreign currency expressed in terms of the domestic currency.

Q_T = the domestic forward martingale measure with respective to the numéraire $P_k(\times T)$.

Based on the insights of Harrison and Kreps (1979), AJ (1991) extended the HJM model to a cross-currency case and specified some conditions on the instantaneous forward rate process. Under these conditions, the market is arbitrage-free and complete and contingent claims can be priced by the risk-neutral valuation method. We present their results in the following proposition.

Proposition 1. The Dynamics under the Domestic Martingale Measure in AJ (1991)

Under the domestic martingale measure Q, for any T Î $[0, T]$, the dynamics of the forward rates, the ZCB prices and the exchange rate are given as follows:

$$df_d(t,T) = \sigma_{fd}(t,T) \cdot \sigma_{Pd}(t,T)dt + \sigma_{fd}(t,T) \cdot dW(t)$$

$$df_f(t,T) = \sigma_{ff}(t,T) \cdot [\sigma_{Pf}(t,T) - \sigma_X(t)]dt + \sigma_{ff}(t,T) \cdot dW(t)$$

$$\frac{dP_d(t,T)}{P_d(t,T)} = r_d(t)dt - s_{Pd}(t,T) \ dW(t)$$

$$\frac{dP_f(t,T)}{P_f(t,T)} = [r_f(t) + s_X(t) ?s_{Pf}(t,T)]dt \quad s_{Pf}(t,T) \ dW(t)$$

$$\frac{dX(t)}{X(t)} = [r_d(t) - r_f(t)]dt + s_X(t) \ dW(t)$$

where $s_{fk}(t,T)$ *and* $s_{Pk}(t,T)$ *denote, respectively, the volatilities of the kth country's forward rate and ZCB, and* $s_X(t)$ *denotes the volatility of the exchange rate. The relationship between* $s_{fk}(t,T)$ *and* $s_{Pk}(t,T)$ *is given as follows:*

$$s_{Pk}(t,T) = \int_t^T s_{fk}(t,u)du$$

The drift and volatility terms in Proposition 1 are subject to some regularity conditions.[4]

It is worth noting that even in the cross-currency environment the drift term of the domestic forward rate under the domestic martingale measure Q is unchanged. Moreover, for the foreign case, the drift has one additional term, $s_{ff}(t,T) \times s_X(t)$, which specifies the instantaneous correlation between the exchange rate and the foreign forward rate. It is also observed that the drift terms of the foreign assets are augmented by the instantaneous correlations between the exchange rate and the assets.

With these arbitrage-free relationships between the volatility and the drift terms in Proposition 1, we can use them to derive the arbitrage-free CLMM and then apply it to pricing Quanto interest rate derivatives.

2.2 The Cross-Currency LIBOR Market Model

In this subsection, we derive the CLMM based on the results as derived in BGM (1997) and AJ (1991). It is important to note that, thereafter, we model the term structure of interest rates by specifying the LIBOR rate dynamics, rather than the instantaneous forward rate dynamics. However, we still use the same economic environment, the same notations and the drift restrictions for no-arbitrage in Proposition 1 to derive the CLMM under the spot martingale measure.

Fix some $\delta > 0$, $T \in [0,T]$ and $k \in \{d,f\}$, define the forward LIBOR rate process $\{F_k(t,T);0\le t\le T\}$ as

$$1+\delta F_k(t,T)=\frac{P_k(t,T)}{P_k(t,T+\delta)}$$

$$=\exp\left(\int_T^{T+\delta} f_k(t,u)du\right) \quad (1)$$

As indicated in BGM (1997), we assume that the volatility structures of the forward LIBOR rate and the exchange rate processes are lognormal volatility structures, but we do not exactly specify the drift term of these processes. Later, we will employ the arbitrage-free relationships between the drifts and the volatility terms in Proposition 1 to determine the specific drift terms of these processes that make the model arbitrage-free.

Assumption 1. A Family of LIBOR Rate Processes

We assume that $F_k(t,T)$, $k\in\{d,f\}$,, *has a lognormal volatility structure and its stochastic process is given by*

$$dF_k(t,T)=\mu_{Fk}(t,T)dt+F(t,T)\gamma_{Fk}(t,T)\cdot dW(t) \quad (2)$$

where $\gamma_{Fk}(\cdot,T):[0,T]\to R^m$ *is deterministic, bounded and piecewise continuous volatility functions and* $\mu_{Fk}(\cdot,T):[0,T]\to R$ *is some unspecified drift function.*

Assumption 2. The Spot Exchange Rate Dynamics

The stochastic process of the spot exchange rate X(t) is given as follows:

$$dX(t)=X(t)\mu_X(t)+X(t)\sigma_X(t)\cdot dW(t) \quad (3)$$

where $\mu_X(t):[0,T\ ]\to R$ *is some unspecified drift function and*

$\sigma_X(t):[0,T\ ]\to R^m$ *is a deterministic process.*

It is important to emphasize that the drift terms of the above stochastic processes are not yet determined. The specific forms of the drift terms must be chosen to make the economy arbitrage-free. We will use the arbitrage-free relationship between the drift and the volatility terms in Proposition 1 to determine the drift terms in (2) and (3).

First, let us determine the drift term, $\mu_{Ff}(t,T)$, in the foreign forward LIBOR rate process. Assume that $H(t)=\int_T^{T+\delta} f_f(t,u)du$ and $F_f(t,T)=(1/\delta)(\exp(H(t))-1)$. Making use of Itô's Lemma, we have

$$dF_f(t,T)=\frac{1}{\delta}\exp\left(\int_T^{T+\delta} f_f(t,u)du\right)\left\{dH(t)+\frac{1}{2}dH(t)dH(t)\right\} \tag{4}$$

and

$$dH(t)=\left(\frac{1}{2}\left\|\sigma_{Pf}(t,T+\delta)\right\|^2-\frac{1}{2}\left\|\sigma_{Pf}(t,T)\right\|^2\right)dt$$

$$-\sigma_X(t)\cdot\left(\sigma_{Pf}(t,T+\delta)-\sigma_{Pf}(t,T)\right)dt$$

$$+\left(\sigma_{Pf}(t,T+\delta)-\sigma_{Pf}(t,T)\right)dW(t) \tag{5}$$

Combining (4) and (5), we have

$$dF_f(t,T)=$$

$$\frac{1}{\delta}\left(1+\delta F_f(t,T)\right)\left(\sigma_{Pf}(t,T+\delta)-\sigma_{Pf}(t,T)\right)\cdot\left(\sigma_{Pf}(t,T+\delta)-\sigma_X(t)\right)dt$$

$$+\frac{1}{\delta}\left(1+\delta F_f(t,T)\right)\left(\sigma_{Pf}(t,T+\delta)-\sigma_{Pf}(t,T)\right)\cdot dW(t) \tag{6}$$

