Reasonable royalty and the division of profit for probabilistic patents[*]

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Abstract

We analyze the incentive to innovate for a patent holder under probabilistic patents. In a sequential innovation framework, actual patent infringement occurs and there exists uncertainty in litigation with the damage award been adjudicated as a reasonable royalty. It is demonstrated that complete profit transfer from the second generation innovator is feasible for medium degree of the second generation innovation. Whereas minuscule and significant innovation induces no pursuit of patent protection through infringement suits and hence no profit transfer. We discuss the effects of varying patent breadth, either by an adjustment in the written law or by application of the doctrine of the equivalents.

I. INTRODUCTION

The division of profit between sequential innovators has attracted recently a great deal of attention not only from economists but also from the legal experts. Despite the fact that economists have been working on issues of intellectual property rights for a long time, the approach taken by the "patent race" literature so far cannot identify all the complex effects of patent protection when innovation is cumulative and the ensuing intellectual property rights are probabilistic (Lemley and Shapiro, 2005). An important research agenda aiming for critical assessment of patents has been pursued by Green and Scotchmer (1995). Their work examines the issue of optimal patent breadth and duration, and the role of different legal mechanisms for cumulative innovations. What emerges from their analysis as the major patent policy concern is the lack of incentive to innovate for the potential patent holder in the shadow of incomplete transfer of social value facilitated by basic research. Hence longer-lasting patents are called for as the remedy for insufficient innovation incentive.

Green and Scotchmer (1995) confine themselves to the "fencepost" interpretation of the patent system upon which majority of the patent literature has been built. For example, Hortsmann et al (1995) study the propensity of a patent for a successful innovator (the sole winner of a patent race) by specifying a "limited but exact coverage" patent system. Almost by definition, important policy implications regarding the litigation process cannot be addressed within a perfect fencepost system. In reality, however, patent breadth is *de facto* a matter of Patent Office and court interpretation.

This article aims to fill the void left by the fencepost patent literature and further dwell into the following important features of probabilistic patents.

Uncertainty in Patent Litigation. Lemley and Shapiro (2005) recently emphasize that "uncertainty about validity and scope are critical when studying the enforcement and litigation of patents." Our "signpost" interpretation of the patent system takes into account the fact that patent protection is imperfect. The patent system often cannot circumscribe its objects, individual patents, in a precise and unquestionable way in the phases of granting and enforcing patents. For instance, a patent is described by Kitti (1978) as a "lottery ticket".

In an earlier attempt to capture this aspect of reality, Waterson (1990) looks at uncertainty in patent infringement litigation and employed the concept of "limited but inexact patent coverage" in a horizontal product differentiation model. Pursuing this line of research one step further in a vertical product differentiation model, Chou and

Haller (2007) find that when the inherent uncertainty of a paten infringement case is taken into consideration, the division of profit between sequential innovators will depend on the degree of the improvement made by the subsequent innovator. In particular, within a wide range of model parameters, full rent extraction by the initial patent holder is possible.

Uncertainty in litigation is a common feature of our papers and the litigation literature. Also, the litigation process assumed here and in Chou and Haller (2007) is similar to the one most frequently used in the litigation literature. Among the precedents are Reinganum and Wilde (1986), Meurer (1989), and Aoki and Hu (1995). In their models, the plaintiff (the harmed party) makes a settlement offer to the defendant (the party who inflicts harm). The defendant then responds by either accepting the offer (take-it) or refusing the settlement proposal (leave-it) after which a court action may be taken. The most crucial modeling differences between Chou and Haller (2007) and the present paper on the one side and the aforementioned litigation literature on the other side are twofold. First of all, the litigation literature focuses on asymmetric information between patentees and infringes about costs (harm, damages) or benefits (surplus, profits).[1] In the presence of asymmetric information, sequential equilibrium is the predominant solution concept. In contrast, our model assumes symmetric information hence renders subgame perfect equilibrium as the appropriate solution concept. Secondly, the conventional litigation literature works with a given invention and a fixed probability that the patentee wins a patent infringement suit whereas in our approach, the degree of improvement derived from a basic innovation determines the patentee's probability of winning infringement litigation.

Asymmetric Bargaining Power. Green and Scotchmer (1995) assume that any agreement between the two parties means a equal split of the available surplus among them. Nevertheless anecdotal evidences and the more frequent use of preliminary injunction point to a paradigmatically shift toward empowering patent holders with substantial bargaining power. For instance, the Kodak-Polaroid dispute has been touted as the "most prominent example of an increasingly pro-patent sentiment in American courts".[2] In terms of procedural and doctrinal changes, the new attitude manifests itself through greater leniency of granting preliminary injunctions. Since a credible threat is at work in halting the subsequent innovator's entire operation by means of the injunctive measure, a scenario in which patent holder makes the first move with a take-it-or-leave-it licensing offer seems worth exploring.

Royalties and Damage. The model of the present paper differs from Chou and Haller (2007) in an unobtrusive, but important detail: how profit is divided, if the

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[1] See, e.g., Bebchuck (1984), Reinganum and Wilde (1986), and Meurer (1989).
[2] Schmitt, E. "Business and the Law: Judicial Shift in Patent Cases," *New York Times*, Jan. 21, 1986.

patent holder prevails in court. In Chou and Haller (2007), after rendering the verdict of infringement, the court always grants the patent holder what she has demanded as a licensing fee at the very beginning of bargaining. By and large, the analysis of that model remains robust by the additional stipulation that the court-imposed royalty must not exceed the total profit. Note that, prior to 1946 a patent holder was allowed to choose between the amount of damages she suffered and the amount of profit earned by the infringer. Since then, it is the courts rather than the plaintiffs who determine royalties and damages paid by infringers. One can argue, though, that in recent years the patent system has been leaning towards patent holders more heavily than ever (Dreyfuss, 1989; Merges, 1995).

If determination of royalties and damages is left to the courts' discretion, then the issue of reasonable royalty arises and ought to be addressed. Since 1946, several doctrines of reasonable royalty have been applied.[3] In the current model, we postulate that after a verdict of infringement the court awards a royalty equal to the profit made by the infringer. Given the other specifications of our model, this assumption proves consistent with the prevailing doctrines. One may contemplate that if the court is capable of assessing the right amount of reasonable royalty, the assumption of uncertain or unsound judgment associated with the legal system could render unjustifiable. Notice, however, that both in our model and in practice the sequencing in infringement suits is such that damages are awarded after the finding of infringement. Therefore, we consider the two functions of the court, verifying the validity of a patent and ruling on infringement versus awarding the damages, as two distinct events.[4] Whereas the model assumes uncertainty about who wins an infringement suit, it does not postulate stochastic damages in case the court finds the defendant guilty of infringement: the court, according to the *ex post* factual findings, always grants the consistent and accurate reasonable royalty.[5]

Doctrine of Equivalents. It has long been recognized that patent breadth is a viable instrument to influence inventors' incentives to innovate. Though not explicitly stated, theorists interested in the design of an optimal patent system seem to suggest establishing the optimal patent breadth through an adjustment in the written law (Klemperer, 1990; Gilbert and Shapiro, 1990; Green and Scotchmer, 1995). However, patent protection can also be expanded (or conversely, contracted) overtime through legal doctrines without resorting to specific modification of the Patent Codes. In particular, the applications of the doctrine of equivalents and the reverse doctrine of

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[3] We provide a brief survey in appendix A.
[4] Typically, these functions are even exercised by two different courts, the CAFC and trial courts, respectively.
[5] Our theoretic underpinng of the reasonable royalty is also consistent with Hypothesis 1 (circularity of damage doctrines and licensing offer) discussed in Schankerman and Scotchmer (2001:205).

equivalents constitute important instances of such flexibility. The doctrine of equivalents apply to the patent representing a "pioneer invention" – which the US Supreme Court has defined as "a patent concerning a function never before performed, a wholly novel device, or one of such novelty and importance as to make a distinct step in the progress in the art," [*Boyden Power-Brake Co. v. Westinghouse*, 170 U.S. 537, 569 (1898)]. As a symmetric counterpart to the doctrine of equivalents, the reverse doctrine of equivalents "is and equitable doctrine invoked in applying properly construed claims to an accused device. Just as the purpose of the 'doctrine of equivalents' is to prevent 'pirating' of the 'reverse doctrine of equivalents' is to prevent unwarranted extension of the claim beyond a fair scope of the patentee's invention" [*Scripps Clinic & Research Fund. v. Genetech, Inc.*, 927 F.2d 1565, 18 U.S.P.Q.2d (BNA) 1001, 18 U.S.P.Q.2d (BNA) 1896 (Fed. Cir. 1991)].

Summary of Results. Three major results in Chou and Haller (2007) are still obtained under the new specification of reasonable royalty. That is, (1) a complete profit transfer equilibrium is attainable under a wide range of model parameters, (2) higher patent infringement litigation cost may dampen the patent holder's incentive to innovate, and (3) a broader patent breadth may not unreservedly improve the patent holder's incentive to innovate.

Unlike in Chou and Haller (2007) where technical intractability forced us to focus exclusively on the sub-interval where a complete-profit-transfer equilibrium exists, we are now able to partition the interval of proclaimed patent protection into three areas that can be characterized by three types of equilibria: complete-profit-transfer Take-it equilibrium, incomplete-profit-transfer Take-it equilibrium, and No-Action equilibrium, respectively. The strength (effectiveness) of the patent system can thus be more comprehensively measured. The range of innovation which corresponds to the No-Action equilibria provides the strongest incentive to the infringing firm to "trespass" the intellectual property of the patent holder. This area consists of an interval adjacent to the basic innovation and an interval adjacent to the upper end point of the interval of proclaimed patent protection. In other words, "inventing around" (Gallini, 1992) with close imitation or "inventing enough" with quite a novel (though still infringing) product may be observed in our model without the patent holder taking legal action[6].

