# Valuation of Quanto Interest Rate Exchange Options

## I、 Introduction

During the past decades, global financial markets have grown rapidly, and thus international company may raise funds from both domestic and foreign countries and get involved with cross-currency interest rate risks. Controlling interest rate risks is thus one of the most central tasks of financial managers and has become more and more complex. To hedge against cross-currency interest rate risks, many typical financial instruments such as quanto caps, quanto floors, quanto collars have been developed. In recent years, quanto interest rate exchange options (hereafter, QIREOs) have been introduced and become popular. The issuance volumes of CMS-linked notes between 2005 and 2008 are presented by type in Table 1.

Table 1: Issuance Volume of CMS-linked Notes between 2005 and 2008

The market data is quoted from Madigan (2008).

| Structure | $ equivalent issuance value | Notes issued | $ |
|---|---|---|---|
| CMS-Linked | 51.7 billion | 1,239 | 47.10 |
| CMS range accrual | 16.6 billion | 932 | 15.10 |
| CMS steepener | 15.3 billion | 382 | 14.00 |
| CMS floating rate note | 5.5 billion | 48 | 5.10 |
| Alternative CMS Note | 3.4 billion | 50 | 3.20 |
| Other | 17.0 billion | 709 | 15.50 |
| Total | 109.7 billion | 3,360 | 100.00 |

QIREOs, also known as quanto interest rate difference options, are options written on the difference between two interest rates (at least one is foreign

$0 \le t \le T \le \Im.$

Notice that (2), (3) and (4) are stochastic processes of the domestic and foreign LIBOR rates and the exchange rate under the spot martingale measure $Q$ . It is sometimes convenient to know the processes under other martingale measures. The following proposition specifies the general rule under which the forward LIBOR rate dynamics is changed following a change of the underlying measure. This rule is useful for deriving the pricing formulas of interest rate derivatives.

**Proposition 2.** The Drift Adjustment Technique in Different Measures

*The dynamic of forward LIBOR rates $F_k(t,T)$ and the exchange rate under an arbitrary forward martingale measure $Q'$ is given as follows:*

$$\frac{dF_d(t,T)}{F_d(t,T)} = \gamma_{Fd}(t,T) \cdot (\sigma_{Bd}(t,T+\delta) - \sigma_{Bd}(t,S))dt + \gamma_{Fd}(t,T) \cdot dZ(t), \tag{6}$$

$$\frac{dF_f(t,T)}{F_f(t,T)} = \gamma_{Ff}(t,T) \cdot (\sigma_{Bf}(t,T+\delta) - \sigma_X(t) - \sigma_{Bd}(t,S))dt + \gamma_{Ff}(t,T) \cdot dZ(t), \tag{7}$$

$$\frac{dX(t)}{X(t)} = (r_d(t) - r_f(t) - \sigma_X(t) \cdot \sigma_{Bd}(t,S))dt + \sigma_X(t) \cdot dZ(t). \tag{8}$$

*where $0 \le t \le \min(S,T)$.* [3]

## 2. A Lognormalization Technique for the CLMM

According to the definition of the bond volatility process in equation (5), $\{\sigma_{Bk}(t,\cdot)\}_{t \in [0,\cdot]}$ is a stochastic rather than deterministic process. Therefore, the stochastic differential equations (6), (7) and (8) are not solvable and the distribution of $F_k(T,T)$ is unknown. However, given a fixed initial time, assumed to be $0$, we can approximate $\sigma_{Bk}(t,T)$ by $\bar{\sigma}_{Bk}^0(t,T)$ which is defined by

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[3] We employ $W(t)$ to denote an independent $d$-dimensional standard Brownian motion under an arbitrary measure without causing any confusion.

$$\bar{\sigma}_{Bk}^0(t,T) = \begin{cases} \sum\limits_{j=1}^{\lfloor \delta^{-1}(T-t) \rfloor} \dfrac{\delta F_k(t,T-j\delta)}{1+\delta F_k(t,T-j\delta)} \gamma_{Fk}(t,T-j\delta), & t \in [0,T-\delta] \\ & \& \ T-\delta > 0, \\ 0 & \text{otherwise,} \end{cases} \qquad (9)$$

where $0 \le t \le T \le \mathfrak{I}$. The calendar time of the process $\{F_k(t,T-j\delta)\}_{t\in[0,T-j\delta]}$ in (9) is frozen at its initial time $0$ and thus the process $\{\bar{\sigma}_{Bk}^0(t,T)\}_{t\in[0,T]}$ becomes deterministic. By substituting $\bar{\sigma}_{Bk}^0(t,T+\delta)$ for $\sigma_{Bk}(t,T+\delta)$ in the drift terms of (6), (7) and (8), their drift and volatility terms become deterministic, so we can solve them and find the approximate distribution of $F_k(T,T)$ to be lognormally distributed.

This technique is so-called the Wiener chaos order 0 approximation which is first used by BGM (1997) for pricing interest rate swaptions. It was developed further in Brace, Dun and Barton (1998) and formalized by Brace and Womersley (2000). The approximation also appeared in Schlögl (2002). The accuracy of this approximation has been shown to be very accurate. We present the result in the following proposition.

**Proposition 3.** The Lognormalized CLMM

*The dynamics of forward LIBOR rate $F_k(t,T)$ and the exchange rate under an arbitrary forward martingale $Q^S$ is given as follows:*

$$\frac{dF_d(t,T)}{F_d(t,T)} = \gamma_{Fd}(t,T) \cdot (\bar{\sigma}_{Bd}^0(t,T+\delta) - \bar{\sigma}_{Bd}^0(t,S))dt + \gamma_{Fd}(t,T) \cdot dZ(t), \qquad (10)$$

$$\frac{dF_f(t,T)}{F_f(t,T)} = \gamma_{Ff}(t,T) \cdot (\bar{\sigma}_{Bf}^0(t,T+\delta) - \sigma_X(t) - \bar{\sigma}_{Bd}^0(t,S))dt + \gamma_{Ff}(t,T) \cdot dZ(t), \qquad (11)$$

$$\frac{dX(t)}{X(t)} = (r_d(t) - r_f(t) - \sigma_X(t) \cdot \bar{\sigma}_{Bd}^0(t,S))dt + \sigma_X(t) \cdot dZ(t). \qquad (12)$$

*where $0 \le t \le \min(S,T)$.*

# 3. An Approximate Distribution of a Swap Rate within the CLMM Framework

Jamshidian (1997) presented the SMM to prices swaption with the Black's swaption formula, which has become the widely-accepted standard pricing formula by the swaption market. However, it is noteworthy that the LMM and the SMM are not compatible in that a swap rate and a LIBOR rate cannot be lognormally distributed under the same measure. Hence, choosing either of the two models as a pricing foundation is a vital problem. Brace, Dun and Barton (1998) suggested to employ the LMM as the central model due to its mathematical tractability. We follow their suggestion.

Following the suggestion of Brace, Dun and Barton (1998), the first problem we encounter is how to find the approximate distribution of swap rates under the CLMM framework. It is known that a swap rate is roughly a weighted average of LIBOR rates. Moreover, LIBOR rates under the CLMM framework are approximately lognormally distributed. Therefore, the distribution of a swap rate is roughly a weighted average of lognormal distributions. This subsection provides a new approach to find the approximate distribution of a (domestic and foreign) swap rate under the CLMM framework.

Define a time-$t$ swap rate with reset dates $\{T_0, T_1, \ldots, T_{n-1}\}$ and payment dates $\{T_1, T_2, \ldots, T_n\}$, as follows:

$$S_k^{(n)}(t,T) = \sum_{i=0}^{n-1} w_k^{(i)}(t) F_k(t,T_i),^4 \tag{13}$$

where

$$w_k^{(i)}(t) = \frac{B_k(t,T_{i+1})}{\sum_{j=0}^{n-1} B_k(t,T_{j+1})}. \tag{14}$$

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[4] $0 \le t \le T, T = T_0$ and $\delta = T_i - T_{i-1}, i = 1, 2, \ldots, n.$

Empirical studies in Brigo and Mercurio (2001) have shown the variability of the $w_k$'s to be small as compared to the variability of the forward LIBOR rates.[5] Therefore, we can freeze the value of the process $w_k^{(i)}(t)$ to its initial value $w_k^{(i)}(0)$ and obtain

$$S_k^{(n)}(t,T) \cong \sum_{i=0}^{n-1} w_k^{(i)}(0) F_k(t,T_i). \tag{15}$$

The distribution of $S_k^{(n)}(T,T)$ is unknown since the weighted average of lognormally distributed variables is not lognormally distributed. To solve this problem, we follow Levy (1992) to employ the Wilkinson approximation to replace the unknown distribution of the weighted average of lognormal random variables by a lognormal distribution which has the correct first two moments. In this way, we can find an approximate lognormal distribution of $S_k^{(n)}(T,T)$.[6]

Assume that $\ln S_k^{(n)}(T,T)$ has a normal distribution with mean $M_{\ln S_k^{(n)}}$ and variance $V_{\ln S_k^{(n)}}^2$. The moment generating function for $\ln S_n^{\delta}(T,T)$ is given by

$$\Phi_{\ln S_k^{(n)}}(h) = E[S_k^{(n)}(T,T)^h] = \exp(M_{\ln S_k^{(n)}} h + \frac{1}{2} V_{\ln S_k^{(n)}}^2 h^2). \tag{16}$$

Taking $h=1$ and $h=2$ in (16), we obtain the following two conditions to solve for $M_k$ and $V_k^2$ and the result is given as follows:[7]

$$M_{\ln S_k^{(n)}} = 2\ln E[S_k^{(n)}(T,T)] - \frac{1}{2}\ln E[S_k^{(n)}(T,T)^2], \tag{17}$$

$$V_{\ln S_k^{(n)}}^2 = \ln E[S_k^{(n)}(T,T)^2] - 2\ln E[S_k^{(n)}(T,T)]. \tag{18}$$

[5] See also Brace and Womersley (2000) for the proof of the low variability.
[6] According to the empirical studies in Brigo and Mercurio (2001), forward swap rates obtained from lognormal forward LIBOR rates are not far from being lognormal under the relevant measure.
[7] $E[S_k^{(n)}(t,T)]$ and $E[S_k^{(n)}(t,T)^2]$ are computed in Appendix A.

# III · Valuation of QIREOs

This section prices five types of QIREOs within the CLMM framework, namely, domestic swap rate vs. foreign swap rate; domestic swap rate vs. foreign LIBOR rate; domestic LIBOR rate vs. foreign swap rate; foreign swap rate vs. foreign swap rate; foreign swap rate vs. foreign LIBOR rate. Their practical applications are also studied.