Equation (6) shows us that the relationship between the drift and the diffusion terms of the foreign forward LIBOR rate processes under the domestic martingale measure Q, and we can employ it to determine $\mu_{Ff}(t,T)$. In Assumption 1, we have assumed that the foreign forward LIBOR rate's volatility structure is lognormal, and thus

$$\frac{1}{\delta}\left(1+\delta F_f(t,T)\right)\left(\sigma_{Pf}(t,T+\delta)-\sigma_{Pf}(t,T)\right)=F_f(t,T)\gamma_{Ff}(t,T) \tag{7}$$

By substituting (7) into the drift term in (6), the foreign forward LIBOR rate dynamics under the measure Q can be expressed as follows:

$$\frac{dF_f(t,T)}{F_d(t,T)}=\gamma_{Ff}(t,T)\cdot\left(\sigma_{Pf}(t,T+\delta)-\sigma_X(t)\right)dt+\gamma_{Ff}(t,T)\cdot dW(t)$$

Under the measure Q , the domestic forward LIBOR rate process can be derived similarly as follows:

$$\frac{dF_f(t,T)}{F_d(t,T)}=\gamma_{Fd}(t,T)\cdot\sigma_{Pd}(t,T+\delta)dt+\gamma_{Fd}(t,T)\cdot dW(t)$$

Note that the bond volatility vector $\sigma_{Pk}(t,T)$, $k\in\{d,f\}$, is not yet specified in the CLMM. For the validity of using the HJM arbitrage-free structure, $\sigma_{Pk}(t,T)$ must be specified by the parameters within the CLMM structure. By the recurrence relationship of equation (7), $\sigma_{Pk}(t,T)$ with $T\in[0,\tau]$ can be represented as follows:

$$\sigma_{Pk}(t,T)=\begin{cases}\displaystyle\sum_{j=1}^{\lfloor\delta^{-1}(T-t)\rfloor}\frac{\delta F_k(t,T-j\delta)}{1+\delta F_k(t,T-j\delta)}\gamma_{Fk}(t,T-j\delta) & t\in[0,T-\delta]\,\&\,T-\delta>0\\ 0 & \text{otherwise}\end{cases} \tag{8}$$

where $\lfloor\delta^{-1}(T-t)\rfloor$ denotes the greatest integer that is less than $\delta^{-1}(T-t)$.

Finally, we make use of the drift-volatility relationships of the exchange rate processes in Proposition 1 to determine the drift functions $\mu_X(t)$ in (3) under the measure Q . Because of the same volatility structure in Proposition 1 and Assumption 2, we have

$$\mu_X(t) = r_d(t) - r_f(t)$$

We conclude the above results in the following Proposition.

Proposition 2. The CLMM under the Martingale Measure

Under the domestic spot martingale measure, the processes of the forward LIBOR rates and the exchange rate are given as follows:

$$\frac{dF_f(t,T)}{F_d(t,T)} = \gamma_{Fd}(t,T)\cdot\sigma_{Pd}(t,T+\delta)dt + \gamma_{Fd}(t,T)\cdot dW(t) \tag{9}$$

$$\frac{dF_f(t,T)}{F_f(t,T)} = \gamma_{Ff}(t,T)\cdot\left(\sigma_{Pf}(t,T+\delta) - \sigma_X(t)\right)dt + \gamma_{Ff}(t,T)\cdot dW(t) \tag{10}$$

$$\frac{dX(t)}{X(t)} = \left(r_d(t) - r_f(t)\right)dt + \sigma_X(t)\cdot dW(t) \tag{11}$$

where $t \in [0,T], T \in [0,T\ ]$ *and* $\sigma_{Pk}(t,T)$, $k \in \{d,f\}$ *is defined in (8).*

Based on the results in AJ (1991) and BGM (1997), we have extend the LMM to the cross-currency LMM. Unlike the abstract short rates in the HW model, the forward LIBOR rates in the CLMM are market-observable. Moreover, the cap pricing formula in the CLMM framework is the Black's formula which is consistent with market practice and make the calibration procedure easier. Therefore, the volatility $\gamma_{Fk}(t,T)$, $T \in [0,T\ ]$ and $k \in \{d,f\}$, can be inverted from the interest rate derivatives traded in the market and $\sigma_{Pk}(t,T)$, $T \in [0,T\ ]$, $k \in \{d,f\}$, can be calculated from equation (8).

According to the definition of the bond volatility process (8), $\{\sigma_{Pk}(t,T+\delta)\}_{t\in[0,T+\delta]}$ is stochastic rather than deterministic. Thus, the stochastic differential equation (9) and (10) are not solvable and the distribution of $F_k(T,T)$ is unknown. However, fixed at initial time 0, we can approximate $\sigma_{Pk}(t,T)$ by $\overline{\sigma}^{0}_{Pk}(t,T)$ which is defined by

$$\overline{\sigma}^{0}_{Pk}(t,T) = \begin{cases} \sum_{j=1}^{\lfloor \delta^{-1}(T-t) \rfloor} \frac{\delta F_k(0,T-j\delta)}{1+\delta F_k(0,T-j\delta)} \gamma_{Fk}(t,T-j\delta) & t \in [0,T-\delta] \& T-\delta > 0 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

where $0 < t < T < T$. It means that the calendar time of the process $\{F_k(t,T-j\delta)\}_{t\in[0,T-j\delta]}$ in (8) is frozen at its initial time 0 and thus the process $\{\overline{\sigma}^{0}_{Pk}(t,T+\delta)\}_{t\in[0,T]}$ becomes deterministic. By substituting $\overline{\sigma}^{0}_{Pk}(t,T+\delta)$ for $\sigma_{Pk}(t,T+\delta)$ into the drift terms of (9) and (10), the drift and the volatility terms become deterministic, so we can solve (9) and (10) and find the approximate distribution of $F_k(T,T)$ to be lognormally distributed.

This argument is the Wiener chaos order 0 approximation which is first used by BGM (1997) for pricing interest rate swaptions. It was developed further in Brace, Dun and Barton (1998) and formalized by Brace and Womersley (2000). The approximation also appeared in Schlögl (2002).