In addition to the length of the complete-profit-transfer interval, the new framework allows for computing alternative measures of the efficacy of the patent system in transferring profits: the average expected payoff for the patent holder, both

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[6] Note that in Gallini (1992) imitation resulting from inventing around is costly but always non-infringing.

in absolute terms and percentage to average expected profit.

The paper is organized as follows. In Section 2 we set forth the model to be used to investigate the profit division between sequential innovation. In Section 3 we perform equilibrium analysis. Section 4 presents the comparative statics of raising and lowering infringement litigation costs. Moreover, we look at the instruments used by the courts to broaden (shorten) the patent breadth: the doctrine of equivalents and the reverse doctrine of equivalents. Some concluding remarks are offered in Section 5. Appendix A elaborates on doctrines of reasonable royalty. Appendix B contains more technical derivations of results.

Ⅱ. The Model

We begin with a development of the basic model. There are one research institution and one firm. The research institution has acquired a patent on its invention with quality x and is called the patent holder (PH) hereafter. We set $x = 0$ without loss of generality. Through disclosure and examination processes, a patent breadth y^* is granted and becomes publicly known. If the firm subsequently develops a product of quality $x + y$ with $y \in [0, y^*]$ then this product is perceived by PH as infringing upon the patent x. Quality x is just a basic research outcome and has no market value *per se*.[7] We analyze the cases where a commercially profitable product with quality y whereas $x \leq y \leq y^*$ has been developed by the firm and thus infringement occurs.[8] The cost of developing quality y is c_y. Once developed, the new product can be produced at zero cost and has potential market value π_y.

The crucial elements of patent litigation can be described as follows: We assume that each party incurs the same litigation cost $L > 0$ for the ease of notations. An objective probability $f(y)$ of PH winning the litigation is assumed to feature imperfect patent protection.[9] While both parties may agree privately whether or not

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[7] This type of invention fits into the categories of "research tools" (Schankerman and Scotchmer, 2001) and "essential inventions" (Encaoua et al., 2006) in the patent literature. As Chou and Haller (2007) find at the industry level, it is a rather typical aspect of R&D in the sense that a commercially profitable innovation often results from basic research.

[8] Although Lemley and Shapiro note that very small portion of patents are ever litigated, they argue for the importance of evaluating the incentive effects of a probabilistic patent litigation process. Our modeling approach focuses on this particular infringement aspect.

[9] Various definitions of "imperfect protection" have been proposed in the patent literature. Waterson (1990) employs a "court cost function" defining litigation costs and damage fees awarded to the patent holder to depict uncertainty in patent infringement litigation. In a sequential framework, Llobet (2003) also adopts this particular view. In addition, he assumes that the patent holder has private information about the size of the innovation and, consequently, the probability of winning in litigation. Crampes and Langinier (2002) consider patents as imperfect protection against entry if the patent holder cannot

an infringement occurs, the court may come to a different judgement. Occasionally, we treat y as variable and $f(y)$ as a decreasing function of $y \in [0, y^*]$ with $f(0) = 1$ and $f(y^*) = 0$. The further away from x a new invention is, the less likely will it get a verdict of infringement. If the court finds infringement in favor of the patent holder (the plaintiff), it assesses the extent of true damage (y), and orders compensation of a <u>reasonable royalty</u>, $R \geq 0$. According to the "analytical method" (see appendix A) and the model parameters, R would amount to the actual profit made by the infringer, that is, $R = \pi_y - c_y$.[10]

We model the strategic interaction as a **strategic game** between PH and the firm. The game lasts one period, which is defined as the time interval beginning when PH initiates the licensing offer and concluding when the infringement and damage award issues are resolved. The two players take several steps during the period. There is no discounting within the period.

Both players enter the game with exogenously given and commonly known y. PH, as a first mover, makes a licensing agreement offer simply by specifying S with $S \in [0, \infty)$, where S is a fixed-fee royalty: S represents the amount to be paid by the firm for the right to profit from the commercialization of the PH-sanctioned technology y. By proposing $S = 0$, PH tolerates the infringement without legal recourse. Knowing the proposed licensing offer, the firm has three strategic alternatives: (i) quit the project; (ii) pay the royalty proposed by PH; (iii) challenge the patent infringement allegation. In the latter contingency, PH has to make one more move: take no action or litigate. In accordance with U.S. practice and the doctrine of "lost profit," we assume that the then-found-infringing firm retains the profit from this application while paying its litigation costs plus lost royalties.[11] Figure 1 summarizes the **extensive form** of the game, showing the order of decisions and the resulting (expected) payoffs.[12]

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observe infringement, cannot identify the infringer, or cannot afford costly enforcement. Anton and Yao (2007) also develop a model allowing for uncertainty in whether the infringement is detected.

[10] This is also compatible with what Anton and Yao (2007) describe as the "best case" enforcement regime.

[11] The payoff structure and the order of bargaining-then-litigating events described here is similar to what Schankerman and Scotchmer (2001:206, Figure 1) consider in their "licensing a research tool" game. However, since their interest is not on assessing probabilistic patents, in the infringement subgame with a given liability damage rule, firm 2 in their model is always deemed infringed if it stays in the market, that is, $f(y) = 1$, for $y \in [0, y^*]$ in our notation.

[12] One crucial difference to Chou and Haller (2007) lies on the award to a winning PH. Chou and Haller (2007) stipulate that PH receives the amount he asks for, i.e. S in our present notation. Notice that in equilibrium, PH never receives more than $\pi_y - c_y$, since the firm has the option to drop out.
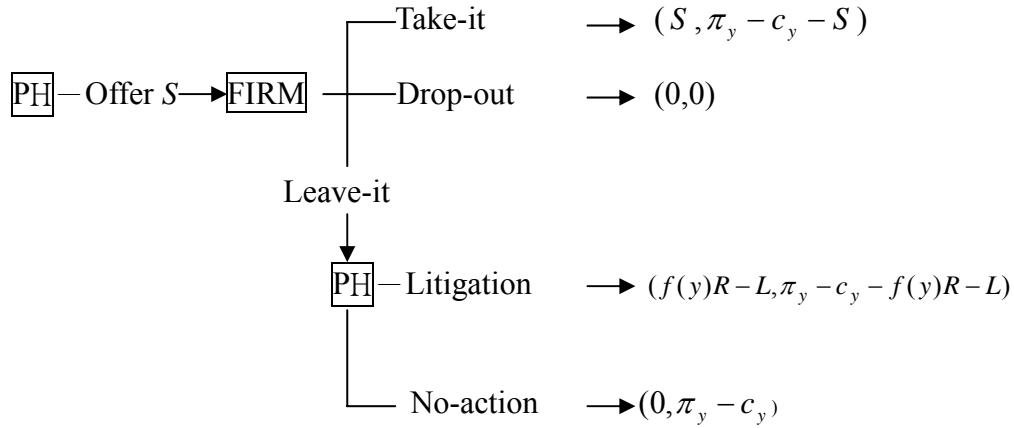
PH—Offer S ➝ FIRM

—Take-it ⟶ $(S, \pi_y - c_y - S)$

—Drop-out ⟶ $(0,0)$

Leave-it

PH—Litigation ⟶ $(f(y)R - L, \pi_y - c_y - f(y)R - L)$

—No-action ⟶ $(0, \pi_y - c_y)$

Figure 1: The Extensive From

Let M={No-action, Litigation} and N={Take-it, Leave-it, Drop-out}. Then the **normal form** of the game has strategy spaces $S_{PH} = \Re \times M^{\Re}$ for PH and $S_F = N^{\Re}$ for the firm. We consider strategy pairs that are Nash equilibria, i.e. each player chooses a strategy that maximizes its expected payoff given the other player's strategy. Moreover, we require **subgame perfection**: Equilibrium pairs of strategies induce equilibrium play in all subgames.

We distinguish four types of pure strategy equilibria. We note that which types emerges as the subgame perfect equilibrium of the game depends on the numerical specification of the model.[13]

1. The **Take-it equilibrium** is characterized by an offer S_t with

   $\pi_y - c_y - S_t \geq 0,$
   $\pi_y - c_y - S_t \geq \pi_y - c_y - f(y)R - L,$ and
   $f(y)R - L \geq 0.$

   The firm responds with Take-it to this offer. Should the firm play Leave-it in response to this offer, then PH would counter with Litigation.

2. The **Leave-it equilibrium** is characterized by an offer S_l with

   $\pi_y - c_y - f(y)R - L \geq 0,$
   $\pi_y - c_y - f(y)R - L \geq \pi_y - c_y - S_l,$ and
   $f(y)R - L \geq 0.$

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[13] PH, as a leader in this game, has the sole interest in proposing the offer R so as to collect the highest possible profit share from the firm. The No-Action and Drop-out equilibria where PH cannot generate any positive gain are hence not the focus of the following analysis. However, in a more complicated many-firm setting, however, these potential types of equilibria may yield impact upon PH's decision-making.

The firm responds with Leave-it to this offer and PH counters with Litigation.

3. The **No-Action equilibrium** is characterized by the damage award R with

$$f(y)R - L \le 0, \text{ and}$$

$$\pi_y - c_y \ge 0.$$

The firm responds with Leave-it to all offers $S > 0$ and PH counters with No-action.

