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On a system of integro-differential equations of parabolic type

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Abstract

In this paper, the existence of the solutions for nonlinear integro-differential systems is discussed. The comparison theorem is first obtained by assuming the existence of weak upper and weak lower solutions for the given problem. We then use monotone method to construct two sequences which converge monotonically to the solution.

1. Introduction

The existence of solutions for a nonlinear integro-differential equation of the form

$$\frac{\partial u}{\partial t} - \Delta u = \Phi(x,t,u,F(u)),$$

is investigated recently by many authors, where $F(u)$ is a nonlinear integral operator. Such equations are used for the description of the physical processes in a nuclear reactor, of those of population, and of other processes. Several specific models in various fields in applied science can be found in [6]. Some classical existence results of parabolic integro-differential equations can be found in [7], [8], [9] and [10]. Carl gave the existence of weak solutions for nonlinear parabolic systems in [2] and [3].

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In this paper, we shall consider the existence of weak solutions of more general systems for nonlinear integro-differential equations of the form

$$\frac{\partial u_k}{\partial t} - L_k u_k = g_k(x,t,u,F(u)),$$

here $L_k = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}^k(x,t) \frac{\partial}{\partial x_j} \right) - \sum_{i=1}^{N} b_i^k(x,t) \frac{\partial}{\partial x_i}$. Constructive monotone-scheme is used. We first give a comparison theorem for weak upper and weak lower solution. By using such a comparison result and monotone method, we then establish an existence result for a system of parabolic integro-differential equations. These results extend the result of [3] to more general systems.

2. Preliminaries and hypotheses

We consider the nonlinear initial-boundary value problem of the form

$$\begin{aligned} &\frac{\partial u_k}{\partial t} - L_k u_k = g_k(x,t,u,F(u)) \text{ in } Q_T, \\ &u_k(x,0) = \varphi_k(x) \text{ in } \Omega, \\ &u_k(x,t) = 0 \text{ on } \Gamma_T, \end{aligned} \tag{2.1}$$

for $k = 1,\dots,n$, where $u \in R^n$, $\Omega \subseteq R^N$ is a bounded domain with smooth boundary $\partial\Omega$, $Q_T = (0,T)\times\Omega$, $\Gamma_T = (0,T)\times\partial\Omega$, $T > 0$. Let $F(u) = (F_1(u_1),\dots,F_n(u_n))$, be an integral operator, for example, $F_i(u_i)(x,t) = \int_\Omega k_i(x,y)u_i(y,t)dy$, $1 \le i \le n$.

Assume that the operator L_k is uniformly elliptic in Q_T,. Let the coefficients a_{ij}^k and b_i^k be real bounded measurable in Q_T, and $\varphi_k \in L^2(\Omega)$. Let $V = L^2(0,T;(W^{1,2}(\Omega))$ and $V^* = L^2(0,T;(W^{1,2}(\Omega))^*)$,. We consider the solutions of (2.1) in the space $W = \{h \mid h \in V, \frac{\partial h}{\partial t} \in V^*\}$ (see [5]), where $\frac{\partial}{\partial t}$ denotes the distributional derivative in V^*. The spaces V, V^* and W are Banach spaces equipped with the norms $\|h\|_V$, $\|h\|_{V^*}$ and $\|h\|_W$ respectively. Denote W_0, V_0, and V_0^* the corresponding spaces if the Sobolev space $W^{1,2}(\Omega)$ in the definitions of W, V, and V^* is replaced by its subspace $W_0^{1,2}(\Omega)$, which is the space of all functions of $W^{1,2}(\Omega)$ with zero traces on $\partial\Omega$ ([1]). Let B be a Banach space, we denote

$B^n = B \times \cdots \times B$ by the n-dimensional Cartesian product of B, which is again a Banach space equipped with the norm $\|(u_1,\ldots,u_n)\|_{B^n} = \sum_{i=1}^{n} \|u_i\|_B$. Let

$$\ell_k(h,\chi) = \int_{Q^T} \left(\sum_{i,j=1}^{N} a_{ij}^k \frac{\partial h}{\partial x_j} \frac{\partial \chi}{\partial x_i} + \sum_{i=1}^{N} b_{ij}^k \frac{\partial h}{\partial x_i} \chi \right) dxdt,$$

and $<\cdot,\cdot>$ denote the scalar product of elements from V^* and V.

