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UNIQUENESS AND ASYMPTOTICS OF TRAVELING WAVES OF MONOSTABLE DYNAMICS ON LATTICES

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Abstract. Established here is the uniquenes of solutions for the traveling wave problem $cU'(x) = U(x+1) + U(x-1) - 2U(x) + f(U(x))$, $x \in \mathbb{R}$, under the monostable non-linearity: $f \in C^1([0,1]), f(0) = f(1) = 0 < f(s)\ \forall\, s \in (0,1)$. Asymptotic expansions for $U(x)$ as $x \to \pm\infty$, accurate enough to capture the translation differences, are also derived and rigorously verified. These results complement earlier existence and partial uniqueness/stability results in the literature (e.g. [3, 4, 18]). New tools are also developed to deal with the degenerate case $f'(0)f'(1) = 0$, about which is the main concern of this article.

Key words. traveling wave, monostable, degenerate, lattice dynamics

AMS subject classification. Primary, 34K05; Secondary, 34E05, 34K25, 34K60

1. Introduction

Consider a system of countably many ordinary differential equations, for $\{u_n(\cdot)\}_{n\in\mathbb{Z}}$,

$$\dot{u}_n(t) = u_{n+1}(t) - 2u_n(t) + u_{n-1}(t) + f(u_n(t)),\ n \in \mathbb{Z}, t > 0, \tag{1.1}$$

where f is a nonlinear forcing term satisfying $f(0) = f(1) = 0$. This system can be embedded into a larger one, for an unknown $\{u(x,\cdot)\}_{x\in\mathbb{R}}$,

$$u_t(x,t) = u(x+1,t) - 2u(x,t) + u(x-1,t) + f(u(x,t)),\ x \in \mathbb{R},\ t > 0. \tag{1.2}$$

A solution of (1.2) or (1.1) is called a **traveling wave with speed** c if there exists a function U defined on $\mathbb{R}$ such that $u(x,t) = U(x+ct)$ or $u_n(t) = U(n+ct)$. Here U is referred to as the **wave profile**. Of interest are solutions taking values in $[0,1]$, specifically, traveling waves connecting the steady states **0** and **1**, i.e, traveling wave solutions $(c,U) \in \mathbb{R}\times C^1(\mathbb{R})$ of the traveling wave problem

$$\begin{cases} c\,U'(\cdot) = U(\cdot+1) + U(\cdot-1) - 2U(\cdot) + f(U(\cdot)) \quad \text{on } \mathbb{R}, \\ U(-\infty) = 0, \qquad U(\infty) = 1, \qquad 0 \leqslant U \leqslant 1 \text{ on } \mathbb{R}. \end{cases} \tag{1.3}$$

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Equation (1.1) can be found in many biological models (e.g. [6, 12, 14]). Also, it can be regarded as a spatial-discrete version of the parabolic partial differential equation

$$u_t = u_{xx} + f(u). \tag{1.4}$$

The existence, uniqueness, and stability of traveling waves of (1.1) have been extensively studied recently under various assumptions on f; see, for example, [1, 3, 4, 5, 7, 8, 15, 16, 17, 18]. The commonly used assumption includes the condition of non-degeneracy $f'(0)f'(1) \neq 0$. For bistable dynamics, i.e., $f'(0) < 0$ and $f'(1) < 0$, the results on traveling waves are quite complete; see, for example, [1, 5, 16, 17] and the references therein. This paper concerns only the monostable dynamics, i.e., f satisfies

(A) $f \in C^1([0,1]), \quad f(0) = f(1) = 0 < f(s)\ \forall\, s \in (0,1).$

Under the non-degeneracy and the condition that $f(s) \leqslant f'(0)s$ for all $s \in [0,1]$, Zinner, Harris, and Hudson established the existence of traveling waves [18]; see also the later developments of Fu, Guo, and Shieh [7] and Chen and Guo [3]. The uniqueness issue was not satisfactorily resolved until a very recent paper of Chen and Guo [4]. For an easy reference, we quote here the following existence and uniqueness result from [4].

Proposition 1. *Assume* **(A)**.

(i) *There exists* $c_{\min} > 0$ *such that* (1.3) *admits a solution if and only if* $c \geqslant c_{\min}$.

(ii) *Given* $c \geqslant c_{\min}$, *there is a speed* c *wave profile satisfying* $U' > 0$ *on* $\mathbb{R}$.