4. The **Drop-out equilibrium** is characterized by an offer S_d with

$$\pi_y - c_y - S_d \le 0,$$

$$f(y)R - L \ge 0, \text{ and}$$

$$\pi_y - c_y - f(y)R - L \le 0.$$

The firm responds with Drop-out to this offer. If the firm responded with Leave-it to this offer, then PH would counter with Litigation.

We proceed with the following simplifying assumption[14]:

(A1) $\pi_y = a \cdot y; c_y = c \cdot y$ where a and c are constants satisfying $a > c \ge 0$.

A constant marginal revenue occurs in a standard vertical (quality) differentiation problem. There consumers have utility functions of the form $U = \theta y - p$ where θ is a taste parameter and p is the price charged for the product of quality y. The distribution of tastes across consumers is given by the uniform distribution along the interval $[0, \theta]$ with $\theta > 0$. Then, given y, the firm maximizes its gross profit by choosing the price level $P_y = \dfrac{y\theta}{2}$. The resulting gross profit is $\pi_y = \dfrac{y\theta^2}{4}$. Put $a = \dfrac{\theta^2}{4}$.

III. EQUILIBRIUM ANALYSIS

We first characterize the types of equilibrium of interest. Specifically the interval $(0, y)$ can be partitioned into three areas each of which mapping to a particular type of equilibrium. Necessary and sufficient conditions for these three types of equilibrium outcomes are provided in Proposition 1.

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[14] Since PH's sole interest is in maximizing the profit transferrable from the firm, he has no incentive to make an offer $S > \pi_y - c_y$. In other words, {Drop-out, Litigate} or {Drop-out, No-Action} can never be an equilibrium outcome since Drop-out is dominated by Take-it for the firm.

Proposition 1. *(i) A Complete-Profit-Transfer Take-it equilibrium emerges, i.e., PH can extract all the profit it facilitates from the firm if and only if*

(a) $y - yf(y) \leq \dfrac{L}{a-c}$ *and*

(b) $yf(y) \geq \dfrac{L}{a-c}$.

(ii) A No-Action equilibrium emerges, i.e., the firm can retain all the profit if and only if

(c) $yf(y) \leq \dfrac{L}{a-c}$.

(iii) An Incomplete-Profit-Transfer Take-it equilibrium emerges, i.e., a licensing offer $S_t^i = (a-c)yf(y) < \pi_y - c_y$ is proposed and accepted if and only if

(b) $yf(y) \geq \dfrac{L}{a-c}$ *and*

(d) $y - yf(y) > \dfrac{L}{a-c}$.

Several forces from various sources are at work in driving the above equilibrium configuration. Firstly inequality (a) and (d) determine how large a licensing fee would make the firm indifferent between acceptance and litigation. Secondly inequality (b) and (c) convey to what extent PH's threat to litigate is credible. Some simple comparative statics can help develop further the intuition for Proposition 1. To begin with, suppose $a, c, y,$ and L are parameterized such that $y \geq \dfrac{L}{a-c}$ and $f(y)$ is treated as a variable. Note then (a) and (b) are simultaneously satisfied if and only if $f(y)$ is sufficiently large (or alternatively speaking, the degree of second generation innovation is sufficiently low). To achieve a full-profit-transfer equilibrium with a licensing offer $S_t = (a-c)y$, PH's threat of litigation in the Leave-it subgame has to be credible, that is, $f(y)R - L > 0$. Since $y \geq \dfrac{L}{a-c}$ can be translated as $(a-c)y > L$ and if $R = (a-c)y$, then the inequality $f(y)R - L > 0$ holds trivially for the boundary case where $f(y) = 1$. By continuity, a high and only a high winning probability $f(y)$ sustains the full rent extraction equilibrium.

Moreover we observe that (a) and (b) together imply $f(y) \geq \dfrac{1}{2}$. This however reveals that a very innovative y associated with a low winning probability for PH will not result in full rent extraction.

On the other hand, when $a, c, y,$ and L are parameterized such that $y < \dfrac{L}{a-c}$, whereas (a) is always satisfied, (b) breaks down for all $f(y) \in [0,1]$. That is, a low degree of innovation (small y) associated with a high winning probability for PH will drive a full-profit-transfer equilibrium either. In a similar vein, one reasonable conjecture is that the existence of an incomplete-profit-transfer Take-it equilibrium may hinge upon some "intermediate" $f(y)$.

And it becomes immediate that when $y \geq \dfrac{L}{a-c}$, only a sufficiently small $f(y)$ could sustain a No-Action equilibrium. Conversely but not surprisingly, for $y < \dfrac{L}{a-c}$, a No-Action equilibrium can be well expected since (d) holds trivially. To sum up, the preceding comments suggest there might exist specific bounds for those intervals of y which permit each type of equilibrium in question.

To extend the analysis further and derive such bounds, we next consider a situation with exogenously given L, a, c, y^*, and a function $f : [0, y^*] \to [0,1]$ satisfying:

(A2) $f(0) = 1, f(y^*) = 0$, and f is twice differentiable with $f' < 0, f'' \leq 0$.

(A2) describes a decreasing winning probability (for PH) function in the innovation parameter y. A less obvious but equally plausible property exhibited in (A2) is that PH's winning probability drops faster as y moves further away from $x = 0$, or the closer y approaches the delimiter of patent protection, y^*. From a different perspective, *ceteris paribus*, the firm is enjoying "increasing returns to litigation" (Chou and Haller, 2007) as y improves.[15]

For expositional purpose, we denote $\bar{y}$ the "median" of $f(y)$, i.e. $\bar{y}$ is implicitly given by the condition $f(\bar{y}) = \dfrac{1}{2}$.

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[15] Waterson (1990), the seminal paper in the "signpost" literature, implicitly imposes an property on the "court cost function" which is equivalent to our concavity of $f(y)$.

Proposition 2. *Suppose (M)* $\dfrac{\bar{y}}{2} \geq \dfrac{L}{a-c}$. *Then there exist* y_l, y_m *and* y_r *with the following properties:*

(i) $0 < y_l \leq y_m \leq y_r < y^*$;

(ii) *The highest-yield equilibrium for PH is a complete-profit-transfer Take-it equilibrium iff* $y \in [y_l, y_m]$.

(iii) *A No-Action equilibrium emerges iff* $y \in [0, y_l] \cup [y_r, y^*]$.

(iv) *The highest-yield equilibrium for PH is an incomplete-profit-transfer Take-it equilibrium with* $S_t^i = (a-c)yf(y) + L$ *iff* $y \in (y_m, y_r]$.

Corollary 1. *Denote* $\hat{y} = \arg \underset{y \in [0, y^*]}{Max\ Max}\ yf(y)$.

(i) *When* $\dfrac{L}{a-c} \in (\hat{y}f(\hat{y}), \infty)$ *there only exist No-Action equilibria for* $y \in [0, y^*]$.

(ii) *When* $\dfrac{L}{a-c} \in (\bar{y}/2, \hat{y}f(\hat{y})]$ *there exist both Take-it and No-Action equilibria for* $y \in [0, y^*]$.

Note that even when litigation is not that costly relative to the profit of the commercialized product, the "full strength" patent protection exists only in the interval $[y_l, y_m]$. Both low and high ends of the improvement would not induce actual patent litigation (No-Action). Because in both cases where "low y hence high $f(y)$" or "high y hence low $f(y)$" PH's expected gain from litigation is less than his litigation cost. In our model, as a consequence, the firm has incentive to invent around with close imitation or invent enough with a quite novel (though still infringing) product because of the lack of effective patent protection.[16]

It is obvious that our conclusions about the division of profit rely crucially on the relevant range that $\dfrac{L}{a-c}$ lies in. Under (M), with a relatively low $\dfrac{L}{a-c}$, three types

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[16] Empirically, as Mansfield et al (1981) point out, 60% of all patented and successful innovations were imitated within 4 years after introduction. Similar observations are also reported in Levin et al (1987).

of equilibria may emerge whereas a relatively high $\dfrac{L}{a-c}$ would only allow No-Action equilibria for all y in the statutory claimed range of patent protection. Generally speaking, two factors counteract in driving PH's expected profit transfer in the analysis so far: what the firm is capable of, i.e., the magnitude of y, and what the odds are for PH to win the infringement case, i.e., the range of $f(y)$. Thus hypothesis (M) and conditions in Corollary 1 correspond to the extent by which PH can master this balancing act and extract profit from the firm.

IV. Comparative Statics

Efficacy of the patent protection in terms of complete profit transfer. In this sub-section we focus on the comparative statics with respect to several key variables - within their most interesting range in our set-up. In the sequel we denote $\lambda \equiv y_m - y_l$ the length of the interval of y where a complete-profit-transfer Take-it equilibrium can be obtained. λ may serve as a measure of the efficacy of the patent system. We first investigate how λ is responding to variations of the litigation cost L, the gross profit parameter a and the product development cost parameter c. It suffices to see how L depends on the compound parameter $k \equiv \dfrac{L}{a-c}$.

Lemma 2. *Suppose* $\dfrac{\bar{y}}{2} \neq \hat{y}f(\hat{y})$, *then* $\dfrac{\partial \lambda}{\partial k} < 0$ *for* $k \in \left(\bar{y}/2, \hat{y}f(\hat{y})\right)$. *Moreover, the corresponding intervals* $\left[y_m(k), y_r(k)\right]$ *are strictly nested.*

Intuitively, a higher litigation cost should have cast a stronger deterring effect upon the opportunistic firm turning down the offer. However, Lemma 2 says that even when the litigation cost is in the "favorable" range where a complete-profit-transfer Take-it equilibrium can be assured, higher litigation cost will diminish the effective patent protection measured by λ. Lemma 2 completes the description of $\lambda(k)$.

