RATE OF CONVERGENCE TO A SINGULAR STEADY STATE FOR A HEAT EQUATION WITH STRONG ABSORPTION

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Abstract. In this paper, we study the initial boundary value problem for a heat equation with strong absorption. We first prove that the solution of this problem converges to the unique singular steady state for a class of initial data. This gives an example of dead-core which is developed in infinite time. Furthermore, we also derive the exact convergence rate (the dead-core rate) by a matching process.

1. Introduction

In this paper, we study the following initial boundary value problem (P) for the heat equation with a strong absorption:

$$u_t = u_{xx} - u^p,\ 0 < x < 1,\ t > 0, \tag{1.1}$$

$$u_x(0,t) = 0,\ u(1,t) = k_p,\ t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x),\ 0 \le x \le 1, \tag{1.3}$$

where $p \in (0,1)$, $k_p := [2\alpha(2\alpha-1)]^{-\alpha}$, $\alpha := 1/(1-p)$, and u_0 is a smooth function defined on $[0,1]$ such that

$$u_0'(0) = 0,\ u_0(1) = k_p,\ u_0'(x) \ge 0,\ U(x) < u_0(x) \le k_p \ \text{ for } x \in [0,1). \tag{1.4}$$

We note that $U(x) := k_p x^{2\alpha}$ is the unique steady state of (1.1)-(1.2). For convenience, by abusing the terminology, we shall call this steady state U as the singular steady state. Notice that $U(0) = 0$ which is "singular" in the sense that the reaction rate becomes infinity there.

Problem (P) arises in the modeling of an isothemal reaction-diffusion process [1, 7] and a description of thermal energy transport in plasma [5, 4]. In the first example, the solution u of (P) represents the concentration of the reactant which is injected with a fixed amount on the boundary $x = \pm 1$ (by a symmetric reflection), and p is the order of reaction.

It is trivial that, for any u_0 as above, problem (P) admits a unique global classical solution.Also, it follows from the strong maximum principle that $u > U$ and $u_x > 0$ in $(0,1) \times (0,\infty)$.

Date: March 5, 2006.

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Fu and Guo are partially supported by the National Science Council of the Republic of China under the grants NSC 93-2115-M-004-001 and NSC 94-2115-M-003-005.

The problem (P) with general boundary value (i.e., any $k > 0$) has been studied extensively. We refer the reader to a recent work of Guo-Souplet [3] and the references cited therein.

Recall that the region where $u = 0$ is called the *dead-core* and the first time when u reaches zero is called the *dead-core time*.

By taking the special constant k_p, we shall show that the solution of (P) is always positive for all $t > 0$ and tends to the singular steady state U uniformly as $t \to \infty$. In particular, we have $u(0,t) \to 0$ as $t \to \infty$. This means that the dead-core occurs at time infinity.

Motivated by the recent works of Dold-Galaktionov-Lacey-Vazquez [2] and Souplet-Vazquez [6], we shall investigate the exact convergence rate of $u(0,t)$ to zero as $t \to \infty$. We call this rate as the *dead-core rate*. We now state the main result of this paper as follows.

Theorem 1.1. *There is a positive constant μ such that*

$$\lim_{t\to\infty} \{\ln[u(0,t)]/t\} = -\mu. \tag{1.5}$$

The main idea to prove this main theorem is the so-called matching process (cf. [2, 6]). In this process, we need to study the inner and outer expansions.

The paper is organized as follows. We first give some preliminary results in §2. In particular, we prove that the dead-core is developed at time infinity. Some properties of the associated steady states to (1.1) are also given. Section 3 is devoted to the study of spectrum of linearized operator around the singular steady state and the regular approximated operators to this linearized operator. Then we derive the so-caller inner and outer expansions rigorously in §4. Finally, in §5, the idea of matching is applied to prove the main theorem on the exact convergence rate of $u(0,t)$ as $t \to \infty$.

2. Preliminaries

In this section, we shall give some preliminary results on the solution u of (P). First, we have the following result of positivity of u. This also implies that the dead-core can only be developed at time infinity.

Theorem 2.1. *We have $u > 0$ for all $0 \le x \le 1$ and $t > 0$.*

Proof. For contradiction, we may assume that

$$T := \sup\{\tau > 0 \mid u(x,t) > 0 \ \ \forall (x,t) \in [0,1] \times [0,\tau]\} < \infty.$$