## 1. Valuation of the First-Type QIREO (QIREO$_1$)

The first type of QIREOs is an option on the difference between a domestic and a foreign swap rate which may have different tenors. Its final payoff is given as follows:

$$QIREO_1(T) = (S_d^{(n)}(T,T) - S_f^{(m)}(T,T))^+ \tag{19}$$

where $S_d^{(n)}(T,T)$ is an $n$-year domestic swap rate observed at time $t_.$; $S_f^{(m)}(T,T)$ is an $m$-year foreign swap rate, $T$ denotes the maturity date of this option and $(a)^+ = \text{Max}(a,0)$.

QIREO$_1$s can be used to take profits if investors have taken accurate views on the spread between the yield curves of two currencies at some specific time point. If a financial manager desires to hedge interest rate risks via a long-period quanto two-way constant maturity swap, he may use a QIREO$_1$ as an ancillary instrument to eliminate the downside risk of some particular payments. As an international company wants to manage its asset and liability which are denominated in different currencies, QIREO$_1$s can be used to enhance the interest return of the asset or reduce the interest cost from the liability. The pricing formula of QIREO$_1$s is presented in the following theorem and its proof is given in Appendix A.

**Theorem 1.** *The pricing formula of QIREO$_1$s with the final payoff specified in (19) is given as follows:*

$$\text{QIREO}_1(0) = B_d(0,T)\left(\exp(M_{S_d^{(n)}} + \frac{1}{2}V_{S_d^{(n)}}^2)N(d_{1,1}) - \exp(M_{S_f^{(m)}} + \frac{1}{2}V_{S_f^{(m)}}^2)N(d_{1,2})\right), \qquad (20)$$

*where*

$$d_{1,1} = \frac{M_{S_d^{(n)}} + \frac{1}{2}V_{S_d^{(n)}}^2 - M_{S_f^{(m)}} - \frac{1}{2}V_{S_f^{(m)}}^2 + \frac{1}{2}\Psi_1^2}{\Psi_1},$$

$$d_{1,2} = d_{1,1} - \Psi_1,$$

$$M_{S_d^{(n)}} = 2\ln E^{Q^T}[S_d^{(n)}(T,T)] - \frac{1}{2}\ln E^{Q^T}[S_d^{(n)}(T,T)^2],$$

$$V_{S_d^{(n)}}^2 = \ln E^{Q^T}[S_d^{(n)}(T,T)^2] - 2\ln E^{Q^T}[S_d^{(n)}(T,T)],$$

$$M_{S_f^{(m)}} = 2\ln E^{Q^T}[S_f^{(m)}(T,T)] - \frac{1}{2}\ln E^{Q^T}[S_f^{(m)}(T,T)^2],$$

$$V_{S_f^{(m)}}^2 = \ln E^{Q^T}[S_f^{(m)}(T,T)^2] - 2\ln E^{Q^T}[S_f^{(m)}(T,T)],$$

$$\Psi_1^2 = V_{S_d^{(n)}}^2 + V_{S_f^{(m)}}^2 - 2\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} w_d^{(i)}(0)w_f^{(j)}(0)\int_0^T \gamma_d(u,T_i)\cdot\gamma_f(u,T_j)du,$$

*and* $E^{Q^T}[S_d^{(n)}(T,T)]$, $E^{Q^T}[S_d^{(n)}(T,T)^2]$, $E^{Q^T}[S_f^{(m)}(T,T)]$ *and* $E^{Q^T}[S_f^{(m)}(T,T)^2]$ *are defined, respectively, in (A.15), (A.16), (A.20) and (A.21).*

The pricing formula (20) bears some resemblance to the Margrabe (1978) formula, but in the CLMM framework. In addition, all parameters appearing in (20) can be easily calibrated from market data, and thereby making the pricing formula more tractable and feasible for market practitioners.

## 2. Valuation of Second-Type QIREOs (QIREO$_2$s)

A QIREO$_2$ is an option on the difference between a domestic swap rate and a foreign LIBOR rate with the final payoff given as follows:

$$QIREO_2(T) = (S_d^{(n)}(T,T) - F_f(T,T))^+ \qquad (21)$$

where $S_d^{(n)}(T,T)$ is an $n$-year domestic forward swap rate observed at time $t$; $F_f(T,T)$ is a foreign forward LIBOR rate observed at time $t$ for the period $[T, T+\delta]$.

For international firms with assets and liabilities in different currencies, QIREO$_2$s can be employed to manage the cross-currency interest rate risk. For example, if companies have assets involved with short-term foreign interest rates and liabilities with long-term domestic interest rates, they may use QIREO$_2$s to hedge interest rate risk. Moreover, QIREO$_2$s can also be employed as ancillary instruments for a differential swaps to remove the downside risks of some particular payments. By employing QIREO$_2$s, investors can make profits from an accurate view on the spread between a domestic swap rate and a foreign LIBOR rate.

The pricing formula of QIREO$_2$s is presented in the following theorem and its proof is similar to Theorem 1 and available upon request from the authors.

**Theorem 2.** *The pricing formula of QIREO$_2$s with the final payoff specified in (21) is given as follows:*

$$QIREO_2(0) = B_d(0,T)\left(\exp(M_{S_d^{(n)}} + \frac{1}{2}V_{S_d^{(n)}}^2)N(d_{2,1}) - \exp(M_{F_f} + \frac{1}{2}V_{F_f}^2)N(d_{2,2})\right), \qquad (22)$$

*where*

$$d_{2,1} = \frac{M_{S_d^{(n)}} + \frac{1}{2}V_{S_d^{(n)}}^2 - M_{F_f} - \frac{1}{2}V_{F_f}^2 + \frac{1}{2}\Psi_2^2}{\Psi_2},$$

$$d_{2,2} = d_{2,1} - \Psi_2,$$

$$M_{F_f} = \ln F_f(0,T) + \int_0^T (\Delta_f^0(u,T;T) - \frac{1}{2}\|\gamma_f(u,T)\|^2)du,$$

$$V_{F_f}^2 = \int_0^T \|\gamma_f(u,T)\|^2 du,$$

$$M_{S_d^{(n)}} = 2\ln E^{Q^f}[S_d^{(n)}(T,T)] - \frac{1}{2}\ln E^{Q^f}[S_d^{(n)}(T,T)^2],$$

$$V_{S_d^{(n)}}^2 = \ln E^{Q^f}[S_d^{(n)}(T,T)^2] - 2\ln E^{Q^f}[S_d^{(n)}(T,T)],$$

$$\Psi_2^2 = V_{S_d^{(n)}}^2 + V_{F_f}^2 - 2\sum_{i=0}^{n-1} w_d^{(i)}(0) \int_0^T \gamma_d(u,T_i) \cdot \gamma_f(u,T)du,$$

and $E^{Q^f}[S_d^{(n)}(T,T)]$ and $E^{Q^f}[S_d^{(n)}(T,T)^2]$, are defined, respectively, in (A.15), (A.16); $\Delta_f^0(\cdot,T;T)$ is defined in (A.4).

The pricing formula (22) also bears some resemblance to the formula in Margrabe (1978), which makes end-users more familiar to employ it. Like equation (20), the parameters in (22) can also be extracted easily from market-quoted prices. It is well-known that a LIBOR rate is a one-period swap rate, and thus a QIREO$_2$ is a special form of QIREO$_1$ with $m=1$.

## 3. Valuation of Third-Type QIREOs (QIREO$_3$s)

A QIREO$_3$ is an option on the difference between a domestic LIBOR rate and a foreign swap rate with the final payoff given as follows:

$$\text{QIREO}_3(T) = (F_d(T,T) - S_f^{(m)}(T,T))^+. \tag{23}$$

QIREO$_3$s can be used to take profits if investors have taken accurate views on the spread between a domestic LIBOR rate and a foreign swap rate. In addition, if companies have assets involved with long-term foreign interest rates

and liabilities with short-term domestic interest rates, they may use QIREO$_3$s to hedge the interest rate risk. An QIREO$_3$ can also be used as a substitute for a differential swap. The pricing formula of QIREO$_3$s is presented as follows and its proof is available upon request from the authors.

**Theorem 3.** *The pricing formula of QIREO$_3$s with the final payoff specified in (23) is given as follows:*

$$QIREO_3(0) = B_d(0,T)\left( \exp(M_{F_d} + \frac{1}{2}V_{S_f^{(m)}}^2)N(d_{3,1}) - \exp(M_{F_d} + \frac{1}{2}V_{S_f^{(m)}}^2)N(d_{3,2}) \right), \qquad (24)$$

*where*

$$d_{3,1} = \frac{M_{F_d} + \frac{1}{2}V_{F_d}^2 - M_{S_f^{(m)}} - \frac{1}{2}V_{S_f^{(m)}}^2 + \frac{1}{2}\Psi_3^2}{\Psi_3},$$

$$d_{3,2} = d_{3,1} - \Psi_3,$$

$$M_{F_d} = \ln F_d(0,T) + \int_0^T (\Delta_d^0(u,T;T) - \frac{1}{2}\|\gamma_d(u,T)\|^2)du,$$

$$V_{F_d}^2 = \int_0^T \|\gamma_d(u,T)\|^2 \, du,$$

$$M_{S_f^{(m)}} = 2\ln E^{\mathcal{Q}^d}[S_f^{(m)}(T,T)] - \frac{1}{2}\ln E^{\mathcal{Q}^d}[S_f^{(m)}(T,T)^2],$$

$$V_{S_f^{(m)}}^2 = \ln E^{\mathcal{Q}^d}[S_f^{(m)}(T,T)^2] - 2\ln E^{\mathcal{Q}^d}[S_f^{(m)}(T,T)],$$

$$\Psi_3^2 = V_{F_d}^2 + V_{S_f^{(m)}}^2 - 2\sum_{j=0}^{m-1} w_f^{(j)}(0)\int_0^T \gamma_d(u,T)\cdot\gamma_f(u,T_j)du,$$

*and* $E^{\mathcal{Q}^d}[S_f^{(m)}(T,T)]$ *and* $E^{\mathcal{Q}^d}[S_f^{(m)}(T,T)^2]$ *are defined, respectively, in (A.20) and*

*(A.21)*; $\Delta_d^0(\cdot, T; T)$ *is defined in (A.3)*.

The parameters in (24) can also be extracted easily from market-quoted prices. Similar to (22), the pricing formula for $QIREO_3$s is also a special case of $QIREO_1$ with $n = 1$.