Proposition 3. The Approximate LIBOR Rate Dynamics under the Martingale Measure Q

The approximate dynamics of the LIBOR rate $F_k(t,T)$ *under the measure* Q *is given as follows:*

$$\frac{dF_d(t,T)}{F_d(t,T)} = \gamma_{Fd}(t,T) \cdot \overline{\sigma}^{0}_{Pd}(t,T+\delta)dt + \gamma_{Fd}(t,T) \cdot dW(t) \tag{13}$$

$$\frac{dF_f(t,T)}{F_f(t,T)} = \gamma_{Ff}(t,T) \cdot \left(\overline{\sigma}^{0}_{Pf}(t,T+\delta) - \sigma_X(t)\right)dt + \gamma_{Ff}(t,T) \cdot dW(t) \tag{14}$$

where $0 < t < T < T$.

The following proposition specifies the general rule under which the LIBOR rate dynamics changes following the change of the underlying measure. This rule is useful for deriving the pricing formulas of interest derivatives.

Proposition 4. The Drift Adjustment Technique in Different Measure

The dynamics of the forward LIBOR rate $F_k(t,T)$ under an arbitrary domestic forward martingale measure Q_S, , where $S > T$, is given as follows:

$$\frac{dF_d(t,T)}{F_d(t,T)} = \gamma_{Fd}(t,T)\cdot\left(\overline{\sigma}^0_{Pd}(t,T+\delta) - \overline{\sigma}^0_{Pd}(t,S)\right)dt + \gamma_{Fd}(t,T)\cdot dW(t)$$

$$\frac{dF_f(t,T)}{F_f(t,T)} = \gamma_{Ff}(t,T)\cdot\left(\overline{\sigma}^0_{Pf}(t,T+\delta) - \overline{\sigma}^0_{Pd}(t,S) - \sigma_X(t)\right)dt + \gamma_{Ff}(t,T)\cdot dW(t)$$ where

$0 \le t \le T$. [5]

With the CLMM and the approximate technique, we can employ them to price quanto interest rate derivatives. In this article, we price QAIROs as an example which is given in the following section.

3. Pricing Quanto Average Interest Rate Option

Consider a generalized QAIRO with initiation at time 0 $(=T_0)$ and expiration at time $T\,(=T_{n-1})$. The underlying average rate of the QAIRO is dependent on n different interest rates observed at $\{T_1, T_2, \ldots, T_n\}$,where $T_i < T_{i+1}$ for i =0,1,2,..., n . For simplicity, we assume $\delta = T_{i+1} - T_i$ for i=0,1,2,..., n and let K denote the exercise rate. The final payoff of the QAIRO at time T is given as follows:

$$\text{Max}\left(A(T) - K, 0\right), \tag{15}$$

$$A(T) = \frac{1}{n}\sum_{i=1}^{n} F_f(T_i, T_i)\cdot \tag{16}$$

As compared with Longstaff (1995), the final payoffs in (15) is dependent on the discrete average of the market-observable foreign LIBOR rates rather than the continuous average of the abstract short rates and thus more consistent with practical implementation. Moreover, in comparison with Cheuk and Vorst (1999) the foreign LIBOR rates involved in the average are directly-modeled rather than transformed from the continuous-time short rate, thereby avoiding a complicated transformation calculation.

Based on the risk-neutral pricing method, the pricing formula of QAIRO can be obtained by calculating the following expectation under the domestic forward martingale measure:

$$P_d(0,T)E^{Q_T}\left[\,\mathrm{Max}\left(A(T)-K,0\right)\right] \tag{17}$$

It is well-known that the distribution of the arithmetic average of lognormally distributed random variables, A(T), is unknown. As such, if the conventional assumptions of geometric diffusion processes are specified for the underlying price processes, analytic pricing formulas fail to exist. In this section, two different approximation approaches are presented to price the QAIRO, i.e. Vorst (1992) and Levy (1992). Vorst (1992) and Levy (1992) have presented, respectively, two approximate methodologies to price analytically the average exchange rate options. We will adopt their approximate methodologies to price the QAIRO.

3.1 Pricing QAIROs

We first employ the approximation presented by Vorst (1992) to derive the closed-form formula of the QAIRO and then by Levy (1992).

3.1.1 The Vorst Approach

Equation (17) cannot be derived analytically as a closed-form solution since the distribution of the arithmetic average of lognormally distributed variables is unknown. Vorst (1992) approximates $A(T)$ by the geometric average of the variables adjusted by the difference between the expectations of the arithmetic and geometric averages, i.e.

$$A(T) \approx G(T) + E^{Q_T}[A(T)] - E^{Q_T}[G(T)], \tag{18}$$

Where

$$G(T) = (\prod_{i=1}^{n} F_f(T_i, T_i))^{1/n} \cdot$$

Thus, equation (17) can be rewritten approximately as follows:

$$P_d(0,T) E^{Q_T} [\text{Max}(G(T) - K^*, 0)], \tag{19}$$

where

$$K^* = K - E^{Q_T}[A(T)] + E^{Q_T}[G(T)]. \tag{20}$$

Since $E^{Q_T}[A(T)]$ and $E^{Q_T}[G(T)]]$ are solvable and the geometric average of lognormally-distributed variables, $G(T)$, is still lognormally distributed, the expectation in equation (19) can be derived as an approximate pricing formula for the QAIRO. The result is presented in the following theorem and the proof is provided in Appendix A.

Theorem 1. The Pricng Formula of the QAIRO via the Vorst Approach

Via the Vorst approach, the price of the QAIRO at its initial time 0 is given as follows:

$$P_d(0,T)\left[G(0) e^{\eta + \frac{1}{2}\xi^2} N\left(\frac{\ln(\frac{G(0)}{K^*}) + \eta + \xi^2}{\xi}\right) - K^* N\left(\frac{\ln(\frac{G(0)}{K^*}) + \eta}{\xi}\right) \right], \tag{21}$$

where

$$G(0) = (\prod_{i=1}^{n} F_f(0,T_i))^{1/n} \quad , \tag{22}$$

$$\eta = \frac{1}{n}\sum_{i=1}^{n} \int_0^{T_i} (\gamma_f(u,T_i)\cdot\Phi_0(u;T_{i+1},T) - \frac{1}{2}\|\gamma_f(u,T_i)\|^2)du \, , \tag{23}$$

$$K^* = K - \frac{1}{n}\sum_{i=1}^{n} F_f(0,T_i)\exp(\int_0^{T_i} \gamma_f(u,T_i)\cdot\Phi_0(u;T_{i+1},T)du) + G(0)e^{\eta+\frac{1}{2}\xi^2} \tag{24}$$

$$\xi^2 = \frac{1}{n^2}(\sum_{i=1}^{n}\int_0^{T_i}\|\gamma_f(u,T_i)\|^2 du + 2\sum_{i=2}^{n}\sum_{j=i}^{n}\int_0^{T_{i-1}}\gamma_f(u,T_{i-1})\cdot\gamma_f(u,T_j)du) \, , \tag{25}$$

$$\Phi_0(u;T_{i+1},T) = (\overline{\sigma}^0_{Pf}(t,T_{i+1}) - \sigma_X(t) - \overline{\sigma}^0_{Pf}(t,T)). \tag{26}$$

