使用整數目標規劃建立指數基金

Construct Index Fund via Integer Goal Programming

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中文摘要

建立追隨指數的一籃子股票組合已逐漸被許多投資者或投資機構所接受，並成為全方位投資策略的一環。實務上，最廣泛被使用的方法有簡易法與優化法。簡易法係結合幾種選取法則所構成(如分層抽樣法)只找到某個可行的組合，其中並未導入優化的概念。優化法則是藉由數學規畫模型尋找一個股票組合使傳統的追蹤誤差為最小，但此方法需用到極長的計算時間才能獲得解答。為使最優化與執行效率兩者得以兼顧，我們提出一種新的追蹤誤差衡量法，其定義為所選取之一籃子股票組合與指數間的絕對值誤差。並提出使用整數目標規劃建立指數基金的數學規畫模型與解法。

Abstract

Creating index-tracking stock baskets has been accepted by more and more investors or institutes as one part of a total investment strategy. In practice, the selection methods widely adopted are some simplified methods (e.g. stratification) combined with some criteria, and some optimization models to minimize the traditional tracking error. Simplified method facilitates for obtaining a feasible answer, optimal in no sense, while the optimization model usually requires larger computational efforts. For bridging the gap between having efficiency and seeking optimality, we propose a new measure of tracking error basing on the absolute difference between the value of the benchmark and the index computed from the portfolio obtained from our model. We also present a goal programming model and develop an efficient solution algorithm.

1. Evolution of the Index Fund

The index fund commenced in 1970 when the Wells Fargo Bank introduced the Stagecoach Fund, designed to track the New York Stock Exchange Composite Index. Perhaps due to the overall poor performance of managed funds and the market in general during the late 1960s and early 1970s, and because of a surging awareness of the efficient-market hypothesis and the random-walk theory, this new type of fund has begun to catch the increasing popularity.

Lodge (1993) mentioned that in United Kingdom (UK) over the past three years, only two general UK equity trusts beat the best UK tracker fund. Willis (1996) also described that over the past ten years in the US, only 28% of all stock funds have outpaced the market indexes they were designed to beat. Growing numbers of investors can't think of a good reason to pay mutual-fund managers for that kind of performance. So they are turning to index funds, which aim only to match the market's performance by investing in a portfolio of securities that mirror a broad index, such as the Standard & Poor's 500 (S&P 500).

Kirby (1993) estimates that of the $1.25 trillion to $1.5 trillion of the total institutional investment in the stock market in 1992, about 25% to 30% was indexed. As of December 1988, about 55% of the defined benefit assets of the 200 largest pension funds were indexed, with 36% of all index fund investments internally managed. In 1997, the Institutional Investor's July Pensionforum also reported that: Overall, respondents tend to index about 1/4 to 1/2 of their equity assets.

Further, more and more investors are looking at international markets outside their country. This encourages some index fund to track international indexes. Givant (1988) reported that there exists some substantial discrepancy in performance among those index funds on account of different strategies they adopted.

2. Available Approaches

Pertaining to the method of constructing an index fund, Black and Scholes (1974), Shapiro (1976), as well as Good, Ferguson, and Treynor (1976) have compendiously commented on the problem without suggesting solutions. Rudd (1980) proposed an optimization model to construct the index fund. The model is based on quadratic programming. The optimization model is transcribed as follows:

$$\min \ w_p^2$$

subject to:

$$\beta_P = \sum_{i=1}^{n} \beta_i u_i = 1$$

$$\sum_{i=1}^{n} u_i = 1, \ \ u_i \geq 0, \ \ \forall i = 1, 2, \ldots, n.$$

where w_p^2 is portfolio residual variance; β_P is the portfolio beta, defined as the weighted sum of the asset betas, where the weights are the asset holdings and u_i is the proportion of the fund to be invested in the i-th asset.

Their model can be paraphrased in another way: Form a portfolio with a systematic risk level equal to that of the S&P 500, that is, with a beta equal to unity relative to the index, and minimum residual risk relative to the S&P 500.

Andrews, Ford, and Mallison (1986) examined and compared three methods of constructing the index fund. These methods are outlined below:

Full replication: All stocks in the index are held in the equivalent proportion as the index. There are high set-up, maintenance and investment costs associated with this strategy. These are atoned for by nice tracking of the index and little demand for managerial decision. This method dwindles its attraction as the number of companies included in the index proliferates.