## 4. Valuation of Fourth-Type QIREOs ( $QIREO_4$s )

The fourth type of QIREOs is an option on the difference between two foreign swap rates with different tenors. Its final payoff is given as follows:

$$QIREO_4(T) = \left( S_f^{(n)}(T,T) - S_f^{(m)}(T,T) \right)^+ \tag{25}$$

where $S_f^{(n)}(T,T)$ and $S_f^{(m)}(T,T)$ are, respectively, an $n$-year and an $m$-year foreign swap rate observed at time $t$; $T$ denotes the maturity date of this option. The pricing formula of $QIREO_4$s is presented as follows and its proof is available upon request from the authors.

**Theorem 4.** *The pricing formula of $QIREO_4$s with the final payoff specified in (25) is given as follows:*

$$QIREO_4(0) = B_d(0,T) \left( \exp\left( M_{S_f^{(n)}} + \frac{1}{2} V_{S_f^{(n)}}^2 \right) N(d_{4,1}) - \exp\left( M_{S_f^{(m)}} + \frac{1}{2} V_{S_f^{(m)}}^2 \right) N(d_{4,2}) \right), \tag{26}$$

*Where*

$$d_{4,1} = \frac{M_{S_f^{(n)}} + \frac{1}{2} V_{S_f^{(n)}}^2 - M_{S_f^{(m)}} - \frac{1}{2} V_{S_f^{(m)}}^2 + \frac{1}{2} \Psi_4^2}{\Psi_4},$$

$$d_{4,2} = d_{4,1} - \Psi_4,$$

$$M_{S_f^{(n)}} = 2 \ln E^{Q^T} \left[ S_f^{(n)}(T,T) \right] - \frac{1}{2} \ln E^{Q^T} \left[ S_f^{(n)}(T,T)^2 \right],$$

$$V_{S_f^{(n)}}^2 = \ln E^{Q^T} \left[ S_f^{(n)}(T,T)^2 \right] - 2 \ln E^{Q^T} \left[ S_f^{(n)}(T,T) \right],$$

$$M_{S_f^{(m)}} = 2\ln E^{Q^T}\left[S_f^{(m)}(T,T)\right] - \frac{1}{2}\ln E^{Q^T}\left[S_f^{(m)}(T,T)^2\right],$$

$$V^2_{S_f^{(m)}} = \ln E^{Q^T}\left[S_f^{(m)}(T,T)^2\right] - 2\ln E^{Q^T}\left[S_f^{(m)}(T,T)\right],$$

$$\Psi^2_4 = V^2_{S_f^{(n)}} + V^2_{S_f^{(m)}} - 2\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} w_f^{(i)}(0)w_f^{(j)}(0)\int_0^T \gamma_f(u,T_i)\cdot\gamma_f(u,T_j)du,$$

and $E^{Q^T}\left[S_f^{(n)}(T,T)\right]$, $E^{Q^T}\left[S_f^{(n)}(T,T)^2\right]$, $E^{Q^T}\left[S_f^{(m)}(T,T)\right]$ and $E^{Q^T}\left[S_f^{(m)}(T,T)^2\right]$ are defined, respectively, in (A.20) and (A.21).

QIREO$_4$s are usually embedded in quanto range-accrual structure notes, which is widely-traded in the financial market. QIREO$_4$s are traded by investors who wish to take a position on future relative changes in different parts of the foreign yield curve. QIREO$_4$s can also be used as ancillary instruments for a two-way quanto constant maturity swap. In addition, the parameters in the pricing formula (26) can be calibrated easily from market data, and thereby the pricing formula for QIREO$_4$s is workable in practice.

## 5. Valuation of Fifth-Type QIREOs ( QIREO$_5$s )

A QIREO$_5$ is an option on the difference between a domestic swap rate and a foreign LIBOR rate with the final payoff given as follows:

$$QIREO_5(T)=\left(S_f^{(n)}(T,T)-F_f(T,T)\right)^+ \tag{27}$$

where $S_f^{(n)}(T,T)$ is an $n$-year foreign forward swap rate observed at time $t$; $F_f(T,T)$ is a foreign forward LIBOR rate observed at time $t$ for the period $[T,T+\delta]$.

As a substitute for quanto constant maturity swaps, QIREO$_5$s can be employed to manage assets and liabilities in a more specific strategy. Moreover, QIREO$_5$s can also be employed as ancillary instruments for a quanto constant maturity swaps to remove the downside risks of some particular payments. Via QIREO$_5$s, investors can make profits from an accurate view on the spread

between a foreign swap rate and a foreign LIBOR rate.

The pricing formula of $QIREO_s s$ is presented in the following theorem and its proof is available upon request from the authors.

**Theorem 5.** *The pricing formula of $QIREO_s s$ with the final payoff specified in (27) is given as follows:*

$$QIREO_s(0) = B_d(0,T)\left( \exp\left( M_{S_f^{(n)}} + \frac{1}{2} V_{S_f^{(n)}}^2 \right) N(d_{s,1}) - \exp\left( M_{F_f} + \frac{1}{2} V_{F_f}^2 \right) N(d_{s,2}) \right), \tag{28}$$

*where*

$$d_{s,1} = \frac{M_{S_f^{(n)}} + \frac{1}{2} V_{S_f^{(n)}}^2 - M_{F_f} - \frac{1}{2} V_{F_f}^2 + \frac{1}{2} \Psi_s^2}{\Psi_s},$$

$$d_{s,2} = d_{s,1} - \Psi_s,$$

$$M_{F_f} = \ln F_f(0,T) + \int_0^T \left( \Delta_f^0(u,T;T) - \frac{1}{2} \| \gamma_f(u,T) \|^2 \right) du,$$

$$V_{F_f}^2 = \int_0^T \| \gamma_f(u,T) \|^2 du,$$

$$M_{S_f^{(n)}} = 2 \ln E^{Q^T} \left[ S_f^{(n)}(T,T) \right] - \frac{1}{2} \ln E^{Q^T} \left[ S_f^{(n)}(T,T)^2 \right],$$

$$V_{S_f^{(n)}}^2 = \ln E^{Q^T} \left[ S_f^{(n)}(T,T)^2 \right] - 2 \ln E^{Q^T} \left[ S_f^{(n)}(T,T) \right],$$

$$\Psi_s^2 = V_{S_f^{(n)}}^2 + V_{F_f}^2 - 2 \sum_{i=0}^{n-1} w_f^{(i)}(0) \int_0^T \gamma_f(u,T_i) \cdot \gamma_f(u,T) du,$$

*and* $E^{Q^T} \left[ S_f^{(n)}(T,T) \right]$ *and* $E^{Q^T} \left[ S_f^{(n)}(T,T)^2 \right]$ *are defined, respectively, in (A.20) and (A.21).*

The aforementioned five QIREOs are widely traded in the financial market for hedging and investing purposes. Besides, they are usually embedded in widely-traded structured notes, such as quanto CMS range accruals and steepeners. In practice, these financial instruments are priced based on Monte Carlo simulations which are too time-consuming. For example, if a financial institution issuing hundreds of structured notes embedded with QIREOs, Monte Carlo simulations are too time-consuming to compute the daily prices of issued structure notes for investors each day. The closed-form pricing formulas as shown in Theorems 1⊔5 are sufficiently and robustly accurate by comparing with Monte Carlo simulation presented in the next section. Therefore, our formulas provide a time-reducing approach to price quanto CMS structured notes and make the management of price quotations easier.

# IV、Calibration Procedure and Numerical Study

In this section, we first present the calibration procedure for the CLMM and then provide some numerical examinations for the approximation formulas presented in Subsection 2.

## 1. Calibration Procedure

The most vital merit of the CLMM is its tractability and feasibility in parameter calibration. We extend the mechanism presented by Rebonato (1999) to engage in a simultaneous calibration of the CLMM to the volatilities and correlation matrix of the domestic and foreign forward LIBOR rates and the exchange rate. Assume that there are $2n$ forward LIBOR rates and an exchange rate in the $d$-factor framework. The steps to calibrate the parameters are given as follows.

First, we assume that each forward LIBOR rate, $F_k(\cdot, T_i)$, has a constant instantaneous volatility, namely for $i = 1, ..., n$, $\gamma_k(\cdot, T_i) = \upsilon_{k,i}$. The setting is as

presented in Table 2.[8] Thus, if the market-quoted volatility for $T_1$-year cap of the $k$ th country is $\xi_{k,1}$, then $\upsilon_{k,1} = \xi_{k,1}$. Next, for $i = 2,...,n$, if the $T_i$-year cap is $\xi_{k,i}$, then $\upsilon_{k,1} = \sqrt{\xi_{k,i}^2 T_i^2 - \xi_{k,i-1}^2 T_{i-1}^2}$. In addition, we also assume that the spot exchange rate has a constant volatility, which can be calibrated from on-the-run currency options' prices in the market.

Table 2: Instantaneous Volatilities of $\{F_k(t,\cdot)\}_{k\in\{d,f\}}$

| Instant. Total Vol. | Time $t \in (T_0,T_1]$ | $(T_1,T_2]$ | $(T_2,T_3]$ | $\cdots$ | $(T_{n-2},T_{n-1}]$ |
|---|---|---|---|---|---|
| Fwd Rate: $F_k(t,T_1)$ | $\upsilon_{k,1}$ | Dead | Dead | $\cdots$ | Dead |
| $F_k(t,T_2)$ | $\upsilon_{k,2}$ | $\upsilon_{k,2}$ | Dead | $\cdots$ | Dead |
| $\vdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $F_k(t,T_{n-1})$ | $\upsilon_{k,n-1}$ | $\upsilon_{k,n-1}$ | $\upsilon_{k,n-1}$ | $\cdots$ | $\upsilon_{k,n-1}$ |

Second, we use the historical data of the forward LIBOR rates and the exchange rate to derive a market correlation matrix $\Gamma$. $\Gamma$ is a $(2n+1)\times(2n+1)$, positive-definite and symmetric matrix with rank $(2n+1)$ and can be written as

$$\Gamma = H\Lambda H,$$

where $H$ is a real orthogonal matrix and $\Lambda$ is a diagonal matrix. Let $A \equiv H\Lambda^{1/2}$ and thus, $\Gamma = AA'$. In this way, we can find a $(2n+1)\times d$ matrix $B$ with rank $d$ ($d \le 2n+1$) so that $\Gamma^B = BB'$ is an approximate correlation matrix for $\Gamma$.

