行政院國家科學委員會補助專題研究計畫成果報告

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※ 封閉式等候網路機率分配的數值計算法 ※
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行政院國家科學委員會專題研究計畫成果報告

封閉式等候網路機率分配的數值計算法

Estimation of Probability Distributions on Closed Queueing Networks

計畫編號：**NSC 89-2118-M-004-012**
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1 Abstract

In this report, we are concerned with the property of a two-stage closed system in which the service times are identically of phase type. We first conjecture that the Laplace-Stieltjes Transforms (LST) of service time distributions may satisfy a system of equations. Then we present that the stationary probabilities on the unboundary states can be written as a linear combination of product-forms. Each component of these products can be expressed in terms of roots of the system of equations. Finally, we establish an algorithm to obtain all the stationary probabilities. The algorithm is expected to work well for relatively large customers in the system.

(Keywords : $PH/G/1$ Queueing, phase type, Matrix-geometric solutions, stationary probabilities, product-form)

中文摘要
在這一篇文章中，我們討論兩個階段的封閉式等候線網路，其中服務時間的機率分配都是 *Phase type* 分配。我們猜測服務時間的機率分配和離開時間間隔的機率分配的拉氏轉換 *(LST)* 滿足一組聯立方程組。然後，我們推導出非邊界狀態的穩定機率可以被表示成 *product-form* 的線性組合，而每個 *product-form* 可以用聯立方程組的根來構成。利用非邊界狀態的穩定機率，我們可以求出邊界狀態的機率。最後我們建立一個求穩定機率的演算過程。利用這個演算方法，可以簡化求穩定機率的複雜度。

(關鍵字 :$/PH/1$ 等候系統，幾何矩陣的解，進入過程的相關變數)

2 Introduction

Luh [7] studied a two-stage multi-server $PH/PH/c_1 \to /PH/c_2$ queueing system where both interarrival times and service times are of the phase type. The system is constructed with an infinite buffer and multiple servers in each stage. He showed that the stationary distribution of the number of customers in the system when all servers are busy is a linear combination of product-forms.

The solution technique is based on a novel approach that was taken to solve a $PH/PH/1$ system by Le Boudec [6]. He showed that all the eigenvectors used in the expression of the stationary probabilities of the $PH/PH/1$ system are Kronecker products and gave a simple formula for computing the stationary probabilities of the number of customers in the system. Because the solution of $PH/PH/1$ can be expressed in terms of roots of the associated characteristic polynomial, we may reduce the state balance equations to a vector difference equation with constant coefficients for a basis of separable solution of the equation of saturated states. Luh [7] use this basis to construct a linear combination that also satisfies the conditions at boundaries of the state space.

Since analysis of the multi-server system is complicated, we restricte our discussion on single-server closed system with phase type service time distribution. In this thesis, we are concerned with a two-stage closed queueing system, denoted by $-/PH/1 \to /PH/1$ in which the service times are identically of phase type. We use a similar method to consider the $-/PH/1 \to /PH/1$ closed system. We solve a $-/PH/1 \to /PH/1$ queueing system by using the result on the Matrix-geometric form solution of a quasi-birth-and-death (QBD) process with a countable number of phases in each level. The result yields a new expression of the stationary distribution which can be written as a linear combination of product-forms and may be used to compute other performance measures, such as the delay probability, the moments of the queue size distribution and the waiting time distribution.

3 Problem statement

We consider a two-stage $-/PH/1 \rightarrow /PH/1$ closed queueing system containing finite number of customers, say N. Each stage has one server. Let k represents the stage number, $k = 1, 2$. The service time of the k-th stage are independent and identically distributed (i.i.d.) random variables subjecting to an irreducible phase-type distribution $PH(\boldsymbol{\beta}_k, \mathbf{S}_k)$ with J_k phases and service rate μ_k. Customers are served under the First-come First-serve discipline (FCFS).

The number of customers already present in the k-th stage (including those in service) is denoted by n_k. The system state is $\mathbf{n} = (n_1, n_2; i_1, i_2)$ where i_k is the phase of the service process at the j-th server of the k-th stage. The index i_k is interpreted to be zero if the corresponding server is idle.

Denote by $\mathbf{I}$ the identity matrix and $\mathbf{e}'$ the column vector of all entries equal to 1. Denote by $S_k^*(s)$ the Laplace-Stieltjes Transforms of the service time distributions.

4 Propositions

We conjecture that the two-stage $-/PH/1 \rightarrow /PH/1$ closed system have the following properties. Base on the solutions of the following equations, we can construct a solution for the stationary probabilities of unboundary states.