Stratification: Stocks are selected to attain the same sector representation and proportional capitalization as the index, but omitting companies below a prescribed level of capital. This method has akin, but scaled down, benefits and drawbacks to full replication.

Sampling: The stocks of a diminutive number of companies are picked to emulate the overall behavior of the market. Low set-up and maintenance costs are its hallmarks. Tracking efficacy hinges on the quality of the database. The crucial assumption is that historical relationships between share prices and index values persist.

Two criteria have been exploited by Toy and Zurack (1989) for tracking the Europe-Pacific(Euro-Pac) Index, an international index that do not include U.S. securities. The first criterion is to measure the risk of each stock in the Euro-Pac Index in relation to the total index. The second criterion is to measure the risk of each stock in relation to its home country stock index. These two criteria were utilized in building stock baskets to track the Euro-Pac Index.

Meade and Salkin (1989) compared a number of different approaches to index fund selection. These approaches used either estimated coefficients (in this case the size of the holding of a particular share is determined according to some statistical criterion) or capitalization weighting (in this case the proportion of the fund invested in a particular company is the ratio of the company's market value to that of all the companies in the fund). The objective was the minimization of historical tracking error. In the first case this objective was pursued using quadratic programming; in the second case a heuristic zero-one selection procedure was used. The effects of adding stratification constraints, where the fund had the same proportion invested in different industrial sectors as the market, was also examined. The index fund based on estimated coefficients was shown to perform better than the capitalization weighted fund. Stratification did not lead to appreciable improvements in tracking performance.

Meade and Salkin (1990) developed a quadratic programming model with the objective of minimizing the expected cost of maintaining an index fund over a given time horizon. They began with proposing an stochastic model for the returns on the individual shares and on the index. Their model is a simplification of the autoregressive conditional heteroscedasticity (ARCH) model described by Taylor (1989). Further, they assumed that the maintaining costs are proportional to the square of the difference between the return on the index and the return on the index fund.

Kornbluth, Meade, and Salkin (1993) formulated a goal-programming duration model to create bond portfolios for the sake of tracking the 5-15 year UK gilts index as closely as possible. They showed that duration moments could be useful for constructing index-tracking bond portfolios.

Tabata and Takeda (1995) considered a bicriteria optimization problem of index fund To simultaneously minimize the tracking error and the number of securities encompassed in the index fund. In attempting to minimize the tracking error under the given number of securities included in the index fund, they formulated it as a quadratic programming problem with zero-one variables. On account of difficulty in solving this combinatorial optimization problem in reality, they submitted an efficient approach to obtain a local optimal solution, which is a compromise answer to this bicriteria optimization problem.

3. Considerations in Tracking Index

Once we have selected the underlying index, our objective remains to minimize the discrepancy in performance between the replicating portfolio and the index. Some aspects concerned about the tracking performance will be discussed later.
Tracking Error

Even in the case of non-indexed equity funds, tracking error my be an important input to money manager and sponsor decisions (see Roll, 1992).

Definition 1. The *tracking error* $\varepsilon : R^n \to R$ is a function which measures the performance discrepancy between the portfolio $\boldsymbol{P}$ and the benchmark.

The performance of a portfolio is measured by its total return (dividends plus change in the market value of the portfolio). So, we let the return of the index and the replicating portfolio at t-th period be R_t^I and R_t^V respectively. The values of R_t^I and R_t^V can be evaluated as:

$$R_t^I = \frac{I_t - I_{t-1}}{I_{t-1}} \quad \text{and} \quad R_t^V = \frac{V_t - V_{t-1}}{V_{t-1}}$$

where I_t is the value of the index (benchmark) and V_t is the value of the portfolio at t-th period.

Traditional tracking error measure:

$$\varepsilon_1(\boldsymbol{P}) = \sqrt{\sum_{t=1}^{T} \frac{(R_t^V - R_t^I)^2}{T}} \qquad (3.1)$$

In indexing, the major goal is to make the tracking error as small as possible. The basic index arbitrage model indicates that the existence of arbitrage position is determined by the absolute difference between the current value and the fair value of the stock index futures. Hence, we propose a new tracking error measure based on the absolute difference between the value of the index and the value of the replicating portfolio.