The advantage of finding $B$ is that we may replace the $(2n+1)$-dimensional original Brownian motion $Z(t)$ with $BW(t)$ where $W(t)$ is a $d$-dimensional Brownian motion. In other words, we change the structure of the market correlation, namely,

$$dZ(t)dZ(t)' = \Gamma dt,$$

---

[8] For other assumptions of volatility structures, please refer to Brigo and Mercurio (2001).

to an approximate correlation structure:

$$BdW(t)(dW(t))' = BdW(t)dW(t)'B' = BB'dt = \Gamma^B dt.$$

The remaining problem is how to find a suitable matrix $B$. Rebonato (1999) proposed a method described as follows. Assume that the $ik$-th element of $B$ for $i=1,...,(2n+1)$ is specified as follows:

$$b_{i,q} = \begin{cases} \cos\theta_{i,q}\Pi_{j=1}^{q-1}\sin\theta_{i,j} & \text{if } q=1,2,...,d-1, \\ \Pi_{j=1}^{q-1}\sin\theta_{i,j} & \text{if } q=d. \end{cases}$$

Thus, $\Gamma^B = BB'$ is a function of $\Theta = \{\theta_{i,q}\}_{i=1,...,(2n+1);q=1,...,d-1}$. We can obtain an optimal solution $\hat{\Theta}$ by solving the following optimization problem:

$$\min_{\Theta} \sum_{i,j=1}^{2n+1} \left|\Gamma_{i,j}^B - \Gamma_{i,j}\right|^2, \tag{29}$$

where $\Gamma_{i,j}$ is the $ij$-th element of $\Gamma$ and $\Gamma_{i,j}^B$ is the $ij$-th element of $\Gamma^B$, specifically defined as follows:

$$\Gamma_{i,j}^B = \sum_{q=1}^{d} b_{i,q}b_{j,q}.$$

By substituting $\hat{\Theta}$ into $B$, we obtain an optimal matrix $\hat{B}$ such that $\hat{\Gamma}^B(=\hat{B}\hat{B}')$ is an approximate correlation matrix for $\Gamma$.

Third, we use $\hat{B}$ to distribute the instantaneous total volatility of the forward LIBOR rate and exchange rate to each Brownian motion without changing the amount of the instantaneous total volatility.[9]

The above procedure is a general calibration method without a constraint on choosing the number of factors, $d$. The number of random shocks, $d$, may depend on the maturity range of interest rates involved in the considered

---

[9] Note the Euclidean norm of each row vector of $B$ is 1.

financial product.[10]  For example, we may use a five-factor model, i.e. $d = 5$. The first two random shocks can be interpreted, respectively, as the short-term and long-term factors of the domestic yield curve. The next two random shocks can be interpreted as the same way for the foreign yield curve. The fifth factor specifies the behavior of the exchange rate. According to this feature, the numerical examples in the following section are based on the five-factor model.

## 2. Numerical Study

This subsection offers practical examples that examine the accuracy of the approximate pricing formulas as presented in the previous section by comparing the results with Monte Carlo simulation. The used market data are presented in Appendix B. The U.S. stands for the domestic country and the U.K. stands for the foreign country. The notional principal is assumed to be $1 and the simulations are based on 10000 paths.

The first case is the $QIREO_1$ on the difference between a 5-year domestic swap rate and a 2-year foreign swap rate; the second case is the $QIREO_2$ between a 5-year domestic swap rate and a 6-month foreign LIBOR rate; the third case is the $QIREO_3$ between a 6-month domestic LIBOR rate and a 5-year foreign swap rate; the fourth case is the $QIREO_4$ between a 2-year foreign swap rate and a 5-year foreign swap rate; the fifth case is the $QIREO_5$ between a 6-month foreign LIBOR rate and a 5-year foreign swap rate. The involved swap rates are reset semiannually. Each QIREO is priced with two times to maturity, namely 1 year and 3 years. To examine the accuracy and robustness of the derived pricing formulas for different market scenarios, the QIREOs are quarterly priced on the dates for the recent two years, namely, 2007/12/03, 2007/09/03, 2007/06/01, 2007/03/01, 2006/12/01, 2006/09/01, 2006/06/01 and 2006/03/01. The results, listed in Table 3 ⊔ 7, show that the approximation formulas are sufficiently accurate by comparing with Monte Carlo simulations and most of the relative errors are under 1%. Therefore, our pricing formulas are worth recommending for practical implementation.

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[10] Please refer to Driessen, Klaassen and Melenberg (2003) and Rebonato (1999) for more details regarding the performance of one- and multi-factor models.

# V、Conclusion

We have developed a new approach to approximate the distribution of a (foreign) forward swap rate under the CLMM and then used the approximation approach to derive the pricing formulas of five types of QIREOs under the CLMM framework, namely, domestic swap rate vs. foreign Swap rate; domestic swap rate vs. foreign LIBOR rate; domestic LIBOR rate vs. foreign swap rate; foreign swap rate vs. foreign swap rate; foreign swap rate vs. foreign LIBOR rate. As compared with Monte Carlo simulation, the numerical examples show that the resulting pricing formulas are sufficiently and robustly accurate. Therefore, the resulting pricing formulas for QIREOs are worth recommending for market practitioners.

## Table 3: The Numerical Examples of Theorem 1

The 1-year and 3-year QIREOs on the difference between the 5-year domestic swap rate and the 2-year foreign swap rate are quarterly priced based on the market data over the past two years. The year fraction of the swap rates is a half year. The market data are listed in Appendix B. The notional value is assumed to be $1. The simulation is based on 10000 paths. MC stands for the result of the Monte Carlo simulation; SE for the standard error; RE=|Theorem 1 - MC|/MC×100% for relative error.

| Date | Time to Maturity | Theorem 1 | MC | SE | RE(%) |
|------|------------------|-----------|-----|-----|-------|
| 2007/12/03 | 1 year | 0.001928 | 0.001932 | 0.000050 | 0.22 |
| | 3 year | 0.005215 | 0.005245 | 0.000092 | 0.58 |
| 2007/09/03 | 1 year | 0.001524 | 0.001520 | 0.000040 | 0.30 |
| | 3 year | 0.004570 | 0.004555 | 0.000080 | 0.33 |
| 2007/06/01 | 1 year | 0.000995 | 0.001008 | 0.000027 | 1.35 |
| | 3 year | 0.004467 | 0.004431 | 0.000071 | 0.81 |
| 2007/03/01 | 1 year | 0.001505 | 0.001518 | 0.000036 | 0.86 |
| | 3 year | 0.005046 | 0.005077 | 0.000081 | 0.61 |
| 2006/12/01 | 1 year | 0.002165 | 0.002151 | 0.000045 | 0.64 |
| | 3 year | 0.006144 | 0.006160 | 0.000096 | 0.26 |
| 2006/09/01 | 1 year | 0.004147 | 0.004172 | 0.000060 | 0.59 |
| | 3 year | 0.008418 | 0.008464 | 0.000106 | 0.55 |
| 2006/06/01 | 1 year | 0.006444 | 0.006495 | 0.000072 | 0.79 |
| | 3 year | 0.009950 | 0.009950 | 0.000112 | 0.01 |
| 2006/03/01 | 1 year | 0.005910 | 0.005957 | 0.000069 | 0.79 |
| | 3 year | 0.009418 | 0.009442 | 0.000111 | 0.26 |

## Table 4: The Numerical Examples of Theorem 2

The 1-year and 3-year QIREOs on the difference between the 5-year domestic swap rate and the 6-month foreign LIBOR rate are quarterly priced based on the market data over the past two years. The year fraction of the swap rates is a half year. The market data are listed in Appendix B. The notional value is assumed to be $1. The simulation is based on 10000 paths. MC stands for the result of the Monte Carlo simulation; SE for the standard error; RE=|Theorem 2 - MC|/MC×100% for relative error.

| Date | Time to Maturity | Theorem 2 | MC | SE | RE(%) |
|------|------------------|-----------|-----|-----|-------|
| 2007/12/03 | 1 year | 0.002378 | 0.002391 | 0.000055 | 0.54 |
| | 3 year | 0.005084 | 0.005123 | 0.000092 | 0.76 |
| 2007/09/03 | 1 year | 0.001643 | 0.001675 | 0.000044 | 1.91 |
| | 3 year | 0.004218 | 0.004265 | 0.000075 | 1.10 |
| 2007/06/01 | 1 year | 0.000834 | 0.000837 | 0.000026 | 0.36 |
| | 3 year | 0.003858 | 0.003846 | 0.000067 | 0.31 |
| 2007/03/01 | 1 year | 0.001311 | 0.001309 | 0.000035 | 0.15 |
| | 3 year | 0.004478 | 0.004506 | 0.000076 | 0.62 |
| 2006/12/01 | 1 year | 0.001947 | 0.001972 | 0.000043 | 1.27 |
| | 3 year | 0.005530 | 0.005538 | 0.000089 | 0.14 |
| 2006/09/01 | 1 year | 0.004865 | 0.004855 | 0.000065 | 0.21 |
| | 3 year | 0.007979 | 0.007984 | 0.000103 | 0.06 |
| 2006/06/01 | 1 year | 0.006763 | 0.006705 | 0.000073 | 0.87 |
| | 3 year | 0.009650 | 0.009692 | 0.000108 | 0.43 |
| 2006/03/01 | 1 year | 0.005973 | 0.005921 | 0.000069 | 0.88 |
| | 3 year | 0.008827 | 0.008855 | 0.000107 | 0.32 |

## Table 5: The Numerical Examples of Theorem 3

The 1-year and 3-year QIREOs on the difference between the 6-month domestic LIBOR rate and the 5-year foreign swap rate are quarterly priced based on the market data over the past two years. The year fraction of the swap rates is a half year. The market data are listed in Appendix B. The notional value is assumed to be $1. The simulation is based on 10000 paths. MC stands for the result of the Monte Carlo simulation; SE for the standard error; RE=|Theorem 3 - MC|/MC×100% for relative error.

| Date | Time to Maturity | Theorem 3 | MC | SE | RE(%) |
|------|------------------|-----------|-----|-----|-------|
| 2007/12/03 | 1 year | 0.000844 | 0.000845 | 0.000031 | 0.12 |
|            | 3 year | 0.005761 | 0.005738 | 0.000132 | 0.40 |
| 2007/09/03 | 1 year | 0.001729 | 0.001705 | 0.000046 | 1.41 |
|            | 3 year | 0.005268 | 0.005259 | 0.000104 | 0.17 |
| 2007/06/01 | 1 year | 0.000980 | 0.000989 | 0.000023 | 0.91 |
|            | 3 year | 0.005497 | 0.005458 | 0.000086 | 0.71 |
| 2007/03/01 | 1 year | 0.001632 | 0.001667 | 0.000034 | 2.10 |
|            | 3 year | 0.006020 | 0.006018 | 0.000098 | 0.03 |
| 2006/12/01 | 1 year | 0.002206 | 0.002209 | 0.000039 | 0.14 |
|            | 3 year | 0.006878 | 0.006883 | 0.000112 | 0.07 |
| 2006/09/01 | 1 year | 0.003949 | 0.003973 | 0.000051 | 0.60 |
|            | 3 year | 0.008601 | 0.008676 | 0.000116 | 0.86 |
| 2006/06/01 | 1 year | 0.005828 | 0.005804 | 0.000062 | 0.41 |
|            | 3 year | 0.009846 | 0.009811 | 0.000121 | 0.36 |
| 2006/03/01 | 1 year | 0.006364 | 0.006385 | 0.000060 | 0.33 |
|            | 3 year | 0.010220 | 0.010254 | 0.000127 | 0.33 |