3.1.2 The Levy Approach

In order to obtain a Black (1976)-type pricing formula, the unknown distribution of $A(T)$ may still be assumed to be lognormal distribution with the correct first two moments, namely,

$$\ln A(T) \sim N(\phi,\psi^2)$$

where

$$\phi = 2\ln E^{Q_T}[A(T)] - \frac{1}{2}\ln E^{Q_T}[A(T)^2] \, , \tag{27}$$

$$\psi^2 = \ln E^{Q_T}[A(T)^2] - 2\ln E^{Q_T}[A(T)] \, . \tag{28}$$

Actually, this is the Wilkinson approximation which replaces the unknown distribution of the arithmetic average of lognormal random variables by a lognormal distribution with the correct first two moments. We adopt his technique to derive the pricing formula of the QAIRO. The result is given in the following theorem and the proof is provided in Appendix B.

Theorem 2. The Pricing Formula of the QAIRO via the Levy Approach

Via the Levy approach, the price of the QAIRO at its initial time 0 is given as follows:

$$P_d(0,T)(e^{\phi+\frac{1}{2}\psi^2}N(\frac{\phi-\ln K+\psi^2}{\psi})-KN(\frac{\phi+\ln K}{\psi})), \tag{29}$$

where ϕ and ψ^2 are defined respectively in (27) and (28), and

$$E^{Q_T}[A(T)]=\frac{1}{n}\sum_{i=1}^{n}F(0,T_i)\exp(\int_0^{T_i}\gamma(u,T_i)\cdot\Phi_0(u;T_{i+1},T_{n+1})du),$$

$$E^{Q_T}[A(T)^2]=\frac{1}{n^2}(\sum_{i=1}^{n}E^{Q_T}[F(T_i,T_i)^2]+2\sum_{i=2}^{n}\sum_{j=i}^{n}E^{Q_T}[F(T_{i-1},T_{i-1})F(T_j,T_j)]),$$

$$E^{Q_T}[F(T_i,T_i)^2]=F(0,T_i)^2\exp(\int_0^{T_i}(2\gamma(u,T_i)\cdot\Phi_0(u;T_{i+1},T_{n+1})+\|\gamma(u,T_i)\|^2)du),$$

$$E^{Q_T}[F(T_{i-1},T_{i-1})F(T_j,T_j)]=F(0,T_{i-1})F(0,T_j)\exp(\int_0^{T_i}\gamma(u,T_i)\cdot\Phi_0(u;T_i,T_{n+1})du$$

$$+\int_0^{T_j}\gamma(u,T_j)\cdot\Phi_0(u;T_{j+1},T_{n+1})du+\int_0^{T_{i-1}}\gamma(u,T_{i-1})\cdot\gamma(u,T_j)du),$$

$$F(T_i,T_i)=F(0,T_i)\exp(\int_0^{T_i}(\gamma(u,T_i)\cdot\Phi_0(u;T_{i+1},T_{n+1})-\frac{1}{2}\|\gamma(u,T_i)\|^2)du$$

$$+\int_0^{T_i}\gamma(u,T_i)\cdot dW(u)).$$

Unlike the pricing formulas of AIROs in Longstaff (1995) and Cheuk and Vorst (1999), the pricing formulas of QAIROs in (21) and (29) are more feasible in practice due to the following reasons. First, the pricing formulas (21) and (29) bear resemblance to the Black (1976) model in the environment of stochastic interest rates. Second, the interest rates in our formulas are LIBOR rates which are market-observable and the parameters in our formulas can be calibrated easily from market data. Unlike the Gaussian term structure models, such as Vasicek (1977) and HW (1990), the CLMM (or LMM) has another advantage that avoids the pricing error arising from negative rates with positive probability. Rogers (1996) indicated that the Gaussian term structure models may cause some pricing error in many cases. However, since the CLMM (or LMM) has a lognormal volatility structure and the underlying LIBOR rates are positive, this pricing error can be avoided.

4. Calibration Procedure and Numerical Analysis

In this section, we first provide the calibration procedure and then examine the accuracy of the approximate formula via Monte Carlo simulation.

4.1 Calibration Procedure

Consider that there are n domestic forward LIBOR rates and n foreign forward LIBOR rates in an m-factor framework. The steps to calibrate the parameters are given as follows. Firstly, as given in Brigo and Mercurio (2001), we assume that the domestic forward LIBOR rate, $F_d(t,\cdot)$, has a piecewise-constant instantaneous total volatility structure depending only on the time-to-maturity. The elements in Table 1 which specify the instantaneous total volatilities applied to each period for each rate can be calculated from the market data. A detailed computational process is also presented in Hull (2003).The case of the foreign forward LIBOR rate, $F_f(t,\cdot)$, can be carried out in a way similar to the domestic case. In addition, we also assume that the spot exchange rate has a piecewise-constant instantaneous total volatility structure. The elements in Table 2 can be calculated from the on-the-run options' prices in the market. However, since the duration of stock options is usually shorter than one year, the market-obtainable elements in Table 2 are usually not sufficient for pricing equity swaps. This problem may be resolved by using the implied (or historical) volatility of the underlying stock index, while assuming that the term structure of the volatility is flat, i.e. $\zeta(t)=\zeta$ for $t\in(t_0,t_n]$.

Table 1: Instantaneous Volatilities of $\{F_k(t,\cdot)\}_{k\in\{d,f\}}$

Instant. Total Vol.	Time t $\in(t_0,t_1]$	$(t_1,t_2]$	$(t_2,t_3]$	...	$(t_{n-2},t_{n-1}]$
Fwd Rate: $F_k(t,t_1)$	$v^k_{1,1}$	Dead	Dead	...	Dead
$F_k(t,t_2)$	$v^k_{2,1}$	$v^k_{2,2}$	Dead	...	Dead
⋮	...	...	...	...	...
$F_k(t,t_{n-1})$	$v^k_{n-1,1}$	$v^k_{n-1,2}$	$v^k_{n-1,3}$	...	$v^k_{n-1,n-1}$

Table 2: Instantaneous Volatilities of the Spot Exchange Rate

Instant. Total Vol.	Time t $\in(t_0,t_1]$	$(t_1,t_2]$	$(t_2,t_3]$	...	$(t_{n-2},t_{n-1}]$
Fwd Rate: $X(t)$	ζ_1	ζ_2	ζ_3	...	ζ_n

Secondly, let us use the historical data of the domestic and foreign forward LIBOR rates and the spot exchange rate to derive a full-rank (2n+1) × (2n+1) instantaneous-correlation matrix Γ. Thus, Γ is a positive-definite and symmetric matrix that can be written as:

$$\Gamma = H\Lambda H',$$

where H is a real orthogonal matrix and Λ is a diagonal matrix. Let $A\equiv H\Lambda^{1/2}$ and thus $AA'=\Gamma$. In this way, we can find a suitable m-rank ($m << 2n+1$) matrix **B** such that $\Gamma^B = BB'$ is a m-rank correlation matrix and can be used to mimic the market correlation matrix Γ.