By the maximum principle, we have $u > U$ in $(0,1) \times [0,T]$. In particular,

$$u(1/2,t) > U(1/2) \ \ \forall t \in [0,T]. \tag{2.1}$$

Let $\{u_n\}_{n\ge1}$ be a sequence of functions defined on $[0,1]$ such that

$$u_n'' = u_n^p \quad \text{on } [0,1]; \quad u_n(0) = 0, \ u_n'(0) = 1/n.$$

It is easy to see that $u_n \geq u_{n+1} \geq U$ on $[0,1]$ for all $n \geq 1$. Furthermore, $u_n \to U$ uniformly on $[0,1]$ as $n \to \infty$. It follows from (2.1) that $u(1/2,t) > U_N(1/2)$ for all $t \in [0,T]$ for some sufficiently large N. By choosing N larger (if necessary), we also have

$$u_0(x) > U_N(x) \ \ \forall x \in [0,1/2].$$

It follows from the maximum principle that $u \geq u_N$ on $[0,1/2]\times[0,T]$. Since $u(0,T) = 0$, we obtain that $u_x(0,T) \geq u_N'(0) > 0$, a contradiction. Hence the theorem is proved. □

The next theorem shows that u converges to the unique singular steady state U as $t \to \infty$. As a consequence, the dead-core does occur at time infinity.

Theorem 2.2. *There holds $u(x,t) \to U(x)$ uniformly for $x \in [0,1]$ as $t \to \infty$.*

Proof. First, we show that u, u_x, u_t are bounded on $[0,1] \times [0,\infty)$. Indeed, the boundedness of u follows from the maximum principle. Since the function $v := u_t$ satisfies

$$\begin{aligned} &v_t = v_{xx} - pu^{p-1}v, \ 0 < x < 1, \ t > 0, \\ &v_x(0,t) = 0, \ v(1,t) = 0, \ t > 0, \\ &v(x,0) = u_0''(x) - u_0^p(x), \ 0 \leq x \leq 1. \end{aligned}$$

It follows from the maximum principle that v (and so u_t) is bounded on $[0,1] \times [0,\infty)$. Now, from (1.1) we see that u_{xx} is bounded on $[0,1] \times [0,\infty)$. Consequently, u_x is also bounded, since $u_x(0,t) = 0$ for all $t > 0$.

Now, we take any sequence $\{t_j\}$ with $t_j \to \infty$ as $j \to \infty$. We define $u_j(x,t) := u(x,t+t_j)$ for any $j \in \mathbb{N}$. From the boundedness of u and u_x it follows that $\{u_j\}$ is uniformly bounded and equi-continuous on $[0,1] \times [0,\infty)$. It follows from the Arzela-Ascoli Theorem that there exists a subsequence, still denoted by u_j, such that $u_j \to w$ uniformly on $[0,1]$ as $j \to \infty$ for some function w satisfying

$$\begin{aligned} &w_t = w_{xx} - w^p, \ 0 < x < 1, \ t > 0, \\ &w_x(0,t) = 0, \ w(1,t) = k_p, \ t > 0. \end{aligned}$$

We claim that $w_t \equiv 0$. To do this, we introduce the energy functional

$$E(t) := \frac{1}{2}\int_0^1 u_x^2 dx + \frac{1}{p+1}\int_0^1 u^{p+1} dx.$$

By a simple computation, we have

$$E'(t) = -\int_0^1 u_t^2 dx.$$

For any fixed $T > 0$, an integration yields

$$\int_0^T \int_0^1 u_t^2 dx dt = E(0) - E(T) \leq E(0) < \infty.$$

It follows that

$$\int_0^\infty \int_0^1 u_t^2 dx dt < \infty.$$

This implies that

$$\int_0^\infty \int_0^1 u_{j,t}^2 dxdt = \int_{t_j}^\infty \int_0^1 u_t^2 dxdt \to 0 \text{ as } j \to \infty.$$

On the other hand, for any $T > 0$, since $\{u_{j,t}\}_{j\in\mathbb{N}}$ is uniformly bounded in $L^2([0,1] \times [0,T])$, it follows that $u_{j,t}$ converges weakly to w_t in $L^2([0,1] \times [0,T])$. This implies that

$$\int_0^T \int_0^1 w_t^2 dxdt \le \liminf_{j\to\infty} \int_0^T \int_0^1 u_{j,t}^2 dxdt = 0.$$

Hence $w_t \equiv 0$ and so $w = U$.

Since the sequence $\{t_j\}$ is arbitrary, the theorem follows. □

The following theorem implies that the convergence of $u(0,t)$ to zero is at least exponentially fast.

Theorem 2.3. *There exist positive constants C and β such that*

$$0 < u(0,t) \le Ce^{-\beta t} \tag{2.2}$$

for all $t > 0$.

Proof. First, we derive the following estimate

$$\int_0^1 [u(x,t) - U(x)]^2 dx \le Ce^{-\gamma t} \tag{2.3}$$

for all $t > 0$ for some positive constants C and γ. To this end, we set $w = u - U$. Then w satisfies

$$\begin{aligned} &w_t = w_{xx} + U^p - u^p \le w_{xx},\ 0 < x < 1,\ t > 0, \\ &w_x(0,t) = 0 = w(1,t),\ t > 0. \end{aligned}$$

It then follows that

$$\int_0^1 ww_t dx \le \int_0^1 ww_{xx} dx.$$

Using an integration by parts and applying the Poincaré Inequality, we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 w^2 dx \le -\int_0^1 w_x^2 dx \le -c\int_0^1 w^2 dx$$

for some positive constant c. Hence (2.3) follows.