## Table 6: The Numerical Examples of Theorem 4

The 1-year and 3-year QIREOs on the difference between the 2-year foreign swap rate and the 5-year foreign swap rate are quarterly priced based on the market data over the past two years. The year fraction of the swap rates is a half year. The market data are listed in Appendix B. The notional value is assumed to be $1. The simulation is based on 10000 paths. MC stands for the result of the Monte Carlo simulation; SE for the standard error; RE=|Theorem 4 - MC|/MC×100% for relative error.

| Date | Time to Maturity | Theorem 4 | MC | SE | RE(%) |
|------|------------------|-----------|-----|-----|-------|
| 2006/03/01 | 1 year | 0.001254 | 0.001259 | 0.000009 | 0.41 |
|            | 3 year | 0.000955 | 0.000953 | 0.000010 | 0.22 |
| 2006/06/01 | 1 year | 0.000779 | 0.000771 | 0.000007 | 1.07 |
|            | 3 year | 0.000823 | 0.000819 | 0.000009 | 0.51 |
| 2006/09/01 | 1 year | 0.001261 | 0.001266 | 0.000008 | 0.34 |
|            | 3 year | 0.000911 | 0.000910 | 0.000010 | 0.19 |
| 2006/12/01 | 1 year | 0.001731 | 0.001739 | 0.000008 | 0.43 |
|            | 3 year | 0.001482 | 0.001490 | 0.000011 | 0.54 |
| 2007/03/01 | 1 year | 0.002022 | 0.002033 | 0.000007 | 0.52 |
|            | 3 year | 0.001656 | 0.001660 | 0.000010 | 0.25 |
| 2007/06/01 | 1 year | 0.002391 | 0.002401 | 0.000008 | 0.44 |
|            | 3 year | 0.001893 | 0.001916 | 0.000011 | 1.18 |
| 2007/09/03 | 1 year | 0.001496 | 0.001471 | 0.000010 | 1.66 |
|            | 3 year | 0.001520 | 0.001532 | 0.000013 | 0.77 |
| 2007/12/03 | 1 year | 0.000645 | 0.000643 | 0.000010 | 0.30 |
|            | 3 year | 0.000718 | 0.000714 | 0.000011 | 0.52 |

## Table 7: The Numerical Examples of Theorem 5

The 1-year and 3-year QIREOs on the difference between the 6-month foreign LIBOR rate and the 5-year foreign swap rate are quarterly priced based on the market data over the past two years. The year fraction of the swap rates is a half year. The market data are listed in Appendix B. The notional value is assumed to be \$1. The simulation is based on 10000 paths. MC stands for the result of the Monte Carlo simulation; SE for the standard error; RE=|Theorem 5 - MC|/MC × 100% for relative error.

| Date | Time to Maturity | Theorem 5 | MC | SE | RE(%) |
|------|------------------|-----------|-----|-----|-------|
| 2006/03/01 | 1 year | 0.001730 | 0.001745 | 0.000017 | 0.88 |
|  | 3 year | 0.001709 | 0.001685 | 0.000016 | 1.39 |
| 2006/06/01 | 1 year | 0.001031 | 0.001032 | 0.000013 | 0.08 |
|  | 3 year | 0.001306 | 0.001298 | 0.000016 | 0.61 |
| 2006/09/01 | 1 year | 0.000971 | 0.000976 | 0.000013 | 0.49 |
|  | 3 year | 0.001531 | 0.001515 | 0.000016 | 1.06 |
| 2006/12/01 | 1 year | 0.002854 | 0.002882 | 0.000022 | 0.96 |
|  | 3 year | 0.002428 | 0.002408 | 0.000017 | 0.81 |
| 2007/03/01 | 1 year | 0.003188 | 0.003205 | 0.000019 | 0.53 |
|  | 3 year | 0.002605 | 0.002596 | 0.000017 | 0.34 |
| 2007/06/01 | 1 year | 0.003495 | 0.003480 | 0.00002 | 0.43 |
|  | 3 year | 0.002962 | 0.002986 | 0.000018 | 0.80 |
| 2007/09/03 | 1 year | 0.002171 | 0.002155 | 0.000016 | 0.75 |
|  | 3 year | 0.002378 | 0.002374 | 0.000024 | 0.19 |
| 2007/12/03 | 1 year | 0.000516 | 0.000513 | 0.00001 | 0.73 |
|  | 3 year | 0.001503 | 0.001486 | 0.000025 | 1.15 |

# Appendix A: The Proof of Theorem 1.

Before starting the derivation of Theorem 1, we present a useful lemma for the following deriving process.

**Lemma 1.** *If* $X \sim N(\mu_X, \sigma_X^2)$ *and* $Y \sim N(\mu_Y, \sigma_Y^2)$, *then we have*

$$E\left[\left(\exp(X - \tfrac{1}{2}\sigma_X^2) - \exp(Y - \tfrac{1}{2}\sigma_Y^2)\right)^+\right] = \exp(\mu_X)N(\eta) - \exp(\mu_Y)N(\eta - \zeta), \tag{A.1}$$

where

$$\eta = \frac{\mu_X - \mu_Y + \tfrac{1}{2}\zeta^2}{\zeta}, \quad and \quad \zeta^2 = Var(X - Y).$$

According to Proposition 3, the dynamics of $\{F_k(t, T_i)\}_{k=d,f}$ under the forward martingale measure $Q^T$ is given as follows:

$$\frac{dF_k(t, T_i)}{F_k(t, T_i)} = \Delta_k^0(t, T_i; T)dt + \gamma_k(t, T_i) \cdot dZ(t), \tag{A.2}$$

where

$$\Delta_d^0(t, T_i; T) = \gamma_{Fd}(t, T_i) \cdot (\bar{\sigma}_{Bd}^0(t, T_i + \delta) - \bar{\sigma}_{Bd}^0(t, T)), \tag{A.3}$$

$$\Delta_f^0(t, T_i; T) = \gamma_{Ff}(t, T_i) \cdot (\bar{\sigma}_{Bf}^0(t, T_i + \delta) - \sigma_X(t) - \bar{\sigma}_{Bd}^0(t, T)). \tag{A.4}$$

Therefore,

$$F_k(T, T_i) = F_k(0, T_i)\exp\left(\int_0^T (\Delta_k^0(u, T_i; T) - \tfrac{1}{2}\|\gamma_k(u, T_i)\|^2)du + \int_0^T \gamma_k(u, T_i) \cdot dZ(u)\right), \tag{A.5}$$

$$\ln F_k(T, T_i) = \ln F_k(0, T_i) + \int_0^T (\Delta_k^0(u, T_i; T) - \tfrac{1}{2}\|\gamma_k(u, T_i)\|^2)du + \int_0^T \gamma_k(u, T_i) \cdot dZ(u), \tag{A.6}$$

$$\ln F_k(T_i, T_i) \sim N(M_{F_k}, V_{F_k}^2), \tag{A.7}$$

where

$$M_{F_k} = \ln F^k(0, T_i) + \int_0^{T_i} (\Delta_k^0(u, T_i; T) - \tfrac{1}{2}\|\gamma_k(u, T_i)\|^2)du, \tag{A.8}$$

$$V_{F_k}^2 = \int_0^{T_i} \left\| \gamma_k(u,T_i) \right\|^2 du. \tag{A.9}$$

The pricing formula of QIREO$_1$s can be derived under forward martingale measure $Q^T$ as follows:

$$\text{QIREO}_1(0) = B_d(0,T)E^{Q^T} \left[ (S_d^{(n)}(T,T) - S_f^{(m)}(T,T))^+ \right]. \tag{A.10}$$

According to the approximate method in Subsection 3 of section II, the distributions of $S_d^{(n)}(T,T)$ and $S_f^{(m)}(T,T)$ are assumed to be lognormal distributions with the correct first two moments, i.e.

$$\ln S_d^{(n)}(T,T) \sim N(M_{S_d^{(n)}}, V_{S_d^{(n)}}^2),$$

$$\ln S_f^{(m)}(T,T) \sim N(M_{S_f^{(m)}}, V_{S_f^{(m)}}^2),$$

where

$$M_{S_d^{(n)}} = 2\ln E^{Q^T}[S_d^{(n)}(T,T)] - \frac{1}{2}\ln E^{Q^T}[S_d^{(n)}(T,T)^2], \tag{A.11}$$

$$V_{S_d^{(n)}}^2 = \ln E^{Q^T}[S_d^{(n)}(T,T)^2] - 2\ln E^{Q^T}[S_d^{(n)}(T,T)], \tag{A.12}$$

$$M_{S_f^{(m)}} = 2\ln E^{Q^T}[S_f^{(m)}(T,T)] - \frac{1}{2}\ln E^{Q^T}[S_f^{(m)}(T,T)^2], \tag{A.13}$$

$$V_{S_f^{(m)}}^2 = \ln E^{Q^T}[S_f^{(m)}(T,T)^2] - 2\ln E^{Q^T}[S_f^{(m)}(T,T)], \tag{A.14}$$