Lemma 3. $\lambda(k)$ *is strictly concave in* k *and there exists a unique* $\widetilde{k} \in \left(0, \bar{y}/2\right)$ *such that* $\lambda(\widetilde{k}) \geq \lambda(k)$ *for all* $k \in R_+$.

In a second type of comparative statics, we investigate how λ, the proposed patent protection measure, is affected by a change of patent protection. Intuition may suggest that the best way to help PH transfer profit from the firm is to grant PH a broad patent protection. Interesting enough, this is a premature conclusion as the next proposition

shows. One more simplifying assumption, (A3) is imposed to establish the result. Prior to that we have to extend the model appropriately by postulating that winning probability takes the more general form $f(y; y^*), 0 \leq y \leq y^*$, where the patent breadth $y^* > 0$ is treated as variable in the sequel. The extended notation $\lambda(k; y^*)$, $\tilde{k}(y^*)$, etc. will be used.

(A3) $f(y; y^*) = f(y/y^*; 1)$, i.e., $f(y; y^*)$ is homogeneous of degree 0.

(A3) stipulates that the winning probability for PH depends only on the ratio y/y^*, not on the absolute magnitude of y or y^*. An extremely high y^* might correspond to a very broad claim such as "All non-human transgenic mammals" or "all hand-use calculators." It is well documented (Lemley and Shapiro, 2005) that a typical defendant's strategy of countering infringement litigation is to argue that the patent is "invalid" in the sense that PTO has failed to identified the existence of "prior art" in the original claim. Hence it appears reasonable that a broader set of claims leave more room for purporting invalid patents.

Thus it is of economic significance to investigate how the specific features of probabilistic patents, i.e. the signpost interpretation of the patent system, respond to variations in patent breadth. In the present paper, imposing (A3) is a simple attempt to capture the effect of broadening patent protection, be it through a change of the written law or the doctrine of equivalents, and the reverse doctrine of the equivalents.[17] Our focus here is to evaluate the impact upon the PH's incentive to innovate from adjusting the patent protection under the doctrine of equivalents or other similar traits.

Lemma 4. *Suppose* $k \in (0, \bar{y}/2)$, *then the functions* $y_l(k; y^*), y_r(k; y^*)$ *and*

$\lambda(k; y^*)$ *are homogeneous of degree 1 in* $(k; y^*)$. *The functions* $\hat{y}(y^*)$ *and* $\bar{y}(y^*)$ *are homogeneous of degree 1 in* y^*.

Let us first state a result that conforms to intuition: as patent protection becomes broader, $\lambda(k; y^*)$ increases, i.e. the size of the interval where PH can extract all the surplus increases.

Proposition 4. *The following three assertions hold:*

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[17] Notice that here we are not concerned with the controversy over the judicial standard for infringement analysis under the doctrine of equivalents, e.g. tests like "element-by-element" and "invention as a whole" See, for example, Lau (1989) and Merges (1992).

(i) $\dfrac{\partial}{\partial k} L(k; y^*)$ is strictly increasing in $y^* > 0$ as long as $0 < k < \bar{y}(y^*)/2$.

(ii) $\lambda(k; y^*)$ is strictly increasing in $y^* > 0$ as long as $0 < k < \bar{y}(y^*)/2$.

(iii) $\tilde{k}(y^*)$ is strictly increasing in $y^* > 0$.

Let us now proceed to the promised, somewhat less intuitive result: As patent protection becomes broader, the relative size of the interval where PH can extract all the surplus decrease.

Proposition 5. *For any* $0 < y^* < y^{**}$, *there exists* $k(y^*, y^{**}) > 0$ *such that*

$$\frac{\lambda(k; y^{**})}{y^{**}} < \frac{\lambda(k; y^*)}{y^*} \text{ for all } 0 < k < k(y^*, y^{**}).$$

Proposition 5 states that even though $\lambda(k; y^*)$ increases as patent protection becomes broader, λ / y^*, that is the relative size of the interval where PH can extract all the surplus, may be falling for certain k. In other words, the efficacy measure of patent enforcement defined as the fraction of the infringing y that provide maximal incentive for PH to innovate, can apparently diminish when the court employs the doctrine of equivalents when the imperfect patent protection exhibits such feature as (A3).

Efficacy of the patent protection in terms of average expected profit transferred. In the sub-section we focus on the comparative statics with respect to $(k; y^*)$. In the sequel we denote

$$E\pi_p \equiv \frac{1}{y^*} \{ \int_{y_l}^{y_m} (a-c) y \, dy + \int_{y_m}^{y_r} [(a-c) y f(y) + L] dy \}.$$

$E\pi_p$ is the expected payoff for PH given that y is uniformly distributed along the interval of proclaimed patent protection $[0, y^*]$. Recall that in Proposition 1 we have shown that the interval $[0, y^*]$ can be partitioned into three areas each of which is characterized by a particular type of equilibrium: the complete-profit-transfer Take-it equilibrium, the incomplete-profit-transfer Take-it equilibrium with

$S_t^i = (a-c)yf(y) + L$ or the No-Action equilibrium. We first investigate how $E\pi_p$ is responding to variations of k.

Lemma 5. *Suppose* $\bar{y}/2 \neq \hat{y}f(\hat{y})$, *then* $\dfrac{\partial E\pi_p}{\partial k} < 0$ *for* $k \in (\bar{y}/2, \hat{y}f(\hat{y}))$. *Moreover the corresponding intervals* $[y_m(k), y_r(k)]$ *are strictly nested.*

Lemma 6. *Suppose* $\bar{y}/2 = \hat{y}f(\hat{y})$, *then there exists a unique* $\mathring{k} \in (\tilde{k}, \bar{y}/2)$ *such that* $E\pi_p(k)$ *is decreasing in* $k \in \left[\mathring{k}, \bar{y}/2\right]$.

Lemma 5 and 6 are in reminiscent of Lemma 2 and 3. In particular, the properties of $\lambda(k)$ and $E\pi_p$ follow directly from the concavity of $f(y)$, the probability of PH winning in litigation. In other words, the phenomenon observed here, that is, shrinkage of the interval of complete profit transfer and decrease in PH's average expected profit with respect to certain k, can be attributed to a particular aspect of imperfect patent protection, increasing returns to litigation for the firm.

Lemma 7. *The efficiency ratio* $\dfrac{E\pi_p(k; y^*)}{E\pi(k; y^*)}$ *is homogeneous degree of zero in* $(k; y^*)$.

Lemma 7, comparable to Proposition 5, reminds us of the potential drawback of an expanded patent protection: an increase in the relative profit transferred to PH as a result broadened patent breadth may not improve the efficacy of patent protection.[18]

Some caution is warranted in interpreting Lemma 7. Namely, two alternative interpretations can be made with respect to the absolute average expected profit transferred from the firm to PH when patent breadth is adjusted. First, observe that $E\pi(k; y^*)$, the average profit to PH under perfect patent protection with y uniformly distributed over $[0, y^*]$ is just $(a-c)\dfrac{y^*}{2}$. Now suppose that under two patent protection regimes y^* and λy^* (w.l.o.g, $\lambda > 1$), the innovation parameter y are distributed with uniform density $\dfrac{1}{y^*}$ on $[0, y^*]$ and $\dfrac{1}{\lambda y^*}$ on $[0, \lambda y^*]$,

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[18] Note that the application of the doctrine of equivalents may be associated with litigation cost in the following sense: (1) enforcement cost goes up as y^* is increased, and (2) the litigants have more burden of proof.

respectively, and suppose that the efficiency ratio $\dfrac{E\pi_p\left(k;y^*\right)}{E\pi\left(k;y^*\right)}$ is equal to $\alpha < 1$. Then Lemma 7 implies that

$$E\pi\left(\lambda k;\lambda y^*\right)=\alpha E\pi_p\left(\lambda k;\lambda y^*\right)=\alpha(a-c)\frac{\lambda y^*}{2}>\alpha(a-c)\frac{y^*}{2}=E\pi_p\left(k;y^*\right)$$

Alternatively, if the patent protection regime y^* can be viewed as a continuous contraction from the λy^* regime, then the innovation parameter y should be distributed with a uniform density $\dfrac{1}{\lambda y^*}$ on $[0,y^*]$ and $[0,\lambda y^*]$. Then Lemma 7 implies

$$E\pi\left(\lambda k;\lambda y^*\right)=\alpha E\pi_p\left(\lambda k;\lambda y^*\right)=\alpha(a-c)\frac{\lambda y^*}{2}>\alpha(a-c)\frac{y^*}{2}\cdot\frac{y^*}{\lambda y^*}=E\pi_p\left(k;y^*\right).$$

While the former interpretation is better suited for the description of an adjustment in written law, the latter is more compatible with the application of the reverse doctrine of equivalents. In either case, PH is enjoying an increase in his absolute average profit transferred from the infringing firm under a broader patent protection.

V. Concluding Remarks

In this article, we investigate the division of profit between a patent holder and a derived product producer in an environment with uncertainty about the outcome of infringement but with certainty about the damage awards, a reasonable royalty granted to the prevailing plaintiff. Our analysis identifies the conditions on model parameters that permit a complete-profit-transfer Take-it equilibrium, an incomplete-profit-transfer Take-it equilibrium or a No-Action equilibrium. The interval of the proclaimed patent protection $[0,y^*]$ can be partitioned into three areas each of which is characterized by a particular type of equilibrium. Whereas the area in which complete-profit-transfer equilibria can be obtained gives a patent holder maximal incentive to innovate, the area in which No-Action equilibria prevail encourage a subsequent innovator to imitate either with a close substitute or with a much advanced but still infringing product. Comparative statics with respect to important policy parameters such as litigation costs and patent breadth is performed.