By a comparison, it suffices to consider the case when $u_0(x) \equiv k_p$. Recall that $u_x > 0$ on $(0,1) \times (0,\infty)$. It implies that

$$u(x,t) \ge u(0,t) \ge U(x) = k_p x^{2\alpha} \ \ \forall x \in [0, h(t)], \tag{2.4}$$

where $h(t) := [u(0,t)/k_p]^{1/(2\alpha)} \le 1$ for $t > 0$. Then it follows from (2.3) and (2.4) that

$$\begin{aligned} Ce^{-\gamma t} &\ge \int_0^1 [u(x,t) - U(x)]^2 dx \\ &\ge \int_0^{h(t)} [u(0,t) - U(x)]^2 dx \\ &= \int_0^{h(t)} k_p^2 [h(t)^{2\alpha} - x^{2\alpha}]^2 dx \\ &= k_p^2 h(t)^{4\alpha+1} \int_0^1 (1 - s^{2\alpha})^2 ds, \end{aligned}$$

by a change of variable $s := x/h(t)$.

Hence the theorem follows by taking $\beta = 2\alpha\gamma/(4\alpha + 1)$. $\square$

Now, for any $\eta \ge 0$, let U_η be the solution of

$$u'' = u^p, \ u > 0 \ \ \forall y > 0; \quad u(0) = \eta, \ u'(0) = 0. \tag{2.5}$$

In particular, $U_0(y) = U(y) = k_p y^{2\alpha}$ for $y \ge 0$. Note that, by a re-scaling, we have

$$U_\eta(y) = \eta U_1(\eta^{(p-1)/2} y) \ \ \forall \eta > 0. \tag{2.6}$$

Also, by a simple comparison, we have $U_{\eta_1} > U_{\eta_2}$ if $\eta_1 > \eta_2 \ge 0$. Moreover, $U_\eta \to U_0$ as $\eta \to 0^+$.

Remark 2.1. *For $\eta = 0$, there are non-negative solutions in the form*

$$U_0^\varepsilon(y) := k_p (y - \varepsilon)_+^{2\alpha}$$

for any $\varepsilon > 0$. And these give all the possible non-negative non-trivial solutions of (2.5).

Concerning the asymptotic behavior of U_η as $\eta \to 0^+$, we have

Lemma 2.4. *As $\eta \to 0^+$,*

$$U_\eta(x) = U_0(x) + a\eta^{(1-p)/2} x^{2\alpha-1}(1 + o(1))$$

for any $x > 0$, where a is a positive constant.

Proof. First, we study the asymptotic behavior of $U_1(y)$ as $y \to \infty$. For this, we write $U_1 = U_0 + v$. Then v satisfies the equation

$$v'' = by^{-2}v + c_2 y^{-2-2\alpha} v^2 + c_3 y^{-2-4\alpha} v^3 + \cdots$$

for some constants c_i, $i \ge 2$, where $b := (2\alpha - 1)(2\alpha - 2)$. Assume that $v(y) \sim y^\gamma$ as $y \to \infty$ for some $\gamma > 0$. Then we must have

$$\gamma(\gamma - 1) = b.$$

By writing $\gamma = 2\alpha - \delta$, we obtain that either $\delta = 1$ or $\delta = 4\alpha - 2 > 2\alpha$ (which is impossible). Hence we obtain that

$$U_1(y) = U_0(y) + ay^{2\alpha-1}(1 + o(1)) \ \text{ as } \ y \to \infty \tag{2.7}$$

for some constant a. The constant a is positive, since $U_1 > U_0$.

Now, for any $x > 0$, from (2.6) and (2.7) it follows that

$$U_\eta(x) = \eta U_1(\eta^{(p-1)/2}x) = U_0(x) + a\eta^{(1-p)/2}x^{2\alpha-1}(1+o(1)) \ \text{ as } \ \eta \to 0^+.$$

The lemma is proved. □

In the sequel, for convenience we denote $\sigma(t) := u(0,t)$.

Lemma 2.5. *For all t sufficiently large, $\sigma(t)$ is strictly decreasing and*

$$u(x,t) < U_{\sigma(t)}(x) \ \text{ in } (0,1]. \tag{2.8}$$

Proof. Define

$$z_\eta(x,t) := u(x,t) - U_\eta(x).$$

Then z_η satisfies

$$(z_\eta)_t = (z_\eta)_{xx} + c_\eta(x,t)z_\eta,$$

where

$$c_\eta(x,t) := -\frac{u^p(x,t) - U_\eta^p(x)}{u(x,t) - U_\eta(x)}.$$

Since $z_\eta(1,t) < 0$ and $(z_\eta)_x(0,t) = 0$ for all $t > 0$, we see that the zero number $J_\eta(t)$ of z_η defined by

$$J_\eta(t) := \#\{x \in [0,1] \mid z_\eta(x,t) = 0\}$$

is non-increasing in t.