(iii) *Given* $c > 0$, (1.3) *admits a solution if there is a super-solution of speed* c.

(iv) *When* $f'(0)f'(1) \neq 0$, *wave profiles are unique up to a translation. In addition,*

$$\lim_{x\to-\infty} \frac{U'(x)}{U(x)} = \lambda, \qquad \lim_{x\to\infty} \frac{U'(x)}{U(x)-1} = \mu \tag{1.5}$$

where $\mu < 0 < \lambda$ *are roots of the* **characteristic equations**

$$c\,\lambda = e^{\lambda} + e^{-\lambda} - 2 + f'(0), \qquad c\,\mu = e^{\mu} + e^{-\mu} - 2 + f'(1). \tag{1.6}$$

In addition, when $c > c_{\min}$, λ *is the smaller root of the characteristic equation.*

Here by a **super-solution of wave speed** c it means a non-constant Lipschitz continuous function Φ from $\mathbb{R}$ to $[0,1]$ satisfying

$$c\,\Phi'(x) \geqslant \Phi(x+1) + \Phi(x-1) - 2\Phi(x) + f(\Phi(x)) \quad \text{a.e.} \quad x \in \mathbb{R}.$$

For example, when $f(s) \leqslant f'(0)s$ for all $s \in [0,1]$, $\Phi(x) := \min\{e^{\lambda x}, 1\}$ is a super-solution of speed c if $c\lambda = e^{\lambda} + e^{-\lambda} - 2 + f'(0)$. This implies that

$$c_{\min} = \min_{\lambda>0} \frac{e^{\lambda} + e^{-\lambda} - 2 + f'(0)}{\lambda}.$$

It is important to observe that a (monotonic) wave profile $U^{\min}$ of the minimum speed is a supersolution of any wave speed $c > c_{\min}$. Since among all wave profiles of all admissible speeds, $U^{\min}$ decays with the largest exponential rate as $x \to -\infty$, it is not always true that near $-\infty$ a supersolution is bigger than a true solution under a certain translation.

Thus, Proposition 1 (iii) is highly non-trivial; its proof in [4] was based on an original idea of the authors of [18], with a simplification that avoids the use of a degree theory.

The purpose of this paper is to remove the non-degeneracy condition $f'(0)f'(1) \neq 0$ made in Proposition 1 (iv); that is, we are mainly concerned with the degenerate case $f'(0)f'(1) = 0$. In terms of the differential equation (1.4), existence, uniqueness, and asymptotic stability of traveling waves have been recently established (cf. [9, 10, 11, 13]). Here we would like to extend the analogous result for (1.4) to (1.1). We summarize our results, for the traveling wave problem (1.3), as follows.

Theorem 1. *Assume* **(A)**. *Wave profiles of a given speed are unique up to a translation.*

Theorem 2. *Assume* **(A)**. *Any wave profile is monotonic; i.e.* $U' > 0$ *on* $\mathbb{R}$.

Theorem 3. *Assume* **(A)**. *Any solution* (c, U) *of* (1.3) *satisfies* (1.5) *and*

$$\lim_{x\to-\infty}\frac{U''(x)}{U'(x)} = \lambda, \qquad \lim_{x\to-\infty}\frac{f(U(x))}{U'(x)} = \begin{cases} c & \text{if } \lambda = 0,\\ f'(0)/\lambda & \text{otherwise,}\end{cases}$$

$$\lim_{x\to\infty}\frac{U''(x)}{U'(x)} = \mu, \qquad \lim_{x\to\infty}\frac{f(U(x))}{U'(x)} = \begin{cases} c & \text{if } \mu = 0,\\ f'(1)/\mu & \text{otherwise,}\end{cases}$$

where $\mu \leqslant 0 \leqslant \lambda$ *are roots of the characteristic equations in (1.6).*

In addition, λ *is the smaller root when* $c > c_{\min}$ *and the larger root when* $c = c_{\min}$.

Note that the root $\mu \leqslant 0$ to (1.6) is unique. In particular, $\mu = 0$ when $f'(1) = 0$. Also, $\lambda = 0$ when $f'(0) = 0$ and $c > c_{\min}$; otherwise, $\lambda > 0$. To our knowledge, it is new in the literature that, as a principle, λ is the larger root (if there are two) of the characteristic equation when $c = c_{\min}$.