Conjecture :

We consider the system of equations for h s_1, and s_2:

$$\begin{cases} S_1^*(s_1) = h^{-1} \\ S_2^*(s_2) = h \\ s_1 + s_2 = 0 \end{cases} \tag{1}$$

Suppose one of solutions is $(h, s_1, s_2) = (\eta, \omega_1, \omega_2)$. Based on the solutions, we construct a solution basis for the stationary probabilities of unboundary states when $\eta \neq 1$.

5 Transition rate matrix

We arrange the states (n_1, n_2, i_1, i_2) in lexicographic order and partition of the state space according to the number of customers, n_1, i.e.

$$\mathcal{L}_m = \{(n_1, n_2, i_1, i_2) | n_1 = m, n_1 + n_2 = N\}$$

Denoted by $\boldsymbol{\pi}$ the stationary probability of row-vector partitioned corresponding to $\mathcal{L}_m$ Then $\mathbf{Q}$ is of the block-tridiagonal form and rewritten as:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{B}_0 & \mathbf{A}_0 & & & & & \\ \mathbf{C}_0 & \mathbf{B} & \mathbf{A} & & & & \\ & \mathbf{C} & \mathbf{B} & \mathbf{A} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \mathbf{C} & \mathbf{B} & \mathbf{A} & \\ & & & & \mathbf{C} & \mathbf{B} & \mathbf{A}_1 \\ & & & & & \mathbf{C}_1 & \mathbf{B}_1 \end{bmatrix}$$

The submatrices could be represented by Kronecker product ($\otimes$) and Kronecker sum ($\oplus$).

$$\begin{array}{lll} \mathbf{A}_0 = \boldsymbol{\beta}_1 \otimes \boldsymbol{\gamma}_2 \boldsymbol{\beta}_2 & \mathbf{B}_0 = \mathbf{S}_2 & \mathbf{C}_0 = \boldsymbol{\gamma}_1 \otimes \mathbf{I}_2 \\ \mathbf{A} = \mathbf{I}_1 \otimes \boldsymbol{\gamma}_2 \boldsymbol{\beta}_2 & \mathbf{B} = \mathbf{S}_1 \oplus \mathbf{S}_2 & \mathbf{C} = \boldsymbol{\gamma}_1 \boldsymbol{\beta}_1 \otimes \mathbf{I}_2 \\ \mathbf{A}_1 = \mathbf{I}_1 \otimes \boldsymbol{\gamma}_2 & \mathbf{B}_1 = \mathbf{S}_1 & \mathbf{C}_1 = \boldsymbol{\gamma}_1 \boldsymbol{\beta}_1 \otimes \boldsymbol{\beta}_2 \end{array}$$

For the balance equations $\boldsymbol{\pi}\mathbf{Q} = 0$, we obtain the following equations:

$$\begin{cases} \boldsymbol{\pi}_0 \mathbf{B}_0 + \boldsymbol{\pi}_1 \mathbf{C}_0 = \mathbf{0} & (2) \\ \boldsymbol{\pi}_0 \mathbf{A}_0 + \boldsymbol{\pi}_1 \mathbf{B} + \boldsymbol{\pi}_2 \mathbf{C} = \mathbf{0} & (3) \\ \boldsymbol{\pi}_{m-1} \mathbf{A} + \boldsymbol{\pi}_m \mathbf{B} + \boldsymbol{\pi}_{m+1} \mathbf{C} = \mathbf{0} \quad 2 \leq m \leq N-2 & (4) \\ \boldsymbol{\pi}_{N-2} \mathbf{A} + \boldsymbol{\pi}_{N-1} \mathbf{B} + \boldsymbol{\pi}_N \mathbf{C}_1 = \mathbf{0} & (5) \\ \boldsymbol{\pi}_{N-1} \mathbf{A}_1 + \boldsymbol{\pi}_N \mathbf{B}_1 = \mathbf{0} & (6) \end{cases}$$

For $2 \leq m \leq N-2$

$$\boldsymbol{\pi}_m (\mathbf{S}_1 \oplus \mathbf{S}_2) = \boldsymbol{\pi}_{m-1} (\mathbf{I}_1 \otimes \mathbf{S}_2 \mathbf{e}_2' \boldsymbol{\beta}_2) + \boldsymbol{\pi}_{m+1} (\mathbf{S}_1 \mathbf{e}_1' \boldsymbol{\beta}_1 \otimes \mathbf{I}_2) \tag{7}$$