In a broader context, while some litigation of the literature assumes litigation is always profitable for the plaintiff to rule out "nuisance" suits (Bebchuk (1984); Reinganum and Wilde, 1986), the current model accommodates inaction as one of the strategies the patent holder may choose (Nalebuff, 1987; Meurer, 1989). That is, a patent holder may not always be willing to use the court system. Consequently,

credible threats have a significant effect on the equilibrium outcome. Recall that which type of equilibrium will prevail depends crucially on (a) how large a licensing fee would make firm indifferent between acceptance and litigation and (b) how credible PH's threat is to litigate in case the infringing firm were to turn down the offer. To maintain a credible litigation threat the patent holder must evaluate the potential gain and loss when the infringing firm refuses to settle. The issue of credibility may severely restrict the patent holder's capacity of achieving the desired outcome - complete profit transfer. In addition, the particular phenomenon observed from the above comparative statics, that is, shrinkage of the interval of complete profit transfer and decrease in PH's average expected profit with respect to certain k, is again a result of the dominance of effect (b) over effect (a).

Our analysis pays special attention to the effects of the variations in patent breadth, either by an adjustment in the written law or by application of the doctrine of the equivalents and the reverse doctrine of equivalents. It is clear that the current paper is not suggesting the "optimal" patent system. Rather it aims at assessing the efficacy of the current patent system when uncertainty about the outcome of infringement litigation is taken into consideration. In particular, we are concerned with the incentive to innovate, not the harm (deadweight loss) caused by blocking patents as Merges and Nelson (1992) have emphasized. Therefore, it is not our intention to determine which y should be allowed to escape the "web of infringement" (Scotchmer, 1996).

Appendix A

A Brief Review of the Doctrines on Reasonable Royalty. Prior to 1946, a successful patent claimant could choose between the amount of damages she suffered and the amount of profits earned by the infringer (Chisum, 1980). However the high cost of determining an infringer's profits eventually led Congress to drop infringer profits as an alternative measure of recovery [Act of August 1, 1946, ch. 726 s 1, 60 stat. 778 (current version at §35 U.S.C. s 284 (1952)). More sophisticated doctrines have been applied since. Despite the revision, an infringer's profits continue to be crucial elements in computing the patent holder's damages - either as an approximation for the patent holder's "lost profits" or as a factor in determining a reasonable royalty. In general, a successful patent claimant is entitled to recover the profits she would have made but for the infringement; if lost profits cannot be proven, she is entitled to a reasonable royalty.

(A) *Lost Profits*:

(1) The Cornerstone case for lost profits: Panduit Corp. v. Stahlin Bros. Fiber Works, Inc.575 F.2d 1152, 197 U.S.P.Q. (BNA) 726 (6th Cir.l978).

Four factors enumerated in Panduit have been used as the primary guidelines for determining whether a patent holder is entitled to recover lost profits.

(B) *Reasonable Royalty*:

(1) The Benchmark case: Georgia-Pacific Corp v. United States Plywood Corp. 318 F. Supp. 1116, 166 U.S.P.Q. (BNA) 235 (S.D.N.Y. 1970), modified, 446 F.2d 295, 170 U.S.P.Q. 369 (2d Cir.), cert. denied, 404 U.S. 870 (1971).
Georgia-Pacific has been relied upon heavily for its fifteen factors, among others, to be verified in determining a reasonable royalty.

(2) Variation of Georgia-Pacific: Hanson v. Alpine Valley Ski Area, Inc. 718 F.2d 1075, 219 U.S.P.Q. (BNA) 679 (Fed. Cir.1983).
In Hanson the Federal Circuit has stated: 'The reasonable royalty may be based upon,..., a hypothetical royalty resulting from arm's length negotiation between a willing licensor and a willing licensee.'
Conceivably, problems with this hypothetical negotiation are manifold: First, the court is required to reconstruct a "fancy contract" based upon fantasy and flexibility [Fromson v. Western Litho Plate & Supply Co. 853 F.2d 1568, 1575-76,7 U.S.P.Q.2d 1606 (Fed Cir. 1988)]. Secondly, it ignores the adverse impact upon converting property rule into liability rule at random wills [See Calabresi and Melamed (1972)]. That is, a flat reasonable royalty with no punitive effect would have reduced the

incentive to innovate at the first place. Though an infringer may not be completely indifferent between an ex-ante licensing agreement and the ex-post damage award since there may be conceivable loss of goodwill and substantial sunk costs.

(3) What flows directly from the willing licensor/willing licensee model is the 'analytical method' (or accounting method). It computes a reasonable royalty based on the infringer's pre-infringement projection of profits. See, for instance, TWM Mfg. Co. v. Dura Corp., 789 F.2d 895, 229 U.S.P.Q. (BNA) 525 (Fed. Cir. 1986).

In reference to the current modeling approach, the following features are worth further investigation.

To begin with, to be eligible for the lost profits compensation, the patent holder has to show she has capacity to produce the volume of potential sales lost due to infringement. However, cases have shown that no stringency has been maintained in this respect by the courts. For example, to prove its capacity to make the infringer's sales, the patent holder does not have to show that it had a plant ready and existing [See Livesay Window Co. v. Livesay Industries, Inc., 251 F.2d 469, 473, 116 U.S.P.Q. 167, 171 (5th Cir. 1958): W. L. Gore & Associated, Inc. v. Carlisle Corp., 198 U.S.P.Q. 353, 367 (D. Del. 1978).J The court even considered the possibility of subcontracting as the patent holder's potential capacity (see Gyromat Corp. v. Champion Spark Plug Co., 735 F.2d 549, 554 222 U.S.P.Q. 4, 7 (Fed. Cir. 1984). As a result, the patent holder in our model, though without marketing power, would still be entitled to lost profits, which figure may be drawn according to the infringer's actual profits.

Secondly, despite the fact that the doctrines and their variations on reasonable royalty have been widely stated in patent damages cases, the analytical method seems to be the dominant guiding principle for computing reasonable royalty [See Conley (1987) citing that the courts giving only 'lip service to the willing licensor/willing licensee model']. Furthermore, as a simple rule, the courts just subtract the infringer's usual profit from the profit earned by the infringement, and award the entire difference to the patent holder. In a sense the analytical approach is a return to awarding to the patent holder the infringer's profits from the use of the invention.

Note that in the context of our model, the value of the second generation product is completely attributable to the basic patented technology. Thus the doctrines of 'entire market value' and 'apportionment' would yield the same compensation figure for the patent holder [See, e.g. Westinghouse v. Wagner, §225 U.S. 604, 614 (1912)]. Specifically, in terms of the model parameters, the analytical approach amounts to a reasonable royalty $R = (a - c)y$ as we assume.

APPENDIX B

Proof of Proposition 1: (i) For a licensing agreement to prevail, i.e., a Take-it equilibrium to exist, the following conditions are necessary and sufficient:

$$\pi_y - c_y - S_t \geq 0, \ \pi_y - c_y - S_t \geq \pi_y - c_y - f(y)R - L, \ \text{and} \ f(y)R - L \geq 0.$$

We first explore the possibility that PH can extract all the profit from the firm, i.e., where and offer $S = (a - c)y$ gets accepted in equilibrium. Thus the previous conditions are equivalent to:

$$S = (a - c)y, \tag{1}$$

$$R \leq \frac{L}{1 - f(y)} \tag{2}$$

$$R \geq \frac{L}{f(y)} \tag{3}$$

(2) and (3) are then equivalent to

$$y[1 - f(y)] \leq \frac{L}{a - c} \ \text{and} \tag{4}$$

$$yf(y) \geq \frac{L}{a - c.} \tag{5}$$

(ii) We proceed with the necessary and sufficient conditions for a No-Action equilibrium, i.e., $\pi_y - c_y \geq 0$, and $f(y)R - L \leq 0$. With (A1) they are equivalent to:

$$\pi_y - c_y \geq 0, \ \text{and}$$

$$f(y)R - L \leq 0.$$

With (A1) they are equivalent to:

$$yf(y) \leq \frac{L}{a - c.} \tag{6}$$

(iii) We commerce with the necessity proof. Recall that for a licensing agreement to prevail, i.e., a Take-it equilibrium to exist, the following conditions must be satisfied:

$$\pi_y - c_y - S_t \geq 0,$$

$$\pi_y - c_y - S_t \geq \pi_y - c_y - f(y)R - L, \ \text{and}$$

$$f(y)R - L \geq 0.$$

Note that in and incomplete-profit-transfer Take-it equilibrium the firm enjoys positive payoff after the transfer of profit through licensing agreement, i.e., $\pi_y - c_y - S_t \geq 0$. We then explore the possibility that PH can extract profit by offering

$S_t = (a-c)yf(y)+L$ which gets accepted in equilibrium. With (A.1) and the assumption on reasonable royalty, the previous conditions can be rewritten as:

$$\pi_y - c_y - [(a-c)yf(y)+L] > 0, \ 0 \geq 0, \ \text{and} \ yf(y) \geq \frac{L}{a-c.}$$

These are just conditions (d) and (b)

$$y[1-f(y)] > \frac{L}{a-c}, \text{ and} \tag{d}$$

$$yf(y) \geq \frac{L}{a-c.} \tag{b}$$

Now we turn to the sufficiency proof. Suppose (b) and (d). Since only the Take-it or the Leave-it equilibrium has potential in generating payoffs for PH, we will focus on these two types of equilibria. We start with the possibility for PH to extract profit via a Take-it equilibrium.