The proof of uniqueness (Theorem 1) relies on the monotonicity (Theorem 2) and the detailed asymptotic behavior (Theorem 3) of wave profiles. Two new techniques are specifically developed here to study the uniqueness of traveling waves of monostable dynamics. One of them, which we call **magnification** and is originated from [4], is to magnify appropriately the difference between two wave profiles U and V by (for the purpose of demonstration only, considering the case $c > c_{\min}$)

$$W(\xi, x) = \int_{V(x)}^{U(x+\xi)} \frac{ds}{f(s)}.$$

Such a magnification has a special property $\lim_{x\to-\infty} W_x(\xi, x) = 0$ for any $\xi \in \mathbb{R}$ and a general property $\inf_{\mathbb{R}^2} W_\xi > 0$. From a basic comparison (for monotonic profiles) which says that if $U > V$ on $[a-1, a)\cup(b, b+1]$ then $U > V$ on $[a, b]$, these two properties prohibit W from any oscillations with non-vanishing magnitudes as $x \to -\infty$; namely, there exists $\lim_{x\to-\infty} W(\xi, x)$ (which maybe infinite). Consequently, any two wave profiles are ordered near $-\infty$; see §4 for more details. An additional advantage of this magnification is that $\lim_{x\to-\infty} W(\xi, x)$ exists even if V is merely a sub or a super solution. This fact will be used in §5 to find asymptotic expansions of wave profiles.

The other technique, which we call **compression**, is developed to include the treatment of the degenerate case $f'(1) = 0$. Traditionally near ∞ one uses $\min\{U + \varepsilon, 1\}$ as a supersolution which works for both monostable and bistable dynamics but needs the assumption that $f' \leqslant 0$ on $[1 - \delta, 1]$ for some $\delta > 0$. To deal with the general case, we use the following compression to obtain (local) supersolutions:

$$Z(\ell, x) = U([1 + \ell]x), \quad x \gg 1, \ell \in (0, 1].$$

The asymptotic behavior of wave profiles implies that Z approaches 1 as $x \to \infty$ at a rate faster than any wave profile. With a limiting $\ell \searrow 0$ process, we can show that near ∞, one wave profile is always bigger than a certain translation of any other wave profile.

The asymptotic behavior (1.5) follows from an analysis similar to that in [4]. After a thorough re-investigation of the method used in [4], we found that the method in [4] can be rephrased into the following quite fundamental theory.

Theorem 4. *Let $c > 0$ be a constant and $B(\cdot)$ be a continuous function having finite $B(\pm\infty) := \lim_{x\to\pm\infty} B(x)$. Let $z(\cdot)$ be a measurable function satisfying*

$$c\, z(x) = e^{\int_x^{x+1} z(s)ds} + e^{-\int_{x-1}^{x} z(s)ds} + B(x) \quad \forall\, x \in \mathbb{R}. \tag{1.7}$$

Then z is uniformly continuous and bounded by quantities in (2.1). *In addition, $\omega^\pm = \lim_{x\to\pm\infty} z(x)$ exist and are roots of the characteristic equation $c\,\omega = e^\omega + e^{-\omega} + B(\pm\infty)$.*

Note that each of $z = U'/U, U'/(U - 1)$ and U''/U' satisfies an equation of the form (1.7). This theory provides a powerful tool to study the asymptotic behavior, as $x \to \pm\infty$, of positive solutions of a variety of semi-linear finite difference-differential equations. In particular, once the monotonicty $U' > 0$ is shown, $z = U''/U'$ is then well-defined and all the limits stated in Theorem 3 follow immediately from the theory.

Now the focus is shifted to show the monotonicity of U. In the non-degenerate case, $\mu < 0 < \lambda$, so that (1.5) and a comparison between $U(x + h)$ and $U(x)$ on a compact interval imply that $U' > 0$ on $\mathbb{R}$. In the degenerate case, $\lambda\mu = 0$, so (1.5) is not sufficient for such an argument. We shall develop a **blow-up** technique, showing that $U' > 0$ on a sequence of intervals $\{[\xi_i - 1, \xi_i + 1]\}$ of two unit length, where $\lim_{i\to\pm\infty} \xi_i = \pm\infty$. Then we develop a **modified sliding method** which enables us to compare $U(x+h)$ and $U(x)$ on any finite interval $[\xi_i - 1, \xi_j + 1]$ $(i < j)$ to prove the monotonicity result.