Definition : A function $u: Q_T \to R^n$ in W_0^n is called a weak solution of (2.1) if the following conditions are fulfilled:

(i) $u(x,0) = \varphi(x)$ in the sense that $\lim_{t\to 0} \|u_k(\cdot,t) - \varphi_k\|_2 = 0$, $k = 1,\ldots,n$.

(ii) $<\frac{\partial u_k}{\partial t}, \chi> + \ell_k(u_k,\chi) = \int_{Q_T} g_k(x,t,u,F(u))\chi dxdt$, for all $\chi \in V_0$, $k = 1,\ldots,n$.

Let $[u,v] = \{z \in (L^2(Q_T))^n \mid u_k \le z_k \le v_k \text{ a.e. } in\ Q_T \text{ for } k = 1,\ldots,n\}$. For $u \in R^n$, denote $[u]_k = (u_1,\ldots,u_{k-1},u_{k+1},\ldots,u_n)$. Let $M = (M_{ij})$, $i,j = 1,\ldots,n$ be any $n \times n$ matrix and denote $[M]_k = (M_{k,1},\ldots,M_{k,k-1},M_{k,k+1},\ldots,M_{k,n}) \in R^{n-1}$ for each $k = 1,2,\ldots,n$.

For $u,v \in R^n$, define

$$G_k(x,t,u_k,[v]_k,F_k(u_k),[F(v)]_k) = g_k(x,t,u_k,[v]_k,F_k(u_k),[F(v)]_k) - [M]_k[v]_k$$

where $[\cdot]_k[\cdot]_k$ is the scalar product in R^{n-1}.

Assume that

(B0) there exist vectors ϕ and $\psi \in W^n$ such that

$$\frac{\partial \phi_k}{\partial t} - L_k\phi_k - [M]_k[\phi]_k \ge G_k(x,t,\phi_k,[\psi]_k,F_k(\phi_k),[F(\psi)]_k) \quad \text{in } Q_T,$$
$$\phi_k(x,0) \ge \varphi_k(x) \quad \text{in } \Omega, \qquad (2.2)$$
$$\phi_k \ge 0 \quad \text{on } \Gamma_T$$

and

$$\frac{\partial \psi_k}{\partial t} - L_k\psi_k - [M]_k[\psi]_k \le G_k(x,t,\psi_k,[\phi]_k,F_k(\psi_k),[F(\phi)]_k) \quad \text{in } Q_T$$
$$\psi_k(x,0) \le \varphi_k(x) \quad \text{in } \Omega \qquad (2.3)$$
$$\psi_k \le 0 \quad \text{on } \Gamma_T$$

for a $n \times n$ matrix $M = (M_{ij}) \geq 0$ such that $G_k(\cdot,\cdot,\zeta,\cdot)$ is monotone nonincreasing in ζ_i, for $i \neq k$.

(B1) $g_k(x,t,u,p)$, $k = 1,2,...,n$, are of Caratheodory type and satisfy the inequality

$$g_k(\cdot,\cdot,u,p) - g_k(\cdot,\cdot,v,q) \leq \sum_{i=1}^{n} \gamma_{ki}|u_i - v_i| - \sum_{i=1,i\neq k}^{n} \gamma_{ki}(p_i - q_i) + \gamma_{kk}|p_k - q_k| ,$$

for $u,v \in R^n$, $p,q \in R^n$ and for some nonnegative constants γ_{ki}, $1 \leq i \leq n$.

(B2) $g_k(\cdot,\ \cdot,\ \cdot,\ \eta)$ are monotone nondecreasing with respect to η_k and monotone nonincreasing with respect to η_i, $i \neq k$.