Define

$$S_t \equiv (a-c)yf(y)+L$$

Thus we can have

$$\pi_y - c_y - S_t \geq \pi_y - c_y - [(a-c)yf(y)+L] \tag{7}$$

Also, by (d), we can infer

$$\pi_y - c_y > (a-c)yf(y)+L \tag{8}$$

or

$$\pi_y - c_y - [(a-c)yf(y)+L] > 0, \text{ that is}$$

$$\pi_y - c_y - S > 0. \tag{9}$$

Next note that By (A.1) and the assumption on the reasonable royalty, (b) can be written as

$$f(y)R - L \geq 0. \tag{10}$$

(7), (9) and (10) establish that, indeed, S_t is a Take-it equilibrium offer. It is also obvious that such an licensing offer yields higher payoff for PH than any Leave-it equilibrium since $S_t = (a-c)yf(y)+L > (a-c)yf(y)-L$. We have shown that the combination of (b) and (d) is equivalent to the necessary and sufficient conditions for an Incomplete-Profit-Transfer equilibrium characterized by $S_t = (a-c)yf(y)+L$. Notice that strict inequality in (d) renders strict inequality in (8), that is incomplete-profit-transfer *strictu sensu*. This completes the proof. ∎∎

Proof of Proposition 2: First, we perform comparative statics with respect to

$y \in [0, y^*]$. For this purpose, we introduce the functions $g_1(y) = y[1 - f(y)]$ and $g_2(y) = yf(y)$ which appear in (4) and (5) and, obviously, play a critical role in our analysis. Notice that $g_1' = 1 - f - y \cdot f' > 0$ and $g_1'' = -2f' - y \cdot f'' > 0$ in the interval $(0, y^*)$. Hence g_1 is strictly increasing and strictly convex in y with $g_1(0) = 0$ and $g_1(y^*) = y^*$.

Further notice that $g_2' = (y \cdot f)' = f + y \cdot f'$ and $g_2'' = (y \cdot f)'' = 2f' + y \cdot f'' < 0$. Thus g_2 is strictly concave in y with $g_2(0) = g_2(y^*) = 0$. Consequently, g_2 has a unique maximizer $\hat{y}$ in $(0, y^*)$. This maximizer is given as the unique solution of the first order condition

$$g_2'(\hat{y}) = f(\hat{y}) + \hat{y}f'(\hat{y}) = 0.$$

Claim: $\hat{y} \leq \bar{y}$ and $0 < y^*/2 < \bar{y} < y^*$.

To show this claim, recall that $g_2(y)$ is strictly concave in y with $g_2(0) = 0$ and $g_2(y^*) = 0$. By Takayama (1985) Theorem 1.C.3: f is concave on $(0, y^*)$ if and only for any $x, y \in (0, y^*)$: $f'(y) \cdot (x - y) \geq f(x) - f(y)$.

Evaluate this inequality at $y = \bar{y}$ and let $x \to 0$. Then $-f'(\bar{y})\bar{y} \geq 1 - \frac{1}{2}$ or

$f'(\bar{y}) \cdot \bar{y} \leq -\frac{1}{2}$. Adding $f(\bar{y}) = \frac{1}{2}$ to the latter inequality yields

$g_2'(\bar{y}) = \bar{y} \cdot f'(\bar{y}) + f(\bar{y}) \leq 0$. Strict concavity of g_2 and $g_2'(\hat{y}) = 0$ imply the assertion $\hat{y} \leq \bar{y}$. Further, (A2) has the immediate implication $0 < y^*/2 < \bar{y} < y^*$. This concludes the proof of the claim.

Next, recall that $g_1(y)$ is strictly increasing and strictly convex in y with $g_1(0) = 0$ and $g_1(y^*) = y^*$. Hypothesis (M) amounts to $g_1(\bar{y}) = g_2(\bar{y}) = \frac{\bar{y}}{2} \geq \frac{L}{a - c}$.

Part (i):
Recall that $\bar{y}$ is defined as the intersection point of $g_1(y)$ and $g_2(y)$, i.e.,

$g_1(\bar{y}) = g_2(\bar{y}) = \frac{\bar{y}}{2}$. By the hypothesis, the continuity and other properties of g_2,

and the intermediate value theorem, there exist $z_l \in (0, \hat{y}]$ and $z_r \in [\hat{y}, y^*)$ such that

$$g_2(\bar{y}) \geq g_2(z_l) = g_2(z_r) = \frac{L}{a - c}.$$

Next note that (M) implies $y^* > g_2(\hat{y}) \geq \frac{L}{a - c} > 0$. Then, by the continuity and

other properties of g_1 and the intermediate value theorem, there exists a unique

$z \in (0, y^*)$ with $g_1(z) = \frac{L}{a - c}$. We then compare the magnitudes of z_l and z.

Now recall that $f(\bar{y}) = \dfrac{1}{2}$ and therefore $\dfrac{L}{a-c} \leq \dfrac{\bar{y}}{2} = g_1(\bar{y}) = g_2(\bar{y})$. Therefore, $\bar{y} \in [z_l, z_r]$, by the strict concavity of g_2. Also, $z \leq \bar{y}$, by the strict monotonictiy of g_1. Hence $z \leq z_r$. Moreover, $0 = g_1(0) = g_2(0)$, $\dfrac{1}{2} \cdot \bar{y} = g_1(\bar{y}) = g_2(\bar{y})$, strict convexity of g_1 and strict concavity of g_2 imply $g_1(y) < \dfrac{1}{2} \cdot y < g_2(y)$ for $y \in (0, \bar{y})$. If $\bar{y} = z_l$, then $z = \bar{y} = z_l$. If $\bar{y} > z_l$, then $g_1(z_l) < g_2(z_l) = \dfrac{L}{a-c}$. Thus $z > z_l$. In any case, therefore, $z \in [z_l, z_r]$.

Now set $y_l = z_l$, $y_m = z$ and $y_r = z_r$. Then (i) is satisfied. $\bullet\bullet\bullet$

Part (ii):

We commence with the sufficiency proof. When condition (M) holds and $y \in [y_l, y_m] = [z_l, z]$, then $y \in [z_l, z_r]$ and the strict concavity of g_2 implies (b) $yf(y) \geq \dfrac{L}{a-c}$. Further $y \in [y_l, y_m]$ implies $y \leq z$. Since $g_1(y)$ is an increasing function in $y \in [0, y^*]$, condition (a) $y[1 - f(y)] \leq \dfrac{L}{a-c}$ holds as well.

Now we turn to the necessity proof. (b) implies that $y \in [z_l, z_r]$. (a) implies that $y \leq z$. Together (a) and (b) imply $y \in [z_l, \min\{z_r, z\}] = [y_l, y_m]$. Note that we know from Proposition 1 that by offering $R_t = (a-c)y$, PH can extract all the profit it facilitates form the firm if and only if (a) and (b) both hold. We have shown that under the hypothesis (c), the combination of (a) and (b) is equivalent to $y \in [y_l, y_m]$. $\bullet\bullet\bullet$

Part (iii):

We commence with the sufficiency proof. When condition (M) holds and $y \in [0, y_l] \cup [y_r, y^*]$, then $y \in [0, y^*] \setminus [z_l, z_r]$ and the strict concavity of g_2 implies (c) $yf(y) \leq \dfrac{L}{a-c}$.

Now we turn to the necessity proof. By the strict concavity of g_2, (c) implies that $y \in [0, y^*] \setminus [z_l, z_r]$ which in turn implies $y \in [0, y_l] \cup [y_r, y^*]$. Note that we know from Proposition 1 that a No-Action equilibrium is attainable if and only if (c) holds. We have shown that under the hypothesis (M), (c) is equivalent to $y \in [0, y_l] \cup [y_r, y^*]$. $\bullet\bullet\bullet$

Part (iv):

We commence with the sufficiency proof. When condition (M) holds and $y \in (y_m, y_r]$, then $y \in (z, z_r]$ and the strict concavity of g_2 implies (b) $yf(y) \ge \dfrac{L}{a-c}$. Further $y \in [y_m, y_r]$ implies $y > z$. Since $g_1(y)$ is an increasing function in $y \in [0, y^*]$, condition (d) $y[1 - f(y)] > \dfrac{L}{a-c}$ holds as well.

Now we turn to the necessity proof. (b) implies that $y \in [z_l, z_r]$. (d) implies that $y > z$. Together (b) and (d) imply $y \in (y_m, y_r]$. Note that we know from Proposition 1 that an incomplete-profit-transfer Take-it equilibrium is attainable if and only if (b) and (d) both hold. We have shown that under the hypothesis (M), the combination of (b) and (d) is equivalent to $y \in [y_m, y_r]$. This completes the proof. ∎∎

Proof of Lemma 2: Suppose a , c , y , and $f(y)$ are given such that $k \in \left(\dfrac{\bar{y}}{2}, \hat{y}f(\hat{y}) \right)$. Together with the supposition $\dfrac{\bar{y}}{2} \ne \hat{y}f(\hat{y})$ they imply that $g_2(\hat{y}) > k$ thus $z_r > \hat{y} > z_l$. It can also be inferred that $\bar{y} \notin [z_l, z_r]$ by the strict concavity of g_2 and $z > \bar{y}$ by the strict monotonictiy of g_1 . By Lemma 1, $\bar{y} \ge \hat{y}$ and $\bar{y} \notin [z_l, z_r]$ imply $\bar{y} > z_r$.