We first claim that there exists $\eta^* > 0$ such that $J_\eta(1) = 1$ for all $\eta \in (0, \eta^*]$. Indeed, since $z_{0,x}(1,1) < 0$, there exists $\delta > 0$ such that $z_{0,x}(x,1) < 0$ for all $x \in [1-\delta, 1]$. Since $z_{\eta,x}(x,1) \to z_{0,x}(x,1)$ uniformly on $[0,1]$ as $\eta \to 0^+$. There is $\eta_0 > 0$ such that

$$z_{\eta,x}(x,1) < 0 \ \ \forall x \in [1-\delta, 1] \ \ \forall \eta \in (0, \eta_0]. \tag{2.9}$$

On the other hand, since $u(x,1) > U(x)$ on $[0, 1-\delta]$ and $U_\eta \to U$ uniformly on $[0, 1-\delta]$ as $\eta \to 0^+$, there exists an $\eta^* \in (0, \eta_0)$ such that

$$z_\eta(x,1) > 0 \ \ \forall x \in [0, 1-\delta] \ \ \forall \eta \in (0, \eta^*]. \tag{2.10}$$

Recall that $z_\eta(1,1) < 0$ for all $\eta > 0$. We conclude from (2.9) and (2.10) that $J_\eta(1) = 1$ for all $\eta \in (0, \eta^*]$.

Next, we fix any $\eta \in (0, \eta^*]$. Note that $J_\eta(t) \leq 1$ for all $t > 1$. We claim that $\sigma(t_0) > \eta$, if $J_\eta(t_0) = 1$ for some $t_0 > 1$. For contradiction, we suppose that $\sigma(t_0) \leq \eta$, i.e., $u(0,t_0) \leq U_\eta(0)$. Note that $u(1,t) < U_\eta(1)$ for all $t > 0$. If $u(0,t_0) = U_\eta(0)$, then $u(x,t_0) < U_\eta(x)$ for all $x \in (0,1]$, since $J_\eta(t_0) = 1$. Since $J_\eta(t) = 1$ for all $t \in [1, t_0]$, there exists $x(t) \in [0,1)$ such that $u(x(t),t) = U_\eta(x(t)$ and $u(x,t) < U_\eta(x)$ for $x \in (x(t), 1]$ for each $t \in [1, t_0]$. By Hopf's Lemma, $u_x(0,t_0) < U_\eta'(0) = 0$, a contradiction. On the other hand, if $u(0,t_0) < U_\eta(0)$, then there exists $t^* \in (1, t_0)$ such that $u(0,s) < U_\eta(0)$ for all $s \in [t^*, t_0]$. Since $u(1,s) < U_\eta(1)$, we can find $x(s) \in (0,1)$ such that $u(x(s),s) = U_\eta(x(s))$ and $u(x,s) < U_\eta(x)$ for $x \neq x(s)$ for all $s \in [t^*, t_0]$. This is a contradiction to the maximum principle. This proves that $\sigma(t_0) > \eta$, if $J_\eta(t_0) = 1$ for some $t_0 > 1$.

Now, since $\sigma(t) \to 0$ as $t \to \infty$, there is t_1 sufficiently large such that $\sigma(t) \le \eta^*$ for all $t \ge t_1$. Hence $J_{\sigma(t)}(t) = 0$ for all $t \ge t_1$. This implies that

$$u(x,t) < U_{\sigma(t)}(x) \quad \text{on } [0,1]$$

for all $t \ge t_1$. Therefore, (2.8) follows. Moreover, $J_{\sigma(t)}(s) = 0$ for all $s > t \ge t_1$. Then $u(x,s) < U_{\sigma(t)}(x)$ for $x \in [0,1]$. In particular,

$$\sigma(s) = u(0,s) < U_{\sigma(t)}(0) = \sigma(t)$$

and the lemma is proved. □

3. Spectrum Analysis

In the matching process, we need to study the following linearized operator

$$\mathcal{L}v := -v'' + \frac{b}{x^2}v,\ b := (2\alpha - 1)(2\alpha - 2)$$

which is from the linearization of (1.1) around the singular steady state U.