$E^{Q^T}[S_d^{(n)}(T,T)]$, $E^{Q^T}[S_d^{(n)}(T,T)^2]$, $E^{Q^T}[S_f^{(m)}(T,T)]$ and $E^{Q^T}[S_f^{(m)}(T,T)^2]$ are

derived as follows:

$$\begin{aligned} E^{Q^T}[S_d^{(n)}(T,T)] &= \sum_{i=0}^{n-1} w_d^{(i)}(0)E^{Q^T}[F_d(T,T_i)] \\ &= \sum_{i=0}^{n-1} w_d^{(i)}(0)F_d(0,T_i)\exp\left(\int_0^T \Delta_d^0(u,T_i;T)du\right), \end{aligned} \tag{A.15}$$

$$
\begin{aligned}
\mathrm{E}^{Q^r}[S_d^{(n)}(T,T)^2] &= \mathrm{E}^{Q^r}\left[\left(\sum_{i=0}^{n-1} w_d^{(i)}(0)F_d(T,T_i)\right)^2\right] \\
&= \sum_{i=0}^{n-1} w_d^{(i)}(0)^2 \mathrm{E}^{Q^r}[F_d(T,T_i)^2] \\
&\quad + 2\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1} w_d^{(i)}(0)w_d^{(j)}(0)\mathrm{E}^{Q^r}[F_d(T,T_i)F_d(T,T_j)],
\end{aligned}
\tag{A.16}
$$

where

$$
\mathrm{E}^{Q^r}[F_d(T,T_i)^2] = F_d(0,T_i)^2 \exp\left(\int_0^T 2\Delta_d^0(u,T_i;T)+\gamma_d(u,T_i)^2\,du\right),
\tag{A.17}
$$

$$
\mathrm{E}^{Q^r}[F_d(T,T_i)F_f(T,T_j)] = F_d(0,T_i)F_d(0,T_j)\exp\left(\int_0^T \xi_d^0(u,T_i,T_j;T)du\right),
\tag{A.18}
$$

$$
\xi_d^0(u,T_i,T_j;T) = \Delta_d^0(u,T_i;T)+\Delta_d^0(u,T_j;T)+\gamma_d(u,T_i)\cdot\gamma_d(u,T_j),
\tag{A.19}
$$

and

$$
\begin{aligned}
\mathrm{E}^{Q^r}[S_f^{(m)}(T,T)] &= \sum_{i=0}^{m-1} w_f^{(i)}(0)\mathrm{E}^{Q^r}[F_f(T,T_i)] \\
&= \sum_{i=0}^{m-1} w_f^{(i)}(0)F_f(0,T_i)\exp\left(\int_0^T \Delta_f^0(u,T_i;T)du\right),
\end{aligned}
\tag{A.20}
$$

$$
\begin{aligned}
\mathrm{E}^{Q^r}[S_f^{(m)}(T,T)^2] &= \mathrm{E}^{Q^r}\left[\left(\sum_{i=0}^{m-1} w_f^{(i)}(0)F_f(T,T_i)\right)^2\right] \\
&= \sum_{i=0}^{m-1} w_f^{(i)}(0)^2 \mathrm{E}^{Q^r}[F_f(T,T_i)^2] \\
&\quad + 2\sum_{i=0}^{m-2}\sum_{j=i+1}^{m-1} w_f^{(i)}(0)w_f^{(j)}(0)\mathrm{E}^{Q^r}[F_f(T,T_i)F_f(T,T_j)],
\end{aligned}
\tag{A.21}
$$

where

$$
\mathrm{E}^{Q^r}[F_f(T,T_i)^2] = F_f(0,T_i)^2 \exp\left(\int_0^T 2\Delta_f^0(u,T_i;T)+\gamma_f(u,T_i)^2\,du\right),
\tag{A.22}
$$

$$
\mathrm{E}^{Q^r}[F_f(T,T_i)F_f(T,T_j)] = F_f(0,T_i)F_f(0,T_j)\exp\left(\int_0^T \xi_f^0(u,T_i,T_j;T)du\right),
\tag{A.23}
$$

$$
\xi_f^0(u,T_i,T_j;T) = \Delta_f^0(u,T_i;T)+\Delta_f^0(u,T_j;T)+\gamma_f(u,T_i)\cdot\gamma_f(u,T_j),
\tag{A.24}
$$

For computing (A.10), we have to further derive

$$
\begin{aligned}
\Psi_1^2 &= \mathrm{Var}\left(\ln S_d^{(n)}(T,T)-\ln S_f^{(m)}(T,T)\right) \\
&= V_{S_d^{(n)}}^2 + V_{S_f^{(m)}}^2 - 2\mathrm{Cov}\left(\ln S_d^{(n)}(T,T),\ln S_f^{(m)}(T,T)\right).
\end{aligned}
\tag{A.25}
$$

While the covariance term in (A.24) is not analytically solvable, we may approximately derive it by replacing $S_d^{(n)}(T,T)$ and $S_f^{(m)}(T,T)$ by $G_d^{(n)}(T,T)$ and $G_f^{(m)}(T,T)$, respectively,

where

$$G_d^{(n)}(T,T) = \prod_{i=0}^{n-1} F_d(T,T_i)^{\omega_d^{(i)}(0)},$$

$$G_f^{(m)}(T,T) = \prod_{i=0}^{m-1} F_f(T,T_i^n)^{\omega_f^{(i)}(0)}.$$

It means that we use the geometric weighted average to replace the arithmetic weighted average which is also used in Vorst (1992).

The covariance term in (A.25) may be approximately derived as follows:

$$
\begin{aligned}
\mathrm{Cov}\left( \ln S_d^{(n)}(T,T), \ln S_f^{(m)}(T,T) \right) &= \mathrm{E}^{Q^f}\left[ \ln S_d^{(n)}(T,T)\ln S_f^{(m)}(T,T) \right] \\
&\quad - \mathrm{E}^{Q^f}\left[ \ln S_d^{(n)}(T,T) \right]\mathrm{E}^{Q^f}\left[ \ln S_f^{(m)}(T,T) \right] \\
&\cong \mathrm{E}^{Q^f}\left[ \ln G_d^{(n)}(T,T)\ln G_f^{(m)}(T,T) \right] \\
&\quad - \mathrm{E}^{Q^f}\left[ \ln G_d^{(n)}(T,T) \right]\mathrm{E}^{Q^f}\left[ \ln G_f^{(m)}(T,T) \right].
\end{aligned}
$$

Thus,

$$\mathrm{Cov}\left( \ln S_d^{(n)}(T,T), \ln S_f^{(m)}(T,T) \right) \cong \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} w_d^{(i)}(0)w_f^{(j)}(0)\mathrm{Cov}(\ln F_d(T,T_i), \ln F_f(T,T_j)) \qquad (A.26)$$

$$= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} w_d^{(i)}(0)w_f^{(j)}(0)\int_0^T \gamma_d(u,T_i)\cdot\gamma_f(u,T_j)du.$$

By observing equation (A.26), the covariance between $\ln S_d^{(n)}(T,T)$ and $\ln S_f^{(m)}(T,T)$ can be approximately considered as the weighted covariances between $\ln F_d(T,T_i)$ and $\ln F_f(T,T_j)$.

With the aforementioned knowledge, the pricing formula of the $QIREO_1$ can be priced via Lemma 1.

# Appendix B: The Market Data

Tables 8 ~ 12 are drawn from the DataStream database which are used for the numerical examples in Section IV.

## Table 8: The Domestic Cap Volatilities Quoted in the U.S. Market

The quoted volatilities of the U.S. caps over the past two years are quarterly presented in this table.

| Year | 2007/12/3 | 2007/9/3 | 2007/6/1 | 2007/3/1 | 2006/12/1 | 2006/9/1 | 2006/6/1 | 2006/3/1 |
|------|-----------|----------|----------|----------|-----------|----------|----------|----------|
| 1 | 26.31 | 22.00 | 7.36 | 11.68 | 12.08 | 10.58 | 10.71 | 9.69 |
| 2 | 31.37 | 23.63 | 11.07 | 15.87 | 17.60 | 14.62 | 13.42 | 13.43 |
| 3 | 30.79 | 22.83 | 12.52 | 16.75 | 18.62 | 16.02 | 14.83 | 15.51 |
| 4 | 29.12 | 21.79 | 13.23 | 17.03 | 19.04 | 16.67 | 15.55 | 16.58 |
| 5 | 27.47 | 20.81 | 13.56 | 17.04 | 19.12 | 17.01 | 15.99 | 17.17 |
| 6 | 26.07 | 20.12 | 13.77 | 16.95 | 19.08 | 17.23 | 16.24 | 17.58 |
| 7 | 24.91 | 19.60 | 13.85 | 16.8 | 19.00 | 17.28 | 16.32 | 17.79 |
| 8 | 23.95 | 19.11 | 13.87 | 16.64 | 18.91 | 17.28 | 16.34 | 17.93 |
| 9 | 23.10 | 18.69 | 13.85 | 16.45 | 18.74 | 17.22 | 16.33 | 17.96 |
| 10 | 22.30 | 18.26 | 13.79 | 16.26 | 18.55 | 17.12 | 16.26 | 17.95 |

## Table 9: The Foreign Cap Volatilities Quoted in the U.K. Market

The quoted volatilities of the U.K. caps over the past two years are quarterly presented in this table.

| Year | 2007/12/3 | 2007/9/3 | 2007/6/1 | 2007/3/1 | 2006/12/1 | 2006/9/1 | 2006/6/1 | 2006/3/1 |
|------|-----------|----------|----------|----------|-----------|----------|----------|----------|
| 1 | 16.59 | 12.95 | 6.87 | 6.71 | 7.25 | 8.95 | 10.05 | 8.49 |
| 2 | 17.85 | 13.6 | 8.72 | 8.83 | 10.24 | 11.3 | 12.05 | 10.94 |
| 3 | 17.12 | 13.5 | 9.48 | 9.59 | 11.17 | 12.38 | 12.86 | 12.21 |
| 4 | 16.32 | 13.22 | 9.9 | 10.01 | 11.81 | 12.9 | 13.2 | 13.05 |
| 5 | 15.57 | 12.92 | 10.2 | 10.31 | 12.27 | 13.29 | 13.46 | 13.62 |
| 6 | 15 | 12.7 | 10.44 | 10.56 | 12.63 | 13.54 | 13.62 | 14.07 |
| 7 | 14.55 | 12.54 | 10.64 | 10.76 | 12.91 | 13.72 | 13.74 | 14.38 |
| 8 | 14.2 | 12.43 | 10.81 | 10.93 | 13.12 | 13.88 | 13.83 | 14.6 |
| 9 | 13.92 | 12.34 | 10.95 | 11.08 | 13.31 | 14.01 | 13.9 | 14.78 |
| 10 | 13.69 | 12.27 | 11.07 | 11.19 | 13.45 | 14.1 | 13.96 | 14.92 |

# Table 10: Initial Domestic Forward LIBOR Rates

The forward LIBOR rates in the U.S. (domestic) market over the past two years are quarterly represented in this table. The rates are obtained from the associated bond prices derived from the zero curves obtained in DataStream.