6 Product form solutions

We look for a solution $\mathbf{w}_m$ of (7), which has the form

$$\mathbf{w}_m = \eta^m (\mathbf{u}_1 \otimes \mathbf{u}_2) \tag{8}$$

where $\mathbf{u}_1 \in C^{J_1}$, $\mathbf{u}_2 \in C^{J_2}$, $\eta \neq 1$

Suppose one of the solutions for (1) is $(\eta, \omega_1, \omega_2)$. Let

$$\mathbf{u}_1 = b_1 \boldsymbol{\beta}_1 (\mathbf{S}_1 - \omega_1 \mathbf{I}_1)^{-1} \text{and } \mathbf{u}_2 = b_2 \boldsymbol{\beta}_2 (\mathbf{S}_2 - \omega_2 \mathbf{I}_2)^{-1}$$

where b_1, b_2 are constants satisfying $\mathbf{u}_1 \mathbf{e}_1' = \mathbf{u}_2 \mathbf{e}_2' = 1$. We obtain

$$b_1 = \frac{\omega_1 \eta}{1 - \eta}, \text{ and } b_2 = \frac{\omega_2}{\eta - 1}, \text{for } \eta \neq 1.$$

We can easily derive

$$\mathbf{u}_1 \mathbf{S}_1 = b_1 \boldsymbol{\beta}_1 + \omega_1 \mathbf{u}_1, \ \mathbf{u}_2 \mathbf{S}_2 = b_2 \boldsymbol{\beta}_2 + \omega_2 \mathbf{u}_2.$$

$$\mathbf{u}_1 \mathbf{S}_1 \mathbf{e}_1' = \frac{b_1}{\eta}, \ \mathbf{u}_2 \mathbf{S}_2 \mathbf{e}_2' = b_2 \eta$$

Now, we want to show $\mathbf{w}_m$, $1 \leq m \leq N-1$, satisfies equation (7) . Inserting (8) into (7) and dividing by η^m, we balance the equation (7).

Now any linear combination of $\mathbf{w}_m$ obviously satisfies the state balance equations for unboundary condition. We have obtained $J_1 + J_2 - 1$ solutions of $(\eta, \omega_1, \omega_2)$ in which we denote with indices $\imath$. We assume unboundary probabilities have linear combination of $\mathbf{w}_m(\imath)$ which has form as:

$$\boldsymbol{\pi}_m = \sum_{\imath=1}^{J_1+J_2-1} c_\imath \mathbf{w}_m(\imath). \qquad (9)$$

where $1 \leq m \leq N-1$ and $c_\imath$ are the coefficients respect to $\mathbf{w}_m(\imath)$. For each choice coefficient c_i, $\boldsymbol{\pi}_m$ satisfies equation (7).

7 Algorithm for the boundary probabilities

In this section, we are interested in finding boundary stationary probabilities $\boldsymbol{\pi}_0$ and $\boldsymbol{\pi}_N$. Recall that $\boldsymbol{\pi}_0$ is a row vector of size J_2 and $\boldsymbol{\pi}_N$ of size J_1. Therefore the total number of unknown variables are $2J_1 + 2J_2 - 1$. Inserting $\boldsymbol{\pi}_m$ into remaining equations (2), (3), (5) and (6). Because those equations form a linear homogeneous system of equations, we should substituted one of them by the normalization condition equation $\boldsymbol{\pi}\mathbf{e}' = 1$. We now have a system of linear nonhomogeneous equations. Total number of equations are $J_1 + J_2 + 2J_1J_2$ which is greater than unknown variables. The solution of this problem may be solved by using some numerical methods, e.g. the least square algorithm. We can get good approximation solutions.

8 A summary of the algorithm

We describe the algorithm for solving stationary probabilities of $-/PH/1 \rightarrow /PH/1$ system with $\rho_1 \geq \rho_2$.

Step 1 Solve system of equations (1).

Step 2 For solutions $(\eta, \omega_1, \omega_2)$, where $\eta \neq 1$. Compute $\mathbf{w}_m(\imath)$, then $\boldsymbol{\pi}_m$ be a linear combination of $\mathbf{w}_m(\imath)$.

Step 3 Set a linear nonhomogeneous system consisting of equations (2), (3), (5) and (6).

Step 4 Solve linear nonhomogeneous system and obtain coefficients of $\boldsymbol{\pi}_m$ and boundary probabilities.

Since all the stationary probabilities for unboundary states are expressed in terms of the product-forms, we can omit equation(7) when solve $\boldsymbol{\pi}\mathbf{Q} = 0$ and $\boldsymbol{\pi}\mathbf{e}' = 1$. It is important to note that no matter how many customers in the closed system, we only need to solve $\boldsymbol{\pi}_0$, $\boldsymbol{\pi}_N$ and coefficients $c_\imath$. Hence the computational complexity is reduced.

9 Conclusion

In this report,we have shown the stationary probabilities of unboundary states can be written as a linear combination of product-forms. Furthermore, we found that each component of these products can be expressed in terms of roots of the system of equations which involve the Laplace-Stieltjes transform of the service time distributions. We provided an efficient computational algorithm for solving stationary probabilities, which is expected to work well for relatively large systems with high traffic intensities. Hence the computational complexity can be reduced.

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