For a solution of (1.2) or (1.4) with initial value $u(x, 0)$, its long time behavior (e.g. approaching a traveling wave) depends on the asymptotic behavior of $u(x, 0)$ as $x \to -\infty$, i.e. tails of which wave profile $U(x)$ that $u(\cdot, 0)$ resembles; see, for example, [2, 3] and the references therein. For this purpose, we shall also provide asymptotic expansions, accurate enough to capture the translation difference of wave profiles near $\pm\infty$. In particular, under the condition that $f(u) = f'(0)u + O(u^{1+\alpha})$ for some $\alpha > 0$ and all small u, we show the following:

(i) *If* $c = c_{\min}$ *and the larger root* λ *of (1.6) is not a double root, then for some* $x_0 \in \mathbb{R}$

$$\lim_{x\to-\infty} e^{-\lambda x} U(x+x_0) = 1. \tag{1.8}$$

(ii) *If* $c = c_{\min}$ *and* λ *is a double root, then for some* $x_0 \in \mathbb{R}$

$$\text{either} \quad \lim_{x\to-\infty} \frac{U(x+x_0)}{|x|e^{\lambda x}} = 1 \qquad \text{or} \quad \lim_{x\to-\infty} \frac{U(x+x_0)}{e^{\lambda x}} = 1. \tag{1.9}$$

(iii) *If* $c > c_{\min}$ *and* $f'(0) > 0$*, then (1.8) holds for some* $x_0 \in \mathbb{R}$ *with* λ *the smaller root of (1.6).*

Note that $\lambda > 0$ in all these cases, so, as we expected from (1.5), that $U(x)$ decays to zero exponentially fast as $x \to -\infty$. Earlier results (e.g. [3, 7, 8, 18]) on this matter depend on the construction of global sub and supersolution pairs that sandwich a wave profile. Such a construction is possible for all large wave speeds for general f and for all non-minimum wave speeds when $f(s) \leqslant f'(0)s$ for all $s \in [0,1]$. We remark that the stability (which implies uniqueness) result in [3] was established under the assumption (1.8). By proving (1.8), the result in [3] then implies that any solution of (1.2) approaches, as $t \to \infty$, a traveling wave of speed c ($> c_{\min}$) if $u(\cdot, 0)$ takes values on $[0,1]$ and

$$\lim_{x\to-\infty} e^{-\lambda x} u(x,0) = 1, \qquad \liminf_{x\to\infty} u(x,0) > 0.$$

On the other hand, $\lambda = 0$ when $f'(0) = 0$ and $c > c_{\min}$, so from (1.5), an exponential decay is impossible and an algebraic decay is to be expected (cf. [9, 10, 11, 13] for (1.4)). Indeed, under certain additional assumptions (cf. **(B1)** in §5) we show the following:

(iv) *If* $c > c_{\min}$ *and* $f'(0) = 0$*, then for some* $x_0 \in \mathbb{R}$

$$\lim_{x\to-\infty} \left\{ \int_{1/2}^{U(x)} \frac{ds}{f(s)[1+f'(s)/c^2]} - \frac{x+x_0}{c} \right\} = 0. \tag{1.10}$$

For example, when $f(u) = \kappa u^2(1-u)^p$ ($\kappa > 0$, $p \geq 1$), the above limit yields

$$U(x) = \frac{c}{\kappa[x_0 - x + o(1)] + (pc - 2\kappa/c)\ln|x|}, \qquad \lim_{x\to-\infty} o(1) = 0.$$

The asymptotic expansion of $U(x)$ as $x \to \infty$ can be treated similarly. Indeed,

$$\lim_{x\to\infty} \left\{ \int_{1/2}^{U(x)} \frac{ds}{f(s)[1+f'(s)/c^2]} - \frac{x+x_0}{\nu} \right\} = 0,$$

for some $x_0 \in \mathbb{R}$, where $\nu = c$ if $f'(1) = 0$ and $\nu = f'(1)/\mu$ if $f'(1) < 0$. Since this limiting behavior has nothing to do with the condition needed on the initial data for the long time behavior of solutions of (1.2), we choose to omit the details here.

In [4], the following general system is considered

$$u_t(x,t) = g(u(x+1,t)) - 2g(u(x,t)) + g(u(x-1,t)) + f(u(x,t))$$

where $g(\cdot)$ is increasing. Under a variable change $v = [g(u) - g(0)]/[g(1) - g(0)]$, the system can be re-written as

$$h(v)v_t = v(x+1,t) + v(x-1,t) - 2v + \tilde{f}(v).$$

Under assumptions that $h \in C^1$ and $h > 0$ on $[0,1]$, all the analysis and results presented in this paper apply to such an extended version.

This paper is organized as follows. In §2, we derive the asymptotic behavior of wave profiles near $\pm\infty$ and prove Theorem 3. We prove the monotonicity of wave profiles (Theorem 2) in §3, by using the method of sliding and a new blow-up technique. In §4, the uniqueness of traveling waves is established. Finally in §5, we construct suitable local super/sub solutions to verify our asymptotic expansions of wave profiles near $x = \pm\infty$.

2. Asymptotic Behavior of Wave Profiles Near $x = \pm\infty$

In the sequel, the assumption **(A)** is always assumed.

2.1. **The idea in** [4]**.** The most important technique developed in [4] can be presented as follows. Suppose that the following quantities

$$\rho(x) := \frac{U'(x)}{U(x)}, \quad \sigma(x) := \frac{U'(x)}{U(x)-1}, \quad \chi(x) := \frac{U''(x)}{U'(x)}$$

are well-defined. This is the case, if $U > 0$, $U < 1$, and $U' > 0$ for ρ, σ, and χ, respectively. Then each of them satisfies an equation of the form (1.7), where $B(\cdot)$ is a continuous function having $\lim_{x\to\pm\infty} B(x) =: B(\pm\infty)$. Set

$$v(x) = e^{mx+\int_0^x z(s)ds} \quad \Rightarrow \quad c\,v'(x) = [cm + B(x)]v(x) + e^{-m}v(x+1) + e^m v(x-1).$$

Taking $m = \|B(x)\|_{L^\infty(\mathbb{R})}/c$, we see that $v'(x) \geq 0$. Consequently,

$$c\,v(x) - c\,v(x-1/2) > \int_{x-1/2}^x e^{-m}v(s+1)ds > \tfrac{1}{2}v(x+1/2)e^{-m}.$$

This implies that $v(x) > v(x+1/2)/(2ce^m) > v(x+1)/(2ce^m)^2$. Therefore,

$$e^{\int_x^{x+1} z(s)ds} = \frac{v(x+1)e^{-m}}{v(x)} \leqslant 4c^2e^m, \quad e^{-\int_{x-1}^x z(s)ds} = \frac{e^m v(x-1)}{v(x)} \leqslant e^m,$$

and so

$$-m < z(x) < m + 4ce^m + e^m/c \quad \forall\, x \in \mathbb{R}, \quad m := \|B\|_{L^\infty(\mathbb{R})}/c. \tag{2.1}$$

The uniform boundedness of z implies that z is uniformly continuous. Hence, for any unbounded sequence $\{x_i\}$, $\{z(x_i + \cdot)\}$ is a bounded and equi-continuous family. Along a subsequence, it converges to a limit r, uniformly in any compact subset of $\mathbb{R}$. In addition, r satisfies the **fundamental equation**

$$c\,r(x) = e^{\int_x^{x+1} r(s)\,ds} + e^{\int_x^{x-1} r(s)\,ds} + b \qquad \forall\, x \in \mathbb{R} \tag{2.2}$$

where $b = B(\infty)$ if $\lim_{i\to\infty} x_i = \infty$ and $b = B(-\infty)$ if $\lim_{i\to\infty} x_i = -\infty$. For the fundamental equation, Chen and Guo established in [4] the following key result:

Proposition 2. *Let $c > 0$, $b \in \mathbb{R}$ and $P(\omega) = c\omega - e^{\omega} - e^{-\omega} - b$. Consider (2.2).*

(1) *When $P(\omega) = 0$ has no real root, there is no solution.*

(2) *When $P(\omega) = 0$ has only one real root λ, $r \equiv \lambda$ is the only solution.*

(3) *When $P(\omega) = 0$ has real roots $\{\lambda, \Lambda\}$ $(\lambda < \Lambda)$, every solution can be written as*

$$r(x) = \frac{u'(x)}{u(x)}, \quad u(x) = \theta e^{\lambda x} + (1-\theta)e^{\Lambda x}, \quad \theta \in [0,1].$$

In particular, any nonconstant solution satisfies $r' > 0$, $r(-\infty) = \lambda$, and $r(\infty) = \Lambda$.

Proof of Theorem 4. We need consider only the case when the characteristic equation has two roots. For this, let λ and Λ be the roots where $\lambda < \Lambda$. Suppose $\lim_{x\to-\infty} z(x)$ does not exist. Then there exist $\omega \notin \{\lambda, \Lambda\}$ and a sequence $\{x_i\}$ satisfying $\lim_{i\to-\infty} x_i = -\infty$, $z(x_i) = \omega$ and $z'(x_i) \leqslant 0$ for all i. Since $\{z(x_i + \cdot)\}$ is uniformly bounded and equi-continuous, a subsequence converges to a limit r which solves (2.2) with $b = B(-\infty)$. In addition, by the definition of r, we have $r(0) = \omega$ and $r'(0) \leqslant 0$. But from Proposition 2, there are no such kind of solutions. Hence, $\lim_{x\to-\infty} z(x)$ exists and is one of the two roots to the characteristic equation. Similarly, one can show that $\lim_{x\to\infty} z(x)$ exists. □

Remark 2.1. *(i) By working on the function $\hat{z}(x) := -z(-x)$ the assertion of the Theorem remains unchanged when $c < 0$.*

(ii) *Theorem 4 extends to a more general equation*

$$z(x) = a_1(x)e^{\int_x^{x+1} z(s)ds} + a_2(x)e^{-\int_{x-1}^{x} z(s)ds} + B(x)$$

where a_1 and a_2 are continuous positive functions having limits

$$a^{\pm} := \lim_{x\to\pm\infty} a_1(x) = \lim_{x\to\pm\infty} a_2(x) > 0.$$

(iii) *Theorem 4 also extends to the case when z is a continuous function defined on* $[-1,\infty)$ (or $(-\infty, 1]$) *and satisfies* (1.7) *on* $[0,\infty)$ (or $(-\infty, 0]$). *The conclusion is that* $\lim_{x\to\infty} z(x)$ (or $\lim_{x\to-\infty} z(x)$) *exists and is the root of the characteristic equation.*

2.2. **The asymptotic behavior.** Now we establish the limits stated in Theorem 3.

1. We begin with the limits in (1.5). First we show that $U > 0$. Suppose on the contrary there exists $y \in \mathbb{R}$ such that $U(y) = 0$. Then it is a global minimum so that $U'(y) = 0$ and from the equation in (1.3), $U(y+1) + U(y-1) = 0$ which implies that $U(y \pm 1) = 0$. An induction gives $U(y+k) = 0$ for all $k \in \mathbb{Z}$, contradicting $U(\infty) = 1$. Thus, $U > 0$. Similarly, $U < 1$. Once we know $0 < U < 1$, we can define

$$\begin{aligned}
\rho(x) &:= \frac{U'(x)}{U(x)} \quad \Rightarrow \quad \int_x^{x+1} \rho(z)dz = \ln\frac{U(x+1)}{U(x)}, \\
\sigma(x) &:= \frac{U'(x)}{U(x)-1} \quad \Rightarrow \quad \int_x^{x+1} \sigma(z)dz = \ln\frac{U(x+1)-1}{U(x)-1}.
\end{aligned}$$

Then ρ and σ satisfy

$$\begin{aligned}
c\rho(x) &= e^{\int_x^{x+1}\rho(z)dz} + e^{\int_x^{x-1}\rho(z)dz} - 2 + B_1(x), \\
c\sigma(x) &= e^{\int_x^{x+1}\sigma(s)\,ds} + e^{\int_x^{x-1}\sigma(s)\,ds} - 2 + B_2(x),
\end{aligned}$$

where $B_1(x) = f(U(x))/U(x)$ and $B_2(x) = f(U(x))/[U(x)-1]$. Since $U(-\infty) = 0$ and $U(\infty) = 1$, we see that $B_1(-\infty) = f'(0)$, $B_1(\infty) = 0$, $B_2(-\infty) = 0$, and $B_2(\infty) = f'(1)$. The limits in (1.5) thus follow from Theorem 4.

2. Next, we establish the remaining limits stated in Theorem 3. Here we shall use the fact $U' > 0$, to be proven in the next section. Note that

$$cU''(x) = U'(x+1) + U'(x-1) + [f'(U(x)) - 2]U'(x).$$

Define

$$\chi(x) := \frac{U''(x)}{U'(x)} \quad \Rightarrow \quad \int_x^{x+1} \chi(z)dz = \ln \frac{U'(x+1)}{U'(x)}.$$

Then

$$c\chi(x) = e^{\int_x^{x+1} \chi(z)dz} + e^{-\int_{x-1}^{x} \chi(z)dz} + f'(U(x