The advantage of doing so is that we may replace the 2n+1-dimensional original Brownian motions $dW(t)$ with $BdZ(t)$, where $dZ(t)$ represents mdimensional Brownian motions. In other words, we change the market instantaneous correlation structure

$$dW(t)dW(t)' = \Gamma dt$$

to a modeled instantaneous correlation structure

$$BdZ(t)(BdW(t))' = BdZ(t)dZ(t)'B' = BB'dt = \Gamma^B dt .$$

The remaining problem is how to choose a suitable matrix B .Rebonato(1999) proposed the following form for the i-th row of B:

$$b_{i,k} = \begin{cases} \cos\theta_{i,k} \prod_{j=1}^{k-1} \sin\theta_{i,j} & \text{If } k = 1,2,\ldots, m\text{-}1, \\ \prod_{j=1}^{k-1} \sin\theta_{i,j} & \text{If } k = m, \end{cases}$$

for i =1,2,..., 2n+1. By finding $\hat{\theta}$ that solves the following optimization problem

$$\min_{\theta} \sum_{i,j=1}^{n} \left|\Gamma_{i,j}^{B} - \Gamma_{i,j}\right|^2,$$

and substituting $\hat{\theta}$ into B, we obtain a suitable matrix $\hat{B}$ such that $\Gamma^B (= \hat{B}\hat{B}')$ is an approximate correlation matrix for Γ.

Thirdly, $\hat{B}$ can be used to distribute the instantaneous total volatility applied to each period for the spot exchange rate or for each rate to each Brownian motions without changing the amount of the instantaneous total volatility. This means that,

$$v_{i,j}^{k}(\hat{B}(i,1), \hat{B}(i,2), \ldots, \hat{B}(i,m)) = (\gamma_{Fk1}(t,t_i), \gamma_{Fk2}(t,t_i), \ldots, \gamma_{Fkm}(t,t_i)),$$

$$\zeta_j(\hat{B}(n,1),\hat{B}(n,2),...,\hat{B}(n,m)) = (\sigma_{X1}(t),\sigma_{X2}(t),...,\sigma_{Xm}(t)),$$

where $i=1,2,...,n-1$ and $t \in (t_{j-1},t_j]$, for each $j=1,2,...,n$.

Under the assumption that the instantaneous total volatility structures are piecewise-constant, the above procedure represents a general calibration method without a constraint on choosing the number of factors. Via the distributing matrix $\hat{B}$, the individual instantaneous volatility applied to each Brownian motion at each period for each process can be derived. With these parameter data calibrated from the market correlation matrix and volatilities, we can employ Monte Carlo simulation to price any associated interest rate derivatives. Moreover, the parameter data can also be used to calculate the prices of the QAIRO derived in Theorems 1 and 2.

4.2 Numerical Analysis

In this subsection we offer a practical implementation example and compare these two approximation approaches with Monte Carlo simulation using the recent two-year market data. Based on the actual market data as shown in Table 5,6,...,10 in Appendix D, the 10-year QAIROs are priced at different quarterly dates and the results are listed in Table 3 and 4. By comparing with Monte Carlo simulation, the approximation formulas via the Vorst and the Levy approach are shown to be accurate and robust for the recent two-year market data.

Table 3: The 10-yr QAIRO

K	VORST	LEVY	MC	s.e
3/31/06				
0.03	0.0097	0.0097	0.0096	5.56×10^{-5}
0.05	0.0018	0.0018	0.0017	3.35×10^{-5}
0.07	1.42×10^{-4}	1.38×10^{-4}	1.67×10^{-4}	1.09×10^{-5}
12/30/05				
0.03	0.0083	0.0083	0.0082	6.10×10^{-5}
0.05	0.0014	0.0015	0.0014	3.23×10^{-5}
0.07	1.39×10^{-4}	1.43×10^{-4}	1.44×10^{-4}	1.01×10^{-5}
9/30/05				
0.03	0.0095	0.0095	0.0095	6.46×10^{-5}
0.05	0.0018	0.0019	0.0018	3.80×10^{-5}
0.07	1.86×10^{-4}	2.04×10^{-4}	2.06×10^{-4}	1.35×10^{-5}

* The prices of the 10-year QAIRO are presented in this table. Here the domestic country is the U.S. and the foreign country is U.K. They are priced via the Vorst approach, the Levy approach and Monte Carlo simulation at different quarterly dates over the past two years. The market data used are listed in Table 5,6,...10 in Appendix D. The flat volatility of the exchange rate is assumed to be 15%. The notional value is assumed to be $1. The simulation is based on 10000 paths.

Table 4: The 10-yr QAIRO

K	VORST	LEVY	MC	s.e
6/30/05				
0.03	0.0091	0.0091	0.0091	6.97×10^{-5}
0.05	0.0017	0.0018	0.0018	3.93×10^{-5}
0.07	1.95×10^{-4}	2.20×10^{-4}	2.31×10^{-4}	1.61×10^{-5}
3/31/05				
0.03	0.0116	0.0116	0.0116	6.28×10^{-5}
0.05	0.0027	0.0027	0.0026	4.27×10^{-5}
0.07	2.86×10^{-4}	2.94×10^{-4}	2.97×10^{-4}	1.51×10^{-5}
12/31/04				
0.03	0.0108	0.0109	0.0109	7.22×10^{-5}
0.05	0.0026	0.0026	0.0026	4.65×10^{-5}
0.07	3.69×10^{-4}	3.93×10^{-4}	3.98×10^{-4}	1.91×10^{-5}
9/30/04				
0.03	0.0130	0.0130	0.0130	7.60×10^{-4}
0.05	0.0037	0.0038	0.0037	5.72×10^{-5}
0.07	6.15×10^{-4}	6.53×10^{-4}	6.59×10^{-4}	2.70×10^{-5}
6/30/04				
0.03	0.0139	0.0140	0.0139	8.39×10^{-5}
0.05	0.0047	0.0048	0.0047	6.56×10^{-5}
0.07	0.0011	0.0011	0.0012	3.64×10^{-5}

* This table continues from Table 3.