Thus $z > \bar{y} > z_r$. So $\lambda = y_m - y_l = \min[z_r, z] - z_l = z_r - z_l$. The strict concavity and the other properties of g_2 imply that for all k_1, k_2 such that $\dfrac{\bar{y}}{2} < k_1 < k_2 < \hat{y}f(\hat{y})$ the corresponding $z_r(k_1), z_l(k_1), z_r(k_2)$ and $z_l(k_2)$ have the following order:

$z_l(k_1) < z_l(k_2) < \hat{y} < z_r(k_2) < z_r(k_1)$ or

$\lambda(k_1) = [z_r(k_1) - z_l(k_1)] > [z_r(k_2) - z_l(k_2)] = \lambda(k_2)$.

This implies the assertion.
We need a technical auxiliary result to proceed:

Lemma A. Suppose that $k = g(y)$ is strictly increasing, concave (convex) and twice continuously differentiable in the interval (a,b) and suppose that $g'(y) \ne 0$ for $y \in (a,b)$. Then $y = g^{-1}(k)$ exists and is monotone, convex (concave), and twice continuously differentiable with respect to k .

Proof: The existence, monotonicity, and twice continuous differentiability of g^{-1} are assured by the inverse function theorem; see Fleet (1966; Th.10.9.5). Moreover, we have $g^{-1\prime}(k) = \dfrac{1}{g'\left(g^{-1}(k)\right)}$.

Now, the only task left is to prove the concavity (convexity) conversion. Differentiation of and application of the chain rule to the foregoing formula for

$$g^{-1\prime\prime}(k) = -\frac{g''\left(g^{-1}(k)\right)}{\left[g'\left(g^{-1}(k)\right)\right]^3},$$

which has sign opposite to that of $g''\left(g^{-1}(k)\right)$. This implies convexity (concavity) of $g^{-1}(k)$.

Proof of Lemma 3: We consider three cases where $\bar{k} = g_2(\dot{y}) = \dfrac{\bar{y}}{2}$.

Case (i): $k \in \left(\hat{y}f(\hat{y}), \infty\right)$. Then trivially $\lambda(k) = 0$ by Corollary 1-(i), that is, there does not exist such interval $[y_l, y_m]$.

Case (ii): $k \in \left(\dfrac{\bar{y}}{2}, \hat{y}f(\hat{y})\right]$. Then, by Lemma 2, $\lambda(\bar{k}) \geq \lambda(k)$. (This is, however, a little more than what Lemma 2 states. When $k = \hat{y}f(\hat{y})$, $\lambda(k)$ is equal to zero since y_l and y_r coincide. So we include this boundary point in the statement.)

Case (iii): $k \in \left(0, \dfrac{\bar{y}}{2}\right)$. Since both g_1 and g_2 are continuous, monotone, and twice differentiable, by the inverse function theorem, the following functions are well defined, unique, and twice differentiable:

$$h_1(k): \left[0, \frac{\bar{y}}{2}\right] \mapsto [0, \bar{y}] \text{ which } h_1(g_1(y)) = y \text{ for all } y \in [0, \bar{y}],$$

$$h_2(k): \left[0, \frac{\bar{y}}{2}\right] \mapsto [0, \dot{y}] \text{ which } h_2(g_2(y)) = y \text{ for all } y \in [0, \dot{y}].$$

Furthermore, by Lemma A, h_1 is monotone and strictly concave while h_2 is monotone and strictly convex. Therefore $\lambda(k) = h_1(k) - h_2(k)$ is strictly concave in k. Notice that h_1' is continuously decreasing from $h_1'(0) = \infty$ to $h_1'(\bar{k}) = \dfrac{1}{g_1'(\bar{y})}$

and h_2' is continuously increasing from $h_2'(0) = 1$ to $h_2'(\bar{k}) = \dfrac{1}{g_2'(\dot{y})}$. By Lemma 1

we already know $\bar{y} \geq \hat{y}$ which implies $g_2'(\bar{y}) = f(\bar{y}) + \bar{y}f'(\bar{y}) \leq 0$. Since $f(\bar{y}) = \dfrac{1}{2}$,

$\bar{y}f'(\bar{y}) \leq -\dfrac{1}{2}$ or $-\bar{y}f'(\bar{y}) \geq \dfrac{1}{2}$. Then $g_1'(y) = 1 - f(\bar{y}) - \bar{y}f'(\bar{y}) \geq 1 - \dfrac{1}{2} + \dfrac{1}{2} = 1$. Thus

$$\frac{1}{g_1'(\bar{y})} = h_1'(\bar{k}) \leq 1 < \frac{1}{g_2'(\dot{y})} = h_2'(\bar{k}).$$

Set $H(k) = h_1'(k) - h_2'(k)$.

H is strictly decreasing and continuous with $H(0) > 0$ and $H(\bar{k}) < 0$. By the

intermediate value theorem, there exists a unique $\tilde{k} \in \left(0, \frac{\bar{y}}{2}\right)$ such that

$H(\tilde{k}) = h_1'(\tilde{k}) - h_2'(\tilde{k}) = 0$, that is $\lambda'(\tilde{k}) = 0$. By strict concavity of $\lambda(k)$, such $\tilde{k}$ will be the unique global maximizer in $k \in \left[0, \frac{\bar{y}}{2}\right]$.

Case (i), (ii), and (iii) together imply $\lambda(\tilde{k}) \geq \lambda(k)$ for all $k \in \Re_+$. This completes the proof.

<u>Proof</u> of <u>Lemma</u> 4: Consider $\alpha > 0$, $y^* > 0$ and $k \geq 0$. Then:

$\eta \in \left[\alpha y_l(y^*; k), \alpha y_r(y^*; k)\right]$

$\Leftrightarrow \eta = \alpha y$ and $y \in \left[y_l(y^*; k), y_r(y^*; k)\right]$

$\Leftrightarrow \eta = \alpha y$ and $y \cdot f(y; y^*) \geq k$ and $y \cdot (1 - f(y; y^*)) \leq k$
$\Leftrightarrow \eta = \alpha y$ and $y \cdot f(\alpha y; \alpha y^*) \geq k$ and $y \cdot (1 - f(\alpha y; \alpha y^*)) \leq k$
$\Leftrightarrow \eta = \alpha y$ and $\alpha y \cdot f(\alpha y; \alpha y^*) \geq \alpha k$ and $\alpha y \cdot (1 - f(\alpha y; \alpha y^*)) \leq \alpha k$
$\Leftrightarrow \eta \cdot f(\eta; \alpha y^*) \geq \alpha k$ and $\eta \cdot (1 - f(\eta; \alpha y^*)) \leq \alpha k$

$\Leftrightarrow \eta \in \left[y_l(\alpha y^*; \alpha k), y_r(\alpha y^*; \alpha k)\right]$.

This shows that in the relevant range, $y_l(y^*; k)$ and $y_m(y^*; k)$ and, consequently, $\lambda(y^*, k)$ are homogeneous of degree 1 in (y^*, k).

Moreover, $f(\lambda \bar{y}(y^*), \lambda y^*) = f(\bar{y}(y^*), y^*) = \frac{1}{2}$ implies $\bar{y}(\lambda y^*) = \lambda \bar{y}(y^*)$.

Finally, (A3) implies $f(y; \lambda y^*) = f(y/\lambda; y^*)$ and, hence, $\frac{\partial}{\partial y} f(y; \lambda y^*) = \frac{1}{\lambda} \frac{\partial}{\partial y} f(y/\lambda; y^*)$. Therefore, $f(\hat{y}(y^*), y^*) + \hat{y}(y^*) \cdot \frac{\partial}{\partial y} f(\hat{y}(y^*), y^*) = 0$

if and only if $f(\lambda \hat{y}(y^*), \lambda y^*) + \lambda \hat{y}(y^*) \cdot \frac{\partial}{\partial y} f(\lambda \hat{y}(y^*), \lambda y^*) = 0$.

That means $\hat{y}(\lambda y^*) = \lambda \hat{y}(y^*)$.

<u>Proof</u> of <u>Proposition</u> 4: With (A3), $g_1(\alpha y; \alpha y^*) = \alpha y \cdot (1 - f(\alpha y; \alpha y^*)) =$
$\alpha y(1 - f(y; y^*)) = \alpha g_1(y; y^*)$, i.e. g_1 is homogeneous of degree 1 in $(y; y^*)$
Similarly, g_2 is homogeneous of degree 1 in $(y; y^*)$. Furthermore h_1, the inverse function of g_1 inherits the homogeneity of degree 1 in $(k; y^*)$, since
$g_1(\alpha y; \alpha y^*) = \alpha g_1(y; y^*) = \alpha k$ implies $h_1(\alpha k; \alpha y^*) = \alpha y = \alpha h_1(k; y^*)$. Similarly, it

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can be demonstrated that h_2, the inverse of g_2, is homogeneous of degree 1 in $(k; y^*)$. Therefore, by Euler's theorem,

$$0 = \frac{\partial^2 h_1}{\partial k \partial y^*} \cdot y^* + \frac{\partial^2 h_1}{\partial k^2} \cdot k,$$

With the strict concavity of h_1, we then have

$$\frac{\partial^2 h_1}{\partial k \partial y^*} = -\frac{\partial^2 h_1}{\partial k^2} \cdot \frac{k}{y^*} > 0. \tag{9}$$

Similarly, with the strict convexity of h_2,

$$\frac{\partial^2 h_2}{\partial k \partial y^*} = -\frac{\partial^2 h_2}{\partial k^2} \cdot \frac{k}{y^*} < 0. \tag{10}$$

Now $\lambda(k; y^*) = h_1(k; y^*) - h_2(k; y^*)$ with $\lambda(0; y^*) = 0$. Clearly, h_1 and h_2 are C^2 so that (9) and (10) imply that $\frac{\partial^2}{\partial y^* \partial k} \lambda(k; y^*) = \frac{\partial^2}{\partial k \partial y^*} \lambda(k; y^*) > 0$; hence (I)

From $\lambda(0; y^*) \equiv 0$ follows $\frac{\partial}{\partial y^*} \lambda(0, y^*) \equiv 0$ which together with

$\frac{\partial^2}{\partial k \partial y^*} \lambda(k; y^*) > 0$ yields $\frac{\partial}{\partial y^*} \lambda(k; y^*) > 0$ for all $k > 0, y^* > 0$. Therefore (II).