| Year | 2007/12/3 | 2007/9/3 | 2007/6/1 | 2007/3/1 | 2006/12/1 | 2006/9/1 | 2006/6/1 | 2006/3/1 |
|---|---|---|---|---|---|---|---|---|
| $L_d(\cdot,0.5)$ | 0.0493 | 0.0564 | 0.0560 | 0.0548 | 0.0547 | 0.0562 | 0.0558 | 0.0515 |
| $L_d(\cdot,1)$ | 0.0369 | 0.0478 | 0.0559 | 0.0527 | 0.0503 | 0.0541 | 0.0564 | 0.0529 |
| $L_d(\cdot,1.5)$ | 0.0371 | 0.0489 | 0.0551 | 0.0507 | 0.0484 | 0.0521 | 0.0563 | 0.0522 |
| $L_d(\cdot,2)$ | 0.0331 | 0.0468 | 0.0545 | 0.0487 | 0.0455 | 0.0498 | 0.0564 | 0.0521 |
| $L_d(\cdot,2.5)$ | 0.0397 | 0.0494 | 0.0551 | 0.0500 | 0.0478 | 0.0519 | 0.0564 | 0.0518 |
| $L_d(\cdot,3)$ | 0.0402 | 0.0493 | 0.0549 | 0.0493 | 0.0469 | 0.0513 | 0.0564 | 0.0515 |
| $L_d(\cdot,3.5)$ | 0.0434 | 0.0511 | 0.0557 | 0.0509 | 0.0484 | 0.0530 | 0.0573 | 0.0521 |
| $L_d(\cdot,4)$ | 0.0449 | 0.0518 | 0.0558 | 0.0508 | 0.0483 | 0.0532 | 0.0575 | 0.0520 |
| $L_d(\cdot,4.5)$ | 0.0461 | 0.0531 | 0.0565 | 0.0518 | 0.0496 | 0.0543 | 0.0584 | 0.0523 |
| $L_d(\cdot,5)$ | 0.0478 | 0.0541 | 0.0567 | 0.0520 | 0.0498 | 0.0547 | 0.0588 | 0.0523 |
| $L_d(\cdot,5.5)$ | 0.0491 | 0.0551 | 0.0568 | 0.0528 | 0.0502 | 0.0543 | 0.0586 | 0.0528 |
| $L_d(\cdot,6)$ | 0.0509 | 0.0562 | 0.0570 | 0.0531 | 0.0506 | 0.0546 | 0.0589 | 0.0528 |
| $L_d(\cdot,6.5)$ | 0.0508 | 0.0557 | 0.0578 | 0.0534 | 0.0507 | 0.0552 | 0.0596 | 0.0526 |
| $L_d(\cdot,7)$ | 0.0526 | 0.0567 | 0.0581 | 0.0537 | 0.0511 | 0.0556 | 0.0600 | 0.0526 |
| $L_d(\cdot,7.5)$ | 0.0520 | 0.0570 | 0.0582 | 0.0536 | 0.0517 | 0.0554 | 0.0599 | 0.0523 |
| $L_d(\cdot,8)$ | 0.0537 | 0.0581 | 0.0586 | 0.0539 | 0.0521 | 0.0557 | 0.0602 | 0.0522 |
| $L_d(\cdot,8.5)$ | 0.053 | 0.0576 | 0.0587 | 0.0546 | 0.0518 | 0.0564 | 0.0600 | 0.0528 |
| $L_d(\cdot,9)$ | 0.0545 | 0.0585 | 0.0590 | 0.0550 | 0.0522 | 0.0569 | 0.0603 | 0.0528 |
| $L_d(\cdot,9.5)$ | 0.0538 | 0.0583 | 0.0593 | 0.0559 | 0.0523 | 0.0565 | 0.0612 | 0.0536 |
| $L_d(\cdot,10)$ | 0.0552 | 0.0592 | 0.0597 | 0.0564 | 0.0527 | 0.0569 | 0.0615 | 0.0536 |

# Table 11: Initial Foreign Forward LIBOR Rates

The forward LIBOR rates in the U.K. (foreign) market over the past two years are quarterly represented in this table. The rates are obtained from the associated bond prices derived from the zero curves obtained in DataStream.

| Year | 2007/12/3 | 2007/9/3 | 2007/6/1 | 2007/3/1 | 2006/12/1 | 2006/9/1 | 2006/6/1 | 2006/3/1 |
|---|---|---|---|---|---|---|---|---|
| $L_f(\cdot,0.5)$ | 0.0651 | 0.0686 | 0.0616 | 0.0580 | 0.0546 | 0.0521 | 0.0492 | 0.0471 |
| $L_f(\cdot,1)$ | 0.0546 | 0.0638 | 0.0642 | 0.0589 | 0.0546 | 0.0529 | 0.0515 | 0.0474 |
| $L_f(\cdot,1.5)$ | 0.0517 | 0.0617 | 0.0642 | 0.0580 | 0.0538 | 0.0511 | 0.0530 | 0.0484 |
| $L_f(\cdot,2)$ | 0.0525 | 0.0614 | 0.0635 | 0.0575 | 0.0535 | 0.0556 | 0.0538 | 0.0489 |
| $L_f(\cdot,2.5)$ | 0.0532 | 0.0610 | 0.0626 | 0.0565 | 0.0528 | 0.0521 | 0.0533 | 0.0485 |
| $L_f(\cdot,3)$ | 0.0520 | 0.0598 | 0.0623 | 0.0559 | 0.0522 | 0.0517 | 0.0539 | 0.0487 |
| $L_f(\cdot,3.5)$ | 0.0526 | 0.0596 | 0.0607 | 0.0550 | 0.0514 | 0.0512 | 0.0528 | 0.0475 |
| $L_f(\cdot,4)$ | 0.0519 | 0.0586 | 0.0600 | 0.0543 | 0.0508 | 0.0508 | 0.0529 | 0.0473 |
| $L_f(\cdot,4.5)$ | 0.0520 | 0.0585 | 0.0589 | 0.0535 | 0.0502 | 0.0505 | 0.0521 | 0.0464 |
| $L_f(\cdot,5)$ | 0.0515 | 0.0577 | 0.0582 | 0.0528 | 0.0495 | 0.0501 | 0.0520 | 0.0461 |
| $L_f(\cdot,5.5)$ | 0.0519 | 0.0571 | 0.0571 | 0.0520 | 0.0489 | 0.0499 | 0.0517 | 0.0459 |
| $L_f(\cdot,6)$ | 0.0515 | 0.0563 | 0.0564 | 0.0514 | 0.0482 | 0.0495 | 0.0516 | 0.0456 |
| $L_f(\cdot,6.5)$ | 0.0514 | 0.0556 | 0.0555 | 0.0507 | 0.0475 | 0.0492 | 0.0511 | 0.0452 |
| $L_f(\cdot,7)$ | 0.0510 | 0.0547 | 0.0546 | 0.0500 | 0.0468 | 0.0488 | 0.0509 | 0.0449 |
| $L_f(\cdot,7.5)$ | 0.0507 | 0.0542 | 0.0543 | 0.0492 | 0.0464 | 0.0482 | 0.0502 | 0.0443 |
| $L_f(\cdot,8)$ | 0.0503 | 0.0533 | 0.0534 | 0.0485 | 0.0457 | 0.0478 | 0.0500 | 0.0440 |
| $L_f(\cdot,8.5)$ | 0.0500 | 0.0529 | 0.0529 | 0.0479 | 0.0454 | 0.0474 | 0.0495 | 0.0435 |
| $L_f(\cdot,9)$ | 0.0496 | 0.0520 | 0.0521 | 0.0472 | 0.0447 | 0.0469 | 0.0492 | 0.0432 |
| $L_f(\cdot,9.5)$ | 0.0496 | 0.0521 | 0.0519 | 0.0470 | 0.0443 | 0.0465 | 0.0489 | 0.0428 |
| $L_f(\cdot,10)$ | 0.0492 | 0.0511 | 0.0512 | 0.0464 | 0.0437 | 0.0461 | 0.0486 | 0.0424 |

## Table 12: The Five-factor B Matrix

Continued.

| | Factor 1 | Factor 2 | Factor 3 | Factor 4 | Factor 5 |
|---|---|---|---|---|---|
| $L_d(\cdot, 0.5)$ | 0.4665 | 0.8176 | 0.1074 | 0.0880 | 0.3077 |
| $L_d(\cdot, 1)$ | 0.1158 | 0.8970 | 0.0020 | 0.4141 | -0.1023 |
| $L_d(\cdot, 1.5)$ | 0.1789 | 0.8499 | -0.1561 | 0.4275 | -0.1962 |
| $L_d(\cdot, 2)$ | 0.1418 | 0.7779 | -0.2446 | 0.4687 | -0.3086 |
| $L_d(\cdot, 2.5)$ | 0.3135 | 0.7540 | -0.3290 | 0.4076 | -0.2425 |
| $L_d(\cdot, 3)$ | 0.3655 | 0.6628 | -0.4326 | 0.3947 | -0.2901 |
| $L_d(\cdot, 3.5)$ | 0.4236 | 0.6583 | -0.4469 | 0.3672 | -0.2295 |
| $L_d(\cdot, 4)$ | 0.4889 | 0.5497 | -0.5412 | 0.3256 | -0.2447 |
| $L_d(\cdot, 4.5)$ | 0.5087 | 0.5503 | -0.5488 | 0.3128 | -0.1982 |
| $L_d(\cdot, 5)$ | 0.5665 | 0.4292 | -0.6285 | 0.2513 | -0.1917 |
| $L_d(\cdot, 5.5)$ | 0.6251 | 0.4366 | -0.5841 | 0.2377 | -0.1445 |
| $L_d(\cdot, 6)$ | 0.6803 | 0.3055 | -0.6362 | 0.1610 | -0.1152 |
| $L_d(\cdot, 6.5)$ | 0.6874 | 0.3243 | -0.6234 | 0.1656 | -0.0794 |
| $L_d(\cdot, 7)$ | 0.7251 | 0.1923 | -0.6546 | 0.0853 | -0.0382 |
| $L_d(\cdot, 7.5)$ | 0.7236 | 0.2684 | -0.6190 | 0.1359 | -0.0516 |
| $L_d(\cdot, 8)$ | 0.7529 | 0.1540 | -0.6367 | 0.0626 | -0.0128 |
| $L_d(\cdot, 8.5)$ | 0.7572 | 0.2055 | -0.6145 | 0.0792 | -0.0229 |
| $L_d(\cdot, 9)$ | 0.7788 | 0.0989 | -0.6192 | 0.0080 | 0.0147 |
| $L_d(\cdot, 9.5)$ | 0.7620 | 0.1761 | -0.6182 | 0.0773 | 0.0143 |
| $L_d(\cdot, 10)$ | 0.7769 | 0.0690 | -0.5978 | 0.0416 | 0.0285 |