Consider the eigenvalue problem

$$\mathcal{L}\phi = \lambda\phi,\ 0 < x < 1; \quad \phi'(0) = 0,\ \phi(1) = 0. \tag{3.1}$$

We introduce the following Hilbert space and quantities:

$$\mathbb{H} := \{\phi \in H^1([0,1]) \mid \int_0^1 \frac{\phi^2(x)}{x^2}dx < \infty,\ \phi(1) = 0\},$$

$$J(\phi) := \int_0^1 \phi_x^2(x)dx + b\int_0^1 \frac{\phi^2(x)}{x^2}dx,$$

$$I(\phi) := \int_0^1 \phi^2(x)dx.$$

Then the principal eigenvalue of (3.1) can be characterized by

$$\lambda := \inf\{J(\phi)/I(\phi) \mid \phi \in \mathbb{H},\ I(\phi) > 0\}. \tag{3.2}$$

It is easy to see that $\lambda > b > 0$. Also, by taking a minimization sequence, we can show that this λ can be attained by a function $\phi^* \in \mathbb{H}$ which is the eigen-function of (3.1) such that

$$\phi^* > 0 \ \text{ in } (0,1), \qquad \int_0^1 (\phi^*(x))^2 dx = 1.$$

Note that $\phi^*(0) = 0$. It is also easy to see that

$$\phi^*(x) = dx^{2\alpha-1}(1 + o(1)) \ \text{ as } x \to 0 \tag{3.3}$$

for some positive constant d.

On the other hand, it is easily seen that for any $\varepsilon \in (0,1)$ there exists the principal eigen-pair $(\lambda_\varepsilon, \phi_\varepsilon)$ of the following regular eigenvalue problem:

$$\mathcal{L}_\varepsilon\phi_\varepsilon = \lambda_\varepsilon\phi_\varepsilon,\ \varepsilon < x < 1; \quad \phi'_\varepsilon(\varepsilon) = \phi_\varepsilon(1) = 0 < \phi_\varepsilon(x)\ \forall x \in (\varepsilon, 1), \tag{3.4}$$

where

$$\mathcal{L}_\varepsilon v := -v'' + \frac{b(1-\varepsilon)}{x^2}v.$$

Without loss of generality, we may further assume that $\phi_\varepsilon(\varepsilon) = \varepsilon$.

Lemma 3.1. *There holds $\lambda_\varepsilon \to \lambda$ as $\varepsilon \to 0^+$.*

Proof. By the characterization of the principal eigenvalue λ_ε of (3.4), we have

$$\lambda_\varepsilon \le \frac{J_\varepsilon(\phi^*)}{I_\varepsilon(\phi^*)},$$

where

$$J_\varepsilon(\phi) := \int_\varepsilon^1 \phi_x^2(x)dx + b(1-\varepsilon)\int_\varepsilon^1 \frac{\phi^2(x)}{x^2}dx,$$
$$I_\varepsilon(\phi) := \int_\varepsilon^1 \phi^2(x)dx.$$

But, $J_\varepsilon(\phi^*) \to \lambda$ and $J_\varepsilon(\phi^*) \to 1$ as $\varepsilon \to 0^+$. We obtain that

$$\limsup_{\varepsilon\to 0^+} \lambda_\varepsilon \le \lambda. \tag{3.5}$$

On the other hand, we introduce a C^∞-function θ by $\theta(s) = 0$ for $s \le 1/2$, $\theta(s) = 1$ for $s \ge 1$, and $\theta' \ge 0$ in $[1/2, 1]$. Let $\theta_\varepsilon(x) := \theta(x/\varepsilon)$ for any $\varepsilon > 0$. Set $\tilde{\phi}_\varepsilon = \phi_\varepsilon$ in $[\varepsilon, 1]$ and $\tilde{\phi}_\varepsilon = \varepsilon$ in $[0, \varepsilon]$. Then for $\psi_\varepsilon := \theta_\varepsilon\tilde{\phi}_\varepsilon$ we have

$$J(\psi_\varepsilon) = J_\varepsilon(\phi_\varepsilon) + b\varepsilon\int_\varepsilon^1 \frac{\phi_\varepsilon^2(x)}{x^2}dx + \varepsilon\left(\int_{1/2}^1 (\theta')^2(s)ds + b\int_{1/2}^1 \frac{\theta^2(s)}{s^2}ds\right),$$
$$I(\psi_\varepsilon) = I_\varepsilon(\phi_\varepsilon) + \varepsilon^3\int_{1/2}^1 \theta^2(s)ds.$$

Since $\lambda \le J(\psi_\varepsilon)/I(\psi_\varepsilon)$ for all $\varepsilon \in (0, 1)$, we conclude that

$$\lambda \le \liminf_{\varepsilon\to 0^+} \lambda_\varepsilon. \tag{3.6}$$

Therefore, the lemma follows by combining (3.5) and (3.6). □

Remark 3.1. *The constant $b(1-\varepsilon)$ in $\mathcal{L}_\varepsilon$ can be replaced by any constant b_ε with $0 < b_\varepsilon < b$ and $b_\varepsilon \uparrow b$ as $\varepsilon \downarrow 0^+$. Lemma 3.1 remains true.*