5. Conclusions

We have extended the single-currency LMM to the cross-currency LMM based on the Amin and Jarrow (1991) structure, and then applied the resulting model to derive the approximate pricing formula of QAIROs via two different approximation approaches, presented by Vorst (1992) and Levy (1992). These two approximation formulas have been examined to be very accurate as compared with Monte Carlo simulation. The advantage of CLMM over the precedent pricing models is its ease to the model calibration and the calibration procedure is discussed in detail in Section 4. Therefore, the resulting model is more tractable and feasible in practice and thus worth recommending for market implementation.

Notes

[1] The fact that hedging by QAIROs is cheaper than by quanto caps is proved in Appendix A.

[2] As examined in Rogers (1996), the Gaussian term structure model has an important theoretical limitation: the rate can attain negative values with positive probability, and thereby leading to some pricing error in many cases.

[3] The filtration $\{F_t\}_{t\in[0,T]}$ is right continuous and F_0 contains all the Q-null sets of F.

[4] See AJ(1991) for more details about the regularity conditions.

[5] We employ $W(t)$ to denote an independent m-dimensional standard Brownian motion under an arbitrary measure if not causing any confusion.

Appendix A: Hedging by QAIROs Is Cheaper than by Quanto Caps.

In Appendix A, we will show that hedging foreign interest rate risk with an QAIRO is cheaper than with the corresponding quanto cap. Assume that, at time 0, we want to hedge the interest rate risks based on the amount $A(T_i)$ at time T_i, for i = 1, 2, ..., n. Consider two different hedging approaches. Firstly, we hedge the overall foreign interest rate risk by employing an QAIRO with the nominal amount $A=\sum_{i=1}^{n}A(T_i)$. Secondly, we hedge the individual foreign interest rate risks involved with the nominal amount $A(T_i)$ using the corresponding caplet. To simplify the following analysis, we assume that $A(T_i)=1$ for $i=1,2,\ldots,n$. Applying Jensen's inequality, we have the following result.

$$\text{QAIRO} = nP_d(0,T)E^{Q_T}\left[\left\{\frac{1}{n}\sum_{i=1}^{n}F_f(T_i,T_i)-K\right\}^+\right]$$

$$= nP_d(0,T)E^{Q_T}\left[\left\{\frac{n-1}{n}\left(\frac{\sum_{i=1}^{n-1}F_f(T_i,T_i)}{n-1}-K\right)+\frac{1}{n}(F_f(T_n,T_n)-K)\right\}^+\right]$$

$$\leq (n-1)P_d(0,T)E^{Q_T}\left[\left\{\frac{\sum_{i=1}^{n-1}F_f(T_i,T_i)}{n-1}-K\right\}^+\right]$$

$$+P_d(0,T)E^{Q_T}\left[\left\{F_f(T_n,T_n)-K\right\}^+\right]$$

......

$$=\sum_{i=1}^{n}P_d(0,T_{i+1})E^{Q_T}\left[\frac{P_d(T_{i+1},T_{i+1})\big/P_d(0,T_{i+1})}{P_d(T_{i+1},T)\big/P_d(0,T)}P_d(T_{i+1},T)\left\{F_f(T_i,T_i)-K\right\}^+\right]$$

$$\leq\sum_{i=1}^{n}P_d(0,T_{i+1})E^{Q_{T_{i+1}}}\left[P_d(T_{i+1},T)\left\{F_f(T_i,T_i)-K\right\}^+\right]$$

$$\leq\sum_{i=1}^{n}P_d(0,T_{i+1})E^{Q_{T_{i+1}}}\left[\left\{F_f(T_i,T_i)-K\right\}^+\right],\qquad P_d(T_{i+1},T)<1$$

=Cap,

which shows that hedging foreign interest rate risks with an QAIRO is cheaper than with the corresponding Quanto cap.

Besides the above mathematical proof, we also present some economic intuitions for this fact. Since the averaging process causes the randomness to be diversified away, an averaged dynamics is typically less volatile than the original underlying dynamics.

Moreover, as indicated in Longstaff (1995), the mean-reverting behavior of interest rates leads its volatility per unit time to converge to zero as the length of the horizon increases. This feature makes the fact more obvious for options on the average interest rate than for options on the average price of a traded asset. The rates modeled in the LMM model are shown to be equipped with mean reverting behavior by BGM (1997) and thus the LMM model reflects the practice.

Appendix B: The Vorst Approach for the QAIROs

By applying Proposition 4, under the measure $Q_T(=Q_{T_{n+1}})$, the dynamics of $F_f(t,T_i)$, $i = 1, 2, ..., n$, is given as follows:

$$\frac{dF_f(t,T_i)}{F_f(t,T_i)} = \gamma_f(t,T_i)\cdot\left(\bar{\sigma}^0_{Pf}(t,T_{i+1}) - \sigma_X(t) - \bar{\sigma}^0_{Pd}(t,T)\right)dt + \gamma_f(t,T_i)\cdot dW(t),$$

$$= \gamma_f(t,T_i)\cdot\Phi_0(t;T_{i+1},T)dt + \gamma_f(t,T_i)\cdot dW(t), \qquad \text{(B.1)}$$

Where

$$\Phi_0(t;T_{i+1},T)dt = \bar{\sigma}^0_{Pf}(t,T_{i+1}) - \sigma_X(t) - \bar{\sigma}^0_{Pd}(t,T). \qquad \text{(B.2)}$$

The pricing formula of the QAIRO at its initial time 0 is derived under the forward measure Q_T as follows.

$$C^V = P_d(0,T)E^{Q_T}\left[(A(T)-K)^+\right] \qquad \text{(B.3)}$$

$$\approx P_d(0,T)E^{Q_T}\left[(G(T)-K^*)^+\right],$$

where

$$K^{*} = K - E^{Q_T}\left[A(T)\right] + E^{Q_T}\left[G(T)\right]. \tag{B.4}$$

$$\begin{aligned} G(T) &= \left(\prod_{i=1}^{n} F_f(T_i, T_i)\right)^{1/n} \\ &= G(0)\exp(\eta + Z) \end{aligned} \tag{B.5}$$

$$G(0) = \left(\prod_{i=1}^{n} F_f(0, T_i)\right)^{1/n},$$

$$\eta = \frac{1}{n}\sum_{i=1}^{n}\int_0^{T_i}\left(\gamma_f(u,T_i)\cdot\Phi_0(u;T_{i+1},T) - \frac{1}{2}\left\|\gamma_f(u,T_i)\right\|^2\right)du,$$

$$Z = \frac{1}{n}\sum_{i=1}^{n}\int_0^{T_i}\gamma_f(u,T_i)\cdot dW(u).$$

Based on Itô Isometry in the stochastic-calculus theory, the distribution of Z is $N(0,\xi^2)$, where

$$\xi^2 = \frac{1}{n^2}\left(\sum_{i=1}^{n}\int_0^{T_i}\left\|\gamma_f(u,T_i)\right\|^2 + 2\sum_{i=2}^{n}\sum_{j=i}^{n}\int_0^{T_{i-1}}\gamma_f(u,T_{i-1})\cdot\gamma_f(u,T_j)du\right).$$

We present a useful lemma for the following derivation process.