(B3) Each F_i, $i = 1,2,...,n$, is Lipschitz continuous on $L^2(Q^+)$, that is, there exists a constant δ such that $\|F_i(u) - F_i(v)\|_2 \leq \delta \|u - v\|_2$, for $u,v \in L^2(Q^+)$, here $Q^+ = \{(x,t) \in L^2(Q_T) |\ u(x,t) \geq v(x,t) \text{ a.e. in } Q_T\}$.

(B4) Each F_i, $i = 1,2,...,n$, is monotone nondecreasing in u, i.e., $u \leq v$ in Q_T implies $F_i(u) \leq F_i(v)$ in Q_T

3. Main results

We first give the following comparison result.

Theorem 3.1 (Comparison): Let (B0), (B1) and (B3) be satisfied for a pair of functions $\phi,\psi \in W^n$ with respect to the order interval $I = [\inf(\phi,\psi),\sup(\phi,\psi)]$. Then $\psi \leq \phi$ in Q_T.

Corollary 3.2: Let $c^k(x,t)$ be real bounded measurable functions in Q_T for $k = 1,...,n$. Replacing L_k by $L_k^c = L_k - c^k(x,t)$ in Theorem 3.1. We also obtain the same conclusion of Theorem 2.1.

The following main result is shown by using the comparison theorem and the existence and uniqueness results for a linear couple system given in [5] therough monotone scheme.

Theorem 3.3 (Existence) : Let (B0)-(B4) be satisfied for a pair of functions $\phi,\psi \in W^n$ with respect to $I=[\inf(\psi,\phi),\sup(\psi,\phi)]$. Then the initial-boundary value problem (2.1) has a solution $u \in W^n$ with $\psi \le u \le \phi$ in Q_T.

Theorem 3.4 (Uniqueness) : Assume that all assumptions of theorem 3.3 are satisfied except that assumption (B1) is replaced by the following condition :

(B1*) $g_k(x,t,u,p)$, $k=1,2,\ldots,n$, are of Caratheodory type and satisfy the inequality

$$g_k(\cdot,\cdot,u_k,[u]_k,p_k,[p]_k)-g_k(\cdot,\cdot,v_k,[v]_k,q_k,[q]_k)$$
$$\le \gamma_{kk}\left(|u_k-v_k|+|p_k-q_k|\right)+[\gamma]_k\left([u-v]_k-[p-q]_k\right),$$

for $u,v \in R^n$, $p,q \in R^n$ and for some nonnegative constants γ_{ki}, $1 \le i \le n$.

Then there exists a unique solution of (2.1).

Example 3.5 : Consider the following initial-boundary value problem :

$$\begin{aligned}
&\frac{\partial u_1}{\partial t}-\Delta u_1=\cos u_1 \sin u_2+\int_\Omega a\cdot u_1(x,t)dx-\int_\Omega b\cdot u_2(x,t)dx \text{ in } Q_T,\\
&\frac{\partial u_2}{\partial t}-\Delta u_2=\sin u_1 \cos u_2-\int_\Omega a\cdot u_1(x,t)dx+\int_\Omega b\cdot u_2(x,t)dx \text{ in } Q_T,\\
&u_1(x,0)=\varphi_1(x)\ ,\ u_2(x,0)=\varphi_2(x) \text{ in } \Omega\\
&u_1(x,t)=u_2(x,t)=0 \text{ on } \Gamma_T \text{ in } \Omega,
\end{aligned} \tag{3.1}$$

here a and b are some nonnegative constants.

Assume that $0 \le \varphi_k(x) \le L$ in Ω.

Let $\phi_k=-\psi_k=\varepsilon\, e^{\lambda t}$, $k=1,2$, where ε and λ are chosen to satisfy that $\varepsilon \ge \max\{1,L\}$ and $\lambda \ge 3+(a+b)Vol(\Omega)$. Then all assumptions (B0)-(B4) are satisfied. By theorem 3.3, there is a solution u of (3.1) lying between $-\varepsilon e^{\lambda t}$ and $\varepsilon e^{\lambda t}$.

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