4. Inner and Outer Expansions

In this section, we shall first derive the convergence of $u(x,t)$ to $U_{\sigma(t)}(x)$ near $x = 0$ as $t \to \infty$, where $\sigma(t) := u(0,t)$. To do this, we make the following transformations:

$$u(x,t) := \sigma(t)\theta(\xi,\tau), \quad \xi := \sigma(t)^{(p-1)/2}x, \quad \tau := \int_0^t \sigma(s)^{p-1}ds. \tag{4.1}$$

Then it is easy to check that θ satisfies the equation

$$\theta_\tau = \theta_{\xi\xi} - \theta^p - g(\tau)\left(\theta - \frac{1-p}{2}\xi\theta_\xi\right), \tag{4.2}$$

where $g(\tau) := \sigma'(t)\sigma(t)^{-p}$. Also, $\theta(0,\tau) = 1$ and $\theta_\xi(0,\tau) = 0$ for all $\tau > 0$. Moreover, it follows from Lemma 2.5 and (2.6) that $\theta(\xi,\tau) < U_1(\xi)$.

We shall study the stabilization of the solution θ of (4.2). First, by considering the function

$$J(x,t) := \frac{1}{2}u_x^2 - Cu^{p+1}$$

for some positive constant C and applying a maximum principle (cf. p. 660 of [3]), we can also derive the following estimate

$$0 \le u_x \le Cu^{(p+1)/2} \ \forall x \in [0,1],\ t > 0, \tag{4.3}$$

where C is a positive constant. Consequently, by an integration, we deduce from (4.3) that

$$u(x,t) \le [\sigma(t)^{(1-p)/2} + cx]^{2\alpha} \ \forall x \in [0,1],\ t > 0, \tag{4.4}$$

for some positive constant c.

Using (4.4), (4.1), and $u_x = \sigma^{(1+p)/2}\theta_\xi$, we obtain the following estimate for the solution θ of (4.2):

$$0 \le \xi\theta_\xi(\xi,\tau),\ \theta(\xi,\tau) \le C(1+\xi)^{2\alpha} \ \ \forall\, \xi \in [0, \sigma^{(p-1)/2}(t)],\ \tau > 0, \tag{4.5}$$

for some positive constant C.

Next, it follows from the Hopf Lemma that $u_{xx}(0,t) > 0$ and so $u_t(0,t) > -u^p(0,t)$ by (1.1). Hence $g(\tau) > -1$ for all $\tau > 0$. We conclude from Lemma 2.5 that $-1 < g(\tau) < 0$ for all $\tau \gg 1$. Note that

$$\int_0^\infty g(\tau)d\tau = -\infty.$$

Nevertheless, we have the following useful lemma.

Lemma 4.1. *There holds* $\lim_{\tau\to\infty} g(\tau) = 0$.

Proof. Otherwise, there is a sequence $\{\tau_n\} \to \infty$ such that $g(\tau_n) \to -\gamma$ as $n \to \infty$ for some constant $\gamma > 0$. By using (4.5) and the standard regularity theory, we can show that there is a subsequence, still denote it by $\{\tau_n\}$, such that

$$\theta(\xi,\tau+\tau_n) \to \tilde\theta(\xi,\tau) \ \text{ as } \ n \to \infty$$

uniformly on any compact subsets, where $\tilde\theta$ solves the equation

$$\tilde\theta_\tau = \tilde\theta_{\xi\xi} - \tilde\theta^p + \gamma(\tilde\theta - \frac{1-p}{2}\xi\tilde\theta_\xi),\ \xi > 0,\ \tau > 0, \tag{4.6}$$

with $\tilde\theta(0,\tau) = 1$ and $\tilde\theta_\xi(0,\tau) = 0$. Moreover, it is easily to check that $\tilde\theta \le U_1$ and $\tilde\theta_\xi \ge 0$.

Furthermore, it follows from the so-called energy argument (cf. the proof of Proposition 3.1 in [3]) that $\tilde\theta(\xi,\tau) \to V(\xi)$ as $\tau \to \infty$ for some V satisfying

$$V'' - V^p + \gamma(V - \frac{1-p}{2}\xi V') = 0,\ \xi > 0,$$
$$V'(0) = 0,\ V(0) = 1.$$

Note that $V \le U_1$ and $V' \ge 0$. Set

$$W(y) := \Big(\frac{\gamma}{\alpha}\Big)^{\alpha} V(\sqrt{\frac{\alpha}{\gamma}}y).$$

Then W satisfies

$$\begin{aligned}&W'' - W^p + \alpha(W - \frac{1-p}{2} yW') = 0,\ y > 0,\\ &W'(0) = 0,\ W(0) = (\gamma/\alpha)^{\alpha}.\end{aligned}$$