Lemma 1. *If* $X \sim N(0,V^2)$,, *then the expectation of* $E\left[(\alpha e^{X} - \beta)^{+}\right]$ *is given as follows:*

$$\alpha e^{\frac{1}{2}V^2} N\left(\frac{\ln\left(\frac{\alpha}{\beta}\right) + V^2}{V}\right) - \beta N\left(\frac{\ln\left(\frac{\alpha}{\beta}\right)}{V}\right),$$

where α *and* β *are constants.*

According to Lemma 1, equation (B.3) can be derived as follows:

$$P_d(0,T)\left[G(0)e^{\eta+\frac{1}{2}\xi^2}N\left(\frac{\ln\left(\frac{G(0)}{K^*}\right)+\eta+\xi^2}{\xi}\right)-K^*N\left(\frac{\ln\left(\frac{G(0)}{K^*}\right)+\eta}{\xi}\right)\right].$$

The remaining task is to derive $E^{Q_T}[A(T)]$ and $E^{Q_T}[G(T)]$ in (B.4). According to (B.5), it is trivial that

$$E^{Q_T}[G(T)]=G(0)e^{\eta+\frac{1}{2}\xi^2}.$$

According to (16) and (B.1), we have

$$E^{Q_T}[A(T)]=\frac{1}{n}\sum_{i=1}^{n}E^{Q_T}[F_f(T_i,T_i)]$$

$$=\frac{1}{n}\sum_{i=1}^{n}F_f(T_i,T_i)\exp\left(\int_0^{T_i}\gamma_f(u,T_i)\cdot\Phi_0(u;T_{i+1},T)du\right). \quad \text{(B.6)}$$

Appendix C: The Levy Approach for the QAIROs

According to Levy (1992), the distribution of $\ln A(T)$ is approximately $N(\phi,\psi^2)$ where ϕ and ψ^2 are given in (27) and (28).

By Lemma 1, the pricing formula of the QAIRO via the Levy approach is given as follows:

$$C^L=P_d(0,T)\left[e^{\phi+\frac{1}{2}\psi^2}N\left(\frac{\phi-\ln K+\psi^2}{\psi}\right)-KN\left(\frac{\phi-\ln K}{\psi}\right)\right]. \quad \text{(C.1)}$$

The remaining task is to compute $E^{Q_T}[A(T)]$ and $E^{Q_T}[A(T)^2]$. We have already derived $E^{Q_T}[A(T)]$ in (B.6). $E^{Q_T}[A(T)^2]$ is derived as follows:

$$E^{Q_T}[A(T)^2] = \frac{1}{n^2} E^{Q_T}\left[\sum_{i=1}^{n} F_f(T_i,T_i)\right]^2$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{n} E^{Q_T}[F_f(T_i,T_i)^2] + 2\sum_{i=2}^{n}\sum_{j=i}^{n} E^{Q_T}[F_f(T_{i-1},T_{i-1})F_f(T_j,T_j)]\right),$$

where

$$E^{Q_T}[F_f(T_i,T_i)^2] = F_f(0,T_i)^2 \exp\left(\int_0^{T_i}\left(2\gamma_f(u,T_i)\cdot\Phi_0(u;T_{i+1},T) + \left\|\gamma_f(u,T_i)\right\|^2\right)du\right)$$

and

$$E^{Q_T}[F_f(T_{i-1},T_{i-1})F_f(T_j,T_j)] =$$

$$F_f(0,T_{i-1})F_f(0,T_j)\exp\left(\int_0^{T_{i-1}} \gamma_f(u,T_{i-1})\cdot\Phi_0(u;T_i,T)du\right.$$

$$\left. + \int_0^{T_j} \gamma_f(u,T_j)\cdot\Phi_0(u;T_{j+1},T)du + \int_0^{T_{i-1}} \gamma_f(u,T_{i-1})\cdot\gamma_f(u,T_j)du\right).$$

Appendix D: The Market Data

Table 5, 6, 7, 8 are drawn from the DataStream database which are used for the numerical example in Section 4.

Table 5: The Domestic Cap Volatilities Quoted in the U.S. Market

Year	3/31/06	12/30/05	9/30/05	6/30/05	3/31/05	12/31/04	9/30/04	6/30/04
1	9.93	11.30	13.93	14.39	13.92	19.38	25.53	35.01
2	13.60	15.62	17.31	19.18	18.31	24.56	31.10	31.70
3	15.21	17.81	19.13	21.35	19.81	26.44	31.86	28.84
4	16.06	18.96	19.89	22.33	20.23	26.53	31.10	26.97
5	16.56	19.48	20.31	22.72	20.24	26.08	29.79	25.40
7	17.01	19.97	20.46	22.60	19.82	24.81	27.58	22.92
10	16.98	19.80	19.85	21.58	18.76	22.19	24.37	20.21

* The quoted volatilities of the caps in the U.S. market over the past two years are quarterly presented in this table. The data for year 6, 8, and 9 can be obtained by an interpolation technique.

Table 6: The Foreign Cap Volatilities Quoted in the U.K. Market

Year	3/31/06	12/30/05	9/30/05	6/30/05	3/31/05	12/31/04	9/30/04	6/30/04
1	7.49	11.58	10.65	12.11	8.47	11.10	10.06	12.42
2	10.28	14.06	13.54	14.75	11.04	14.22	13.40	15.77
3	11.71	14.75	14.59	15.68	12.37	15.23	14.73	16.86
4	12.50	15.13	14.96	16.04	13.02	15.71	15.41	17.08
5	13.01	15.24	15.04	16.08	13.45	15.92	15.70	17.05
7	13.65	15.27	14.90	15.82	13.92	15.95	15.72	16.82
10	14.09	15.13	14.62	15.42	14.03	15.62	15.40	16.32

* The quoted volatilities of the caps in the U.K. market over the past two years are quarterly presented in this table. The data for year 6, 8, and 9 can be obtained by an interpolation technique.