Finally, $\frac{\partial}{\partial k} \lambda(\tilde{k}(y^*); y^*) = 0$ together with $\frac{\partial^2}{\partial y^* \partial k} \lambda(k; y^*) > 0$ and strict concavity

of λ in k implies that $\tilde{k}(y^*) < \tilde{k}(y^{**})$ for $0 < y^* < y^{**}$, i.e. (III).

Proof of Proposition 5. We divide the proof into three parts:

(i)   From Lemma 3 and its proof, we know that for any $y^* > 0$, there exists a

unique $\tilde{k}(y^*) \in \left(0, \bar{y}(y^*)/2\right)$ such that $\lambda(n; y^*) < \lambda(m; y^*)$ for

$0 \le n \le m \le \tilde{k}(y^*)$.

(ii)   For $0 < y^* < y^{**}$, set $k(y^*, y^{**}) \equiv \tilde{k}(1) \cdot y^*$. Then $0 < k < k(y^*, y^{**})$ implies

$0 < k/y^{**} < k/y^* < \tilde{k}(1)$. Hence by (i), $\lambda\!\left(k/y^{**};1\right) < \lambda\!\left(k/y^*;1\right)$.

(iii)   Let $0 < y^* < y^{**}$ and $0 < k < k(y^*, y^{**})$. Then by Lemma 4 and (ii)
$\lambda(k; y^{**})/y^{**} = \lambda\!\left(k/y^{**};1\right) < \lambda\!\left(k/y^*;1\right) = \lambda(k; y^*)/y^*$

<u>Proof of Lemma 5</u>: By Corollary 1, when $k \in (\frac{\bar{y}}{2}, \hat{y}f(\hat{y})]$, only
complete-profit-transfer Take-it equilibria yield positive payoffs for PH. Then

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$$E\pi_p = \frac{1}{y^*}\int_{y_l}^{y_m}(a-c)ydy = \frac{1}{2y^*}(a-c)(y_m^2 - y_l^2).$$ Moreover,

$$\frac{\partial E\pi_p}{\partial k} = \frac{1}{y^*}(a-c)\left(y_m\frac{\partial y_m}{\partial k} - y_l\frac{\partial y_l}{\partial k}\right)$$

$$= \frac{1}{y^*}(a-c)\left(\frac{y_m}{g_2'(y_m)} - \frac{y_l}{g_2'(y_l)}\right) < 0$$

given that $y_m \geq \hat{y} \geq \bar{y} \geq y_l$ and other properties of $g_2(y)$, i.e., $g_2'(y_m) < 0$ and $g_2'(y_l) > 0$.

<u>Proof</u> <u>of</u> <u>Lemma</u> <u>6</u>: When $k \in \left[0, \frac{\bar{y}}{2}\right]$, only complete-profit-transfer and incomplete-profit-transfer Take-it equilibria yield positive payoffs for PH, Thus

$$E\pi_p = \frac{1}{y^*}\{\int_{y_l}^{y_m}(a-c)ydy + \int_{y_m}^{y_r}[(a-c)yf(y)+L]dy\}$$

$$= \frac{1}{y^*}\{\frac{1}{2}(a-c)(y_m^2 - y_l^2) + (a-c)[\int_{y_m}^{y_r}yf(y)dy + k(y_r - y_m)]\}$$

It suffices to show $\frac{1}{2}(a-c)(y_m^2 - y_l^2)$ and $(a-c)[\int_{y_m}^{y_r}yf(y)dy + k(y_r - y_m)]$ are both decreasing in k to prove the claim.
Denote

$$D(k) \equiv \frac{\partial(y_m^2 - y_l^2)}{\partial k} = 2\left(y_m\frac{\partial y_m}{\partial k} - y_l\frac{\partial y_l}{\partial k}\right) = 2\left(\frac{y_m}{g_2'(y_m)} - \frac{y_l}{g_2'(y_l)}\right).$$

By the convexity of g_1 and concavity of g_2 we can infer $g_1''(y_m) > 0, g_2''(y_l) < 0$ and $\frac{g_1'(y_m)}{[g_1'(y_m)]^3} - \frac{g_2'(y_l)}{[g_2'(y_l)]^3} < 0$. Thus it is clear that $D(k)$ is decreasing in $k \in \left[0, \frac{\bar{y}}{2}\right]$:

$$D'(k) = \frac{g_1'(y_m) - y_m g_1''(y_m)}{[g_1'(y_m)]^3} - \frac{g_2'(y_l) - y_l g_2''(y_l)}{[g_2'(y_l)]^3} < 0$$

Recall that $g_1'[y_m(\tilde{k})] = g_2'[y_l(\tilde{k})]$ at $\tilde{k}$, then

$$D(\tilde{k}) = \frac{2}{g_1'(y_m)}(y_m - y_l) > 0.$$

By lemma 3 we know $g_1'(\bar{y}) \geq 1$ and $g_2'(\bar{y}) = 0$ so that

$$D\left(\frac{\bar{y}}{2}\right) = 2\bar{y}\cdot(\frac{1}{g_1'(\bar{y})} - \frac{1}{g_2'(\bar{y})}) < 0$$

Thus by the intermediate value theorem, there exists a $\overset{\circ}{k} \in \left(\tilde{k}, \frac{\bar{y}}{2}\right)$ such that

$D\left(\overset{\circ}{k}\right)=0$ and $D(k)\le 0$ for $k\in\left(\overset{\circ}{k},\dfrac{\bar{y}}{2}\right)$. It is then immediate that

$\dfrac{1}{2}(a-c)(y_m^2-y_l^2)$ is decreasing in $\left[\overset{\circ}{k},\dfrac{\bar{y}}{2}\right]$ since its first order derivative is non-positive in the relevant range.

Now we verify that the second term, $(a-c)[\int_{y_m}^{y_r}yf(y)dy+k(y_r-y_m)]$ is decreasing in k. Observe that

$$\frac{\partial E\pi_p}{\partial k}=\frac{1}{y^*}\{\frac{1}{2}(a-c)D(k)+(a-c)\{\frac{g_2(y_r)}{g_2'(y_r)}-\frac{g_2(y_m)}{g_1'(y_m)}+k[\frac{1}{g_2'(y_r)}-\frac{1}{g_1'(y_m)}]\}\}.$$

It is easy to verify that both $\dfrac{g_2(y_r)}{g_2'(y_r)}-\dfrac{g_2(y_m)}{g_1'(y_m)}$ and $[\dfrac{1}{g_2'(y_r)}-\dfrac{1}{g_1'(y_m)}]$ are both negative by the convexity (concavity) of $g_1(g_2)$ for k in the relevant range. This, together with the term $\dfrac{1}{2}(a-c)(y_m^2-y_l^2)$ decreasing, implies that there exist $\overset{\circ}{k}\in\left(0,\dfrac{\bar{y}}{2}\right)$ such that $E\pi_p(k)$ is decreasing in $\left[\overset{\circ}{k},\dfrac{\bar{y}}{2}\right]$. ∎∎

Proof of Lemma 7:
Denote

$$E(k;y^*)\equiv\frac{E\pi_p(k;y^*)}{E\pi(k;y^*)}\text{, then}$$

$$E(k;y^*)=\frac{\frac{1}{y^*}\{\int_{y_l}^{y_m}(a-c)ydy+\int_{y_m}^{y_r}[(a-c)yf(y;y^*)+L]dy\}}{\frac{1}{y^*}\int_0^{y^*}(a-c)ydy}$$

$$=\frac{\frac{1}{2}(a-c)(y_m^2-y_l^2)+(a-c)[\int_{y_m}^{y_r}yf(y;y^*)dy+k(y_r-y_m)]}{\frac{1}{2}(a-c)(y^*)^2}$$

$$=\frac{(y_m^2-y_l^2)+2[\int_{y_m}^{y_r}yf(y;y^*)dy+k(y_r-y_m)]}{(y^*)^2}$$

Let $\lambda>0$. Then by the homogeneity of y_l, y_m, y_r and $f(y;y^*)$ from Lemma 4, we know that

$$E\left(\lambda k;\lambda y^*\right)=\frac{\lambda^2(y_m^2-y_l^2)+2[\int_{\lambda y_m}^{\lambda y_r}yf(y;\lambda y^*)dy+\lambda^2 k(y_r-y_m)]}{\lambda^2(y^*)^2}$$

$$=\frac{(y_m^2-y_l^2)+2[\int_{y_m}^{y_r}yf(y;y^*)dy+k(y_r-y_m)]}{(y^*)^2}$$

$$=E\left(k;y^*\right)\ \blacksquare\blacksquare$$

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