Since $W > 0$, $W' \ge 0$ for $y > 0$, and $V \le U_1$ gives the polynomial boundedness of W, it follows from Proposition 3.3 of [3] that either $W = U$ or $W \equiv \alpha^{-\alpha}$. The first case is impossible, since $U(0) = 0$. The second case is also impossible, since θ is unbounded by Theorem 2.2. Hence the lemma follows. □

Again, by the standard limiting process with the estimate (4.5) and Lemma 4.1, for any given sequence $\{\tau_n\} \to \infty$ we can show that there is a limit $\tilde{\theta}$ satisfying

$$\begin{aligned}&\tilde{\theta}_\tau = \tilde{\theta}_{\xi\xi} - \tilde{\theta}^p,\ \xi > 0,\ \tau > 0,\\ &\tilde{\theta}(0, \tau) = 1,\ \tilde{\theta}_\xi(0, \tau) = 0,\end{aligned}$$

such that $\theta(\xi, \tau + \tau_n) \to \tilde{\theta}(\xi, \tau)$ as $n \to \infty$ uniformly on compact subsets. Since we also have

$$\tilde{\theta}(\xi, \tau) \le U_1(\xi),\ \tilde{\theta}(0, \tau) = U_1(0),\ \tilde{\theta}_\xi(0, \tau) = (U_1)_\xi(0),$$

the Hopf Lemma implies that $\tilde{\theta} \equiv U_1$. Since this limit is independent of the given sequence $\{\tau_n\}$, we see that $\theta(\xi, \tau) \to U_1(\xi)$ as $\tau \to \infty$ uniformly on any compact subsets. Returning to the original variables and using the relation (2.6), we thus have proved the following so-called inner expansion.

Theorem 4.2. *As $t \to \infty$, we have*

$$u(x, t) = U_{\sigma(t)}(x)(1 + o(1))$$

uniformly in $\{0 \le \sigma^{(p-1)/2}(t)x \le C\}$ for any positive constant C.

For the outer expansion, we first derive the following lower bound estimate.

Lemma 4.3. *There exists a small positive constant δ such that*

$$u(x, t) - U(x) \ge \delta e^{-\lambda t}\phi^*(x),\ x \in [0, 1],\ t > 1. \tag{4.7}$$

Proof. Write $w = u - U$. Then $w(0, t) > 0$, $w(1, t) = 0$, and w satisfies the equation

$$w_t = w_{xx} - \frac{b}{x^2} w + F(x, w), \tag{4.8}$$

where

$$F(x, w) := U^p - (w + U)^p + \frac{b}{x^2} w = \frac{1}{2} p(1 - p)\tilde{U}^{p-2} w^2, \tag{4.9}$$

for some $\tilde{U} \in (U, U+w)$. Note that $F \geq 0$. Set $\hat{w}(x,t) := \delta e^{-\lambda t}\phi^*(x)$, where δ is a positive constant to be determined later. Then

$$\hat{w}_t = \hat{w}_{xx} - \frac{b}{x^2}\hat{w},\ x \in (0,1),\ t > 0,$$
$$\hat{w}(0,t) = 0,\ \hat{w}(1,t) = 0,\ t > 0.$$

Recall that $(\phi^*)'(1) < 0$. Also, note that $u_x(1,1) - U'(1) < 0$, by the Hopf Lemma. By the continuity, there exist positive constants δ and η such that

$$u_x(x,1) - U'(x) - \delta e^{-\lambda}(\phi^*)'(x) < 0 \tag{4.10}$$

for all $x \in [1-\eta, 1]$. It follows from (4.10) that $w(x,1) \geq \hat{w}(x,1)$ for all $x \in [1-\eta, 1]$. Using $u(\cdot,1) > U(\cdot)$ in $[0, 1-\eta]$ and by choosing smaller positive δ (if necessary), we obtain that $w(x,1) \geq \hat{w}(x,1)$ for all $x \in [0,1]$. Therefore, by the comparison principle, the estimate (4.7) follows. □

Now, following the proof of Lemma 3.2 in [3], we can also show that

$$u_x(x,t) \geq \hat{\varepsilon} x u^p(x,t)\ \forall x \in [0,1],\ t > 0, \tag{4.11}$$

for some small constant $\hat{\varepsilon} > 0$. It then follows from (4.11) that

$$u(x,t) \geq [\sigma(t)^{1-p} + cx^2]^\alpha\ \forall x \in [0,1],\ t > 0, \tag{4.12}$$

for some positive constant c.

Recall the principal eigen-pair $(\lambda_\varepsilon, \phi_\varepsilon)$ of (3.4) for any $\varepsilon \in (0,1)$. Then we have the following lemma which is another part of the outer expansion.