Table 7: Initial Domestic Forward LIBOR Rates

Year	3/31/06	12/30/05	9/30/05	6/30/05	3/31/05	12/31/04	9/30/04	6/30/04
0	5.478	4.976	4.562	3.970	3.917	3.162	2.505	2.491
1	5.456	5.075	4.899	4.207	4.804	3.946	3.602	4.088
2	5.449	5.007	4.809	4.317	5.017	4.269	4.196	4.905
3	5.532	5.064	4.868	4.408	5.155	4.631	4.644	5.427
4	5.627	5.118	4.926	4.536	5.288	4.989	4.999	5.805
5	5.651	5.118	5.022	4.630	5.387	5.269	5.308	6.109
6	5.658	5.143	5.053	4.710	5.544	5.469	5.581	6.311
7	5.698	5.168	5.117	4.874	5.587	5.730	5.764	6.417
8	5.738	5.313	5.242	4.946	5.689	5.866	5.945	6.580
9	5.851	5.300	5.325	5.069	5.843	6.072	6.096	6.708
10	5.831	5.382	5.410	5.118	5.772	6.092	6.188	6.745

* The domestic forward LIBOR rates in the U.S. market over the past two years are quarterly represented in this table. The rates are obtained from the associated bond prices derived from the zero curves obtained in DataStream.

Table 8: Initial Foreign Forward LIBOR Rates

Year	3/31/06	12/30/05	9/30/05	6/30/05	3/31/05	12/31/04	9/30/04	6/30/04
0	4.877	4.684	4.645	4.586	5.234	5.020	5.193	5.434
1	5.111	4.669	4.623	4.449	5.238	5.005	5.321	5.624
2	5.101	4.745	4.710	4.527	5.210	5.038	5.358	5.767
3	5.014	4.687	4.716	4.580	5.005	5.055	5.403	5.813
4	4.934	4.667	4.790	4.652	5.193	5.073	5.419	5.787
5	4.916	4.600	4.856	4.719	5.198	5.073	5.463	5.749
6	4.841	4.548	4.869	4.752	5.192	5.042	5.463	5.707
7	4.771	4.511	4.837	4.728	5.160	5.041	5.470	5.688
8	4.689	4.443	4.834	4.732	5.117	5.042	5.436	5.657
9	4.597	4.397	4.788	4.725	5.096	5.008	5.417	5.580
10	4.545	4.385	4.756	4.682	5.075	4.996	5.306	5.499

* The foreign forward LIBOR rates in the U.K. market over the past two years are quarterly represented in this table. The rates are obtained from the associated bond prices derived from the zero curves obtained in DataStream.

Table 9: The Correlation Matrix

	D1	D2	D3	D4	D5	D6	D7	D8	D9
D1	1	0.9821	0.9637	0.8911	0.8365	0.8264	0.7375	0.6668	0.5901
D2	0.9821	1	0.9933	0.9309	0.8922	0.9072	0.8392	0.7761	0.7087
D3	0.9637	0.9933	1	0.9425	0.9109	0.9373	0.879	0.8235	0.7605
D4	0.8911	0.9309	0.9425	1	0.7578	0.9116	0.8668	0.8206	0.7549
D5	0.8365	0.8922	0.9109	0.7578	1	0.9117	0.8841	0.8471	0.8179
D6	0.8264	0.9072	0.9373	0.9116	0.9117	1	0.9771	0.9505	0.9162
D7	0.7375	0.8392	0.879	0.8668	0.8841	0.9771	1	0.9721	0.9556
D8	0.6668	0.7761	0.8235	0.8206	0.8471	0.9505	0.9721	1	0.9576
D9	0.5901	0.7087	0.7605	0.7549	0.8179	0.9162	0.9556	0.9576	1
F1	-0.0966	0.0446	0.1157	0.1775	0.2316	0.3663	0.4576	0.5000	0.5335
F2	0.2801	0.4168	0.4811	0.5182	0.5479	0.6822	0.7402	0.7580	0.7646
F3	0.3643	0.5001	0.5620	0.5949	0.6148	0.7510	0.8015	0.8149	0.8152
F4	0.3357	0.4770	0.5400	0.5780	0.5979	0.7409	0.7973	0.8156	0.8197
F5	0.2636	0.4116	0.4763	0.5227	0.5462	0.6969	0.7640	0.7912	0.8030
F6	0.1754	0.3287	0.3944	0.4496	0.4776	0.6352	0.7134	0.7501	0.7711
F7	0.0823	0.2383	0.3037	0.3666	0.3996	0.5609	0.6492	0.6949	0.7248
F8	-0.0103	0.1459	0.2103	0.2796	0.3172	0.4794	0.5763	0.6300	0.6681
F9	-0.0954	0.0590	0.1217	0.1960	0.2373	0.3982	0.5019	0.5625	0.6074
X	-0.6746	-0.5758	-0.5297	-0.4391	-0.3885	-0.2757	-0.1810	-0.1111	-0.0583

* Two-year data (2004/4/1 - 2006/3/31) are used to calculate this correlation matrix of the relevant forward LIBOR rates and the exchange rate. DN and FN mean, respectively, the domestic and foreign forward LIBOR rates $F_d(\cdot, N)$ and $F_f(\cdot, N)$, and X denotes the exchange rate.

Table 10: The Correlation Matrix

	F1	F2	F3	F4	F5	F6	F7	F8	F9
F1	1	0.9918	0.8614	0.8610	0.8675	0.8616	0.8418	0.8119	0.7762
F2	0.9118	1	0.9917	0.9827	0.9637	0.9273	0.8752	0.8130	0.7473
F3	0.8614	0.9917	1	0.9953	0.9769	0.9402	0.8871	0.8232	0.7553
F4	0.8610	0.9827	0.9953	1	0.9924	0.9670	0.9243	0.8693	0.8085
F5	0.8675	0.9637	0.9769	0.9924	1	0.9908	0.9638	0.9226	0.8734
F6	0.8616	0.9273	0.9402	0.9670	0.9908	1	0.9909	0.9661	0.9309
F7	0.8418	0.8752	0.8871	0.9243	0.9638	0.9909	1	0.9920	0.9714
F8	0.8119	0.8130	0.8232	0.8693	0.9226	0.9661	0.9920	1	0.9936
F9	0.7762	0.7473	0.7553	0.8085	0.8734	0.9309	0.9714	0.9936	1
X	0.6439	0.3774	0.2982	0.3121	0.3574	0.4083	0.4563	0.4980	0.5318

This table continues Table 9.

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