Proposed tracking error measure:

$$\varepsilon_2(\boldsymbol{P}) = \sum_{t=1}^{T} |V_t - I_t| = \sum_{t=1}^{T} |\sum_{i=1}^{n} p_{it} x_i - I_t| \quad (3.2)$$

As this new measure is concerned, we would like to give the next theorem on which our model is founded.

Inevitable Errors

Completely matching the underlying index is not pragmatic in reality. Even though a replicating portfolio can be created by purchasing all the stock issues included in the benchmark, tracking error still persists.

First, on account of odd-lot purchases are cumbersome, replicating portfolios usually comprise round lots, and as such the number of shares of each stock in the portfolio is rounded to the nearest round-lot multiple from the exact number of shares indicated by the computer programs that have been developed to build the optimal replicating portfolio. This rounding may affect the ability of smaller replicating portfolios (less than NT$2.5 million) to accurately track the index. To handle this problem, we contrive the variable x_i which represents the multiple of the round-lot unit of stock issue i. Therefore, x_i is a positive integer, $\forall i = 1, 2, \ldots, n$.
This design can enhance the realism of our model.

Second, and more importantly, the maintenance of a replicating portfolio is a dynamic process. Since most indexes are capitalization-weighted, the relative weights of individual issues are constantly changing. In addition, the stocks that compose the index often change. Thus, the cost of continually adjusting the portfolio, as well as timing differences, hampers an indexer's ability to accurately track a benchmark. The former problem is eliminated by holding all stocks in the benchmark. The portfolio is then self-replicating, which simply means the weights are self-adjusting. If, however, the replicating portfolio contains fewer stocks than the benchmark, the weights are not self-adjusting and may require periodic rebalancing.

With a view to constructing the optimal replicating portfolio, you may select all the stock issues in the benchmark or a subset of those issues. Meanwhile, the number of stock issues in the replicating portfolio would generally affect administrative expenditure and transactions costs. Nevertheless, in Taiwan the transactions costs charged in one trade is gauged by fixed proportion in the total amount of money involved in this trade. This implies that the transactions costs are immaterial to the number of stock issues in the replicating portfolio.

On the other hand, retaining fewer stock issues than those in the benchmark may trigger a greater tracking error. The trade-off between tracking error and number of issues held must also be considered in terms of administrative expenditure, which will be augmented by the increase in the number of issues encompassed in the replicating portfolio.

To control the number of issues in the portfolio, we introduce the binary variables defined as:

$$y_i = \begin{cases} 1 & \text{if } x_i > 0 \\ 0 & \text{otherwise} \end{cases} \quad \forall i = 1, 2, \ldots, n.$$

This relation can be converted into another form in the formulations.

$$x_i \le My_i, \quad \forall i = 1, 2, \ldots, n$$

x_i is a positive integer, $y_i \in \{0, 1\}$, $\forall i = 1, 2, \ldots, n$.

By imposing the upper bound N_0 on the sum of y_i, the number of issues in the portfolio will be confined. The following constraint serves for this purpose.

$$\sum_{i=1}^{n} y_i \le N_0$$

4. The Model Formulations

The major target is to create an index fund minimizing the tracking error with the stocks of as few different companies as possible.

Absolute-Deviation Model:

Model ***A-D*:**

$$\min \sum_{t=1}^{T} \left| \sum_{i=1}^{n} p_{it} x_i - I_t \right| \qquad (4.1)$$

subject to:

$$\sum_{i=1}^{n} y_i \le N_0 \qquad (4.2)$$

$$x_i \le M y_i, \ \forall i = 1, 2, \ldots, n. \qquad (4.3)$$

x_i is a positive integer, $y_i \in \{0, 1\}$, $\forall i = 1, 2, \ldots, n.$ (4.4)

Objective function (4.1) is equivalent to our proposed tracking error $t\varepsilon_2(\boldsymbol{P})$. Constraints (4.2) ~ (4.4) have been explained in section 3.3.

Charnes, Cooper, and Ferguson (1955) have shown that the optimal values of the variables x_i ($i = 1, 2, \ldots, n$) in objective function (4.1) can be reached through a linear programming formulation. Taking advantage of their result, we can convert Model *A-D* into the consecutive goal programming model. For transforming the Model *A-D*, we introduce new variables d_t^+, d_t^- ($d_t^+, \ d_t^- \ge 0$).

Goal Programming Model:

Model ***GP*:**

$$\min \ z = \sum_{t=1}^{T} (d_t^+ + d_t^-) \qquad (4.5)$$

subject to:

$$\sum_{t=1}^{n} p_{it} x_i + d_t^+ - d_t^- = I_t,$$

$$\forall t = 1, 2, \ldots, T. \qquad (4.6)$$

$$d_t^+, \ d_t^- \ge 0, \ \forall t = 1, 2, \ldots, T. \qquad (4.7)$$

(4.2) ~ (4.4).

By (4.6) in Model *GP*, d_t^+ and d_t^- can be interpreted as the negative and positive amount by which the portfolio deviates from the current index at *t*-th period.

From constraint (4.4), we can readily perceive the huge complexity of model *GP*. Take the MSCI for instance: 77 issues are included in MSCI, which means there are at lease 2^{77} combinations of all variables y_i, let alone taking the combinations of all variables x_i into real computations. Due to this difficulty, our programs are not solvable within 2 full days even for the simplified case when $n = 40$ and $T = 30$. Such a situation coerced us to do some preprocessing and develop a heuristic to approximate the optimal solution. Thereby, we derive a lemma for generating some naive cuts.

Lemma 3. *Let* $U_i = \min_{1 \le t \le T} \{\frac{I_t}{p_{it}}\}$, $\forall i = 1, 2, \ldots, n$, *then we have naive cuts*

$x_i \le U_i$, $\forall i = 1, 2, \ldots, n$.

Concentrating at t-th period, suppose that we invest total value I_t solely on stock issue i, we can have the equation $p_{it}x_i = I_t$. This equation gave rise to the largest value $\frac{I_t}{p_{it}}$ which x_i can be at t-th period. Aggregating these constraints $x_i \le \frac{I_t}{p_{it}}$, $\forall t = 1, 2, \ldots, T$, we can generate the naive cut $x_i \le U_i$ for each stock issue i.

The constraint (4.8) is actually a valid equality for the feasible set of model *GP*. However, this constraint is occasionally not tight enough. Thus, we use the traits of the problem to bring forth another constraint.

Let c_{it} be the capital of stock issue i at t-th period, $\bar{c}_i = \frac{\sum_{t=1}^{T} c_{it}}{T}$, and $\bar{p}_i = \frac{\sum_{t=1}^{T} p_{it}}{T}$. Therefore the weight w_{it} of stock issue i in the index is equal to $\frac{c_{it}}{\sum_{i=1}^{n} c_{it}}$. By the formula of the index, the stock issue with the biggest $p_{it}c_{it}$ has the strongest connection to I_t at t-th period. In the overall horizon, the issue with largest $\bar{p}_i \cdot \bar{c}_i$ has the most influence on the tracking performance of the index fund. This trait motivates us to generate another cut. (4.8)

Suppose $\bar{p}_k \cdot \bar{c}_k = \max_{1 \le i \le n} \{\bar{p}_i \cdot \bar{c}_i\}$ and set $U = U_k$, we can derive heuristic cuts

$x_i \le U$, $\forall i = 1, 2, \ldots, n$.

Constraints (4.8) and (4.9) can be merged into one constraint as below:

$x_i \le \min\{U_i, U\}$, $\forall i = 1, 2, \ldots, S$.

Our computation results presented in section 5.2 showed that only half of all issues are needed to reduce the tracking error to a minimal value. This result entitle us to do some preprocessing before the model is actually solved. Since the weight of each issue in the index is proportional to its capital, we can set a selection criterion of stock issues by their capital. Thus, we select about half of all issues which have more capital than others. This criterion is carried out in step 1 of our algorithm.

Algorithm:

Step 1

Preprocessing: Set $S = \left\lfloor \frac{n}{2} \right\rfloor$, and select S issues which have more capital than other stocks. Reset $n = S$. ($\lfloor m \rfloor$ is the largest integer smaller than m)

Step 1.1

Generate cuts by (4.10): $x_i \le \min\{U_i, U\}$, $\forall i = 1, 2 \ldots, S$.

Step 2

Relaxation: Relax each x_i as a positive real number. Solve model *GP* under constraint (4.10) to get the relaxed optimal solution x_i^R, $\forall i = 1, 2, \ldots, S$.

Step 3

Heuristic Procedures:

Set $CE = 0$ and let $x_i^R = z_i + \Delta_i$, where $z_i \in Z^+$, $0 \le \Delta_i < 1$, $\forall i = 1$, $2, \ldots, S$. $SE = \sum_{i=1}^{S} \bar{p}_i \cdot \Delta_i$.

Step 3.1

Rank $\bar{p}_i$ in an increasing order. Without loss of generality, we assume that: $\bar{p}_i \le \bar{p}_{i+1}$, $\forall i = 1$, $2, \ldots, (S-1)$.

Step 3.2

For $i = 1, 2, \ldots, S$: If $(CE - \bar{p}_i) \le SE$ and $x_i^R > 0$, set $x_i^H = z_i + 1$, and $CE = CE + \bar{p}_i$. Else $x_i^H = z_i$.

Step 3.3

Stopping Criterion:

Set $RE = SE - CE$. If $RE \ge \bar{p}_1$, go to Step 3.2. Else go to Step 4.

Step 4

Reassign x_1^H an integer closest to $(x_1^H + \dfrac{RE}{\bar{p}_1})$ and we have the heuristic solution: x_i^H, $\forall i = 1$, $2, \ldots, S$.

The basic idea behind the heuristic is to get the relaxed optimal solution x_i^R (step 2 of our algorithm), set the initial x_i^H to be the value from rounding x_i^R down, and enlarge the values of some x_i^H to rectify the sum of average error per day (*SE*) incurred by rounding. For retaining the original number of issues in relaxed solution, only those x_i with $x_i^R > 0$ can be enlarged. For correcting *SE*, we first rank $\bar{p}_i$ in an increasing order (step 3.1), and then enlarge the value x_i (with $x_i^R > 0$) by 1 unit per time according to this order (step 3.2). Whenever x_i increase by one unit, *SE* is corrected by the amount $\bar{p}_i$. So we set a number *CE* to represent the cumulative corrected error and use residual error $RE = SE - CE$ to check if the heuristic procedures should stop.

5. Conclusion

This report proposed a method for constructing an index fund. We have built a goal programming model which is concise, flexible, and realistic. Unlike other optimization models in the literature, the linear objective function and linear constraints in model *GP* are substantially easier to solve. This merit greatly improves the efficiency of our optimization model. On the other hand, model *GP* provided us a ground for ensuring the solution is optimal in view of the objective function adopted here, i.e. tracking error function. This feature eliminates the drawbacks of other simplified methods.

In controlling the tracking ability of the index fund, we submitted a brand new measure ε_2 (***P***). This new measure can reflect the discrepancy between the index and the created portfolio more properly. We also proved ε_2 (***P***) is a convex function which promises an optimal solution for model *AD* and model *GP*. The other benefit of ε_2 (***P***) lies in the fact that when ε_2 (***P***) in model *AD* is transformed into model *GP*, the objective function becomes a linear function which is more readily to be handled and solved.

Model *GP* presented here is a mixed integer linear programming. Its enormous complexity pushes us to develop a solution algorithm for actually dealing the real cases of this problem. The solution algorithm consists primarily of three parts: Preprocessing, generating cuts, and the relaxation-based heuristic. Empirical results express that the algorithm can produce the solution efficiently while maintaining the good tracking ability of its solution. We use five different horizons of data to construct five portfolios by the proposed algorithm. The performance of each portfolio is satisfactory during the simulating periods. More amazingly, the portfolio created by the data of only 15 days has the best performance. More data are needed to see if this is still true for longer horizon of the simulating periods.

From our computational results, we can see that each portfolio tracks the index precisely during the first month after its construction. On account of this phenomenon, when rebalancing is allowed, we suggest that once the index fund is set up, exploit the newly arriving data together with model *GP* to engender a revised portfolio every month, and subsequently rebalance the