Lemma 4.4. *For each $\varepsilon > 0$, there exists a positive constant c_ε such that*

$$u(x,t) - U(x) \leq c_\varepsilon e^{-\lambda_\varepsilon t}\phi_\varepsilon(x),\ x \in [\varepsilon, 1], \tag{4.13}$$

for all t sufficiently large.

Proof. Again, we set $w = u - U$. We first estimate F as follows. Since $\tilde{U} \in (U, U+w)$, we compute from (4.9) that

$$F(x,w) \leq \frac{1-p}{2}[U^{-1}w][pU^{p-1}w] = \frac{1-p}{2}[U^{-1}w]\left(\frac{b}{x^2}w\right).$$

By Theorem 2.2, there is t_0 sufficiently large such that

$$\frac{1-p}{2}[U^{-1}(x)w(x,t)] \leq \varepsilon\ \ \forall x \in [\varepsilon, 1],\ t \geq t_0.$$

Consequently, we obtain from (4.8) that w satisfies the following inequality

$$w_t \leq w_{xx} - \frac{b(1-\varepsilon)}{x^2}w\ \ \forall x \in (\varepsilon, 1),\ t \geq t_0 \tag{4.14}$$

and $w(1,t) = 0$ for all $t > 0$.

Now, set $\hat{w}(x,t) := c_\varepsilon e^{-\lambda_\varepsilon t}\phi_\varepsilon(x)$, where c_ε is a positive constant to be determined. Then

$$\hat{w}_t = \hat{w}_{xx} - \frac{b(1-\varepsilon)}{x^2}\hat{w},\ x \in (\varepsilon, 1),\ t > 0,$$
$$\hat{w}_x(\varepsilon, t) = 0,\ \hat{w}(1,t) = 0,\ t > 0.$$

Recall that $(\phi_\varepsilon)'(1) < 0$. Then by the continuity there exist a small positive constant η and a large positive constant c_ε such that

$$u_x(x,t_0) - U'(x) - c_\varepsilon e^{-\lambda_\varepsilon t_0}(\phi_\varepsilon)'(x) > 0 \ \ \forall x \in [1-\eta, 1]. \tag{4.15}$$

It follows from (4.15) that $w(x,t_0) \le \hat{w}(x,t_0)$ for $x \in [1-\eta, 1]$. Then, by choosing c_ε larger (if necessary), we obtain that $w(x,t_0) \le \hat{w}(x,t_0)$ for $x \in [\varepsilon, 1]$.

It remains to show that $w_x(\varepsilon, t) \ge 0$ for all $t \ge t_0$.

Q: can this be deduced from (4.12)?? or, need other ideas?? Check!!

Therefore, the lemma follows by applying the comparison principle. □

5. Rate of Convergence: Matching

In this section, we shall using the idea of matching to derive the exact convergence rate of $\sigma(t) := u(0,t)$ to zero.

For the lower bound, we recall from Lemmas 2.4 and 2.5 that for any $x > 0$:

$$u(x,t) \le U_{\sigma(t)}(x) = U(x) + a\sigma^{(1-p)/2}(t)x^{2\alpha-1}(1+o(1)) \ \text{ as } \ t \to \infty. \tag{5.1}$$

On the other hand, by (4.7) and (3.3), we have

$$u(x(t),t) \ge U(x(t)) + d\delta e^{-2\alpha\lambda t}(1+o(1)) \ \text{ as } \ t \to \infty, \tag{5.2}$$

where $x(t) := e^{-\lambda t}$. Consequently, there exists a positive constants d_1 such that

$$e^{-\lambda t} \le d_1 \sigma^{(1-p)/2}(t)(1+o(1)) \ \text{ as } \ t \to \infty$$

i.e.,

$$\sigma(t) \ge d_2 e^{-2\alpha\lambda t}(1+o(1)) \ \text{ as } \ t \to \infty \tag{5.3}$$

for some positive constant d_2. This gives the lower bound estimate of σ.

For the upper bound, we apply Theorem 4.2 and (4.13) to deduce that for $x \ge \varepsilon$

$$U(x) + c_\varepsilon e^{-\lambda_\varepsilon t}\phi_\varepsilon(x) \ge u(x,t) = U(x) + a\sigma^{(1-p)/2}(t)x^{2\alpha-1}(1+o(1)) \ \text{ as } \ t \to \infty.$$

It follows that

$$\sigma(t) \le d_3 e^{-2\alpha\lambda_\varepsilon t}(1+o(1)) \ \text{ as } \ t \to \infty \tag{5.4}$$

for some positive constant d_3 for any $\varepsilon \in (0,1)$. Also, from Lemma 3.1, we have $\lambda_\varepsilon \to \lambda$ as $\varepsilon \to 0^+$. From this, we conclude the proof of Theorem 1.1 with $\mu = 2\alpha\lambda$.

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