## Table 12: The Five-factor B Matrix

The matrix B is computed based on the correlation matrix of the relevant variables calculated from the 2-year data (2006/01/02 - 2007/12/31 ).

| | Factor 1 | Factor 2 | Factor 3 | Factor 4 | Factor 5 |
|---|---|---|---|---|---|
| $L_f(\cdot, 0.5)$ | 0.6984 | -0.4938 | 0.2460 | -0.4495 | 0.0768 |
| $L_f(\cdot, 1)$ | 0.8953 | -0.0615 | 0.3845 | -0.2165 | 0.0012 |
| $L_f(\cdot, 1.5)$ | 0.9349 | 0.0798 | 0.3209 | -0.0982 | -0.0831 |
| $L_f(\cdot, 2)$ | 0.9503 | 0.0564 | 0.2410 | -0.1270 | -0.1397 |
| $L_f(\cdot, 2.5)$ | 0.9663 | -0.0347 | 0.2163 | -0.1010 | -0.0897 |
| $L_f(\cdot, 3)$ | 0.9789 | 0.0320 | 0.1598 | -0.0435 | -0.1156 |
| $L_f(\cdot, 3.5)$ | 0.9785 | -0.0660 | 0.1663 | -0.0825 | -0.0613 |
| $L_f(\cdot, 4)$ | 0.9884 | -0.0452 | 0.1222 | -0.0467 | -0.0617 |
| $L_f(\cdot, 4.5)$ | 0.9854 | -0.0964 | 0.1216 | -0.0654 | -0.0251 |
| $L_f(\cdot, 5)$ | 0.9912 | -0.0943 | 0.0825 | -0.0367 | -0.0200 |
| $L_f(\cdot, 5.5)$ | 0.9875 | -0.1449 | 0.0390 | -0.0466 | 0.0133 |
| $L_f(\cdot, 6)$ | 0.9879 | -0.1525 | -0.0067 | -0.0168 | 0.0233 |
| $L_f(\cdot, 6.5)$ | 0.9806 | -0.1886 | -0.0213 | -0.0293 | 0.0398 |
| $L_f(\cdot, 7)$ | 0.9754 | -0.2012 | -0.0723 | -0.0014 | 0.0534 |
| $L_f(\cdot, 7.5)$ | 0.9710 | -0.2177 | -0.0666 | -0.0031 | 0.0727 |
| $L_f(\cdot, 8)$ | 0.9618 | -0.2289 | -0.1156 | 0.0260 | 0.0922 |
| $L_f(\cdot, 8.5)$ | 0.9541 | -0.2569 | -0.1237 | 0.0125 | 0.0907 |
| $L_f(\cdot, 9)$ | 0.9396 | -0.2727 | -0.1727 | 0.0393 | 0.1063 |
| $L_f(\cdot, 9.5)$ | 0.9392 | -0.3000 | -0.1309 | * 0.0025 | 0.1039 |
| $L_f(\cdot, 10)$ | 0.9266 | -0.3147 | -0.1674 | 0.0199 | 0.1179 |
| X | 0.6943 | -0.3685 | 0.2286 | -0.5444 | 0.1833 |

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interest rate) and settled in the domestic currency. Depending on the needs for controlling the interest rate risks due to relative change in the shapes of the yield curves of two currencies, the interest rates involved in the difference may be a (domestic and foreign) short rate, an intermediate rate and a long rate. Some specific applications of QIREOs are described as follows.

In the global financial market, international companies may raise funds from one currency and make an investment in another currency and thus their assets and liabilities may be denominated in different currencies. In order to hedge interest rate risks of the two currencies, they can use QIREOs to lock in a current interest rate spread. Second, as investors find an abnormal interest rate spread between yield curves of two currencies, they can employ QIREOs to enhance profits from a change in the spread. Third, as compared with differential swaps and quanto constant maturity swaps, QIREOs are more flexible to meet specific trading strategies. They can be designed to provide payoffs that depend on whether the spread of two interest rates in different currencies is above or below a specified level, or within or outside a specified range on a specific date in the future. Fourth, QIREOs can be used as ancillary instruments for differential swaps and quanto constant maturity swaps. For example, end-users can use differential swaps to capitalize on anticipated yield curve movements of two currencies while purchasing QIREOs to eliminate the downside risk.

There are several past researches on the pricing of interest rate exchange options (IREOs). Within the framework of an extended version of the Cox, Ingersoll and Ross (1985) interest rate model, Longstaff (1990) priced IREOs on a yield spread. Fu (1996) derived the pricing formulas for the yield-spread options within the Gaussian Heath, Jarrow and Morton (1992, HJM) framework and for the exchange options on two LIBOR rates with different tenors. Miyazaki and Yoshida (1998) also derived the pricing formulas for the yield-spread options within the multi-factor Gaussian HJM (1992) model. Wu and Chen (2009) priced three types of IREOs within the multi-factor LIBOR market model (LMM) and pointed out that the pricing formula of LIBOR-for-LIBOR IREOs have a drift effect which is different to the result given in Fu (1996).

As mentioned previously, QIREOs have become popular in recent years while to our best knowledge no studies have ever tried to derive the pricing formulas of QIREOs in the previous literature. The main purpose of this article is to price QIREOs under the cross-currency LIBOR market model (CLMM). It is well-known that the LMM and the swap market model (hereafter, SMM) are not compatible and thus the distribution of swap rates within the LMM framework is unknown. As another contribution of this paper, a new approach is presented to approximate the distribution of foreign swap rates in the CLMM. Under this approximation approach, the resulting pricing formulas are shown to be sufficiently accurate by comparing with Monte Carlo simulation.

The LMM was developed by Musiela and Rutkowski (1997), Miltersen, Sandmann and Sondermann (1997), and Brace, Gatarek and Musiela (1997, BGM). Schlögl (2002) and Wu and Chen (2007) extended, respectively in different approaches, the LMM from a one-currency framework to a cross-currency case (called the CLMM). The CLMM has been widely used by market practitioners to price quanto interest rate derivatives due to the following advantages. The LIBOR rates specified in the CLMM are market-observable, which makes the calibration for current yield curve automatically. Furthermore, the pricing formulas for caps and floors within the CLMM framework are the Black's formula which has been consistent with market practice and also makes the calibration for volatilities easier.

Moreover, the distribution of the LIBOR rate in the CLMM is a lognormal distribution rather than a Gaussian distribution, which avoids the pricing error due to the negative rates with positive probabilities, induced by the traditional Gaussian interest rate models.[1] In addition, most popular and actively-traded quanto interest rate products can be priced within the CLMM framework so that interest rate risk management can be conducted consistently and efficiently.

This paper is organized as follows. Section II reviews the CLMM and introduces a change-of-measure technique and some useful approximation methods. Section III presents five types of QIREOs and their pricing formulas.

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[1] As examined in Rogers (1996), the Gaussian term structure model has an important theoretical limitation: the rate can attain negative values with positive probability which may cause some pricing error in many cases.

In Section IV, we examine the accuracy of the approximate formulas via Monte Carlo simulation. The conclusion is made in the last section.

## II · The CLMM and an Approximation for Swap Rates

This section reviews briefly the CLMM, a drift-adjustment technique associated with a change of measure and a lognormalization technique for LIBOR rates under different measure. Moreover, we also introduce a new lognormalization approach for swap rates under the CLMM.

### 1. Review of the CLMM

Assume that trading takes place continuously in time over an interval $[0,\Im]$, $0 < \Im < \infty$. The uncertainty is described by the filtered spot martingale probability space $(\Omega, F, Q, \{F_t\}_{t\in[0,\Im]})$ and a $d$-dimensional independent standard Brownian motion $Z(t) = (Z_1(t), Z_2(t), \ldots, Z_d(t))$ is defined on the probability space. The flow of information accruing to all agents in the economy is represented by the filtration $\{F_t\}_{t\in[0,\Im]}$ which satisfies the usual hypotheses.[2] Note that $Q$ denotes the domestic spot martingale probability measure and $k$ denotes the country index with $k = d$ for domestic and $k = f$ for foreign. The notations are introduced as follows:

$F_k(t,T)$ = the kth country's forward LIBOR rate contracted at time $t$ for borrowing and lending during the period $[T, T+\delta]$ with $0 \le t \le T \le \Im$, where $k \in \{d, f\}$.

$B_k(t,T)$ = the time $t$ price of the kth country's zero coupon bond paying one dollar at time $T$.

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[2] The filtration $\{F_t\}_{t\in[0,\Im]}$ is right continuous and $F_0$ contains all the $Q$-null sets of $F$.

$r_k(t)$ = the kth country's risk-free short rate at time $t$.

$X(t)$ = the spot exchange rate at time $t \in [0, \Im]$ for one unit of the foreign currency expressed in terms of the domestic currency.

$Q^T$ = the domestic forward martingale measure with respective to the numéraire $B_d(\cdot, T)$.

The relationship between $F_k(t,T)$ and $B_k(t,T)$ can be expressed as follows:

$$F_k(t,T) = \frac{1}{\delta}(B_k(t,T) - B_k(t,T+\delta)) / B_k(t,T+\delta). \tag{1}$$

Based on the arbitrage-free conditions in Amin and Jarrow (1991), Wu and Chen (2007) extended the LMM to the CLMM and their results are briefly specified as follows.

**Proposition 1.** The CLMM under the Martingale Measure $Q$

*Under the domestic spot martingale measure $Q$, the processes of the forward LIBOR rates and the exchange rate are given as follows:*

$$\frac{dF_d(t,T)}{F_d(t,T)} = \gamma_{Fd}(t,T) \cdot \sigma_{Bd}(t,T+\delta)dt + \gamma_{Fd}(t,T) \cdot dZ(t), \tag{2}$$

$$\frac{dF_f(t,T)}{F_f(t,T)} = \gamma_{Ff}(t,T) \cdot (\sigma_{Bf}(t,T+\delta) - \sigma_X(t))dt + \gamma_{Ff}(t,T) \cdot dZ(t), \tag{3}$$

$$\frac{dX(t,T)}{X(t,T)} = (r_d(t) - r_f(t))dt + \sigma_X(t) \cdot dZ(t). \tag{4}$$

*where $\sigma_{Bk}(t,T), k \in \{d, f\}$ is defined as follows:*

$$\sigma_{Bk}(t,T) = \begin{cases} \sum\limits_{j=1}^{\lfloor \delta^{-1}(T-t) \rfloor} \dfrac{\delta F_k(t,T-j\delta)}{1+\delta F_k(t,T-j\delta)} \gamma_{Fk}(t,T-j\delta), & \begin{array}{l} t \in [0, T-\delta] \\ \& \, T-\delta > 0, \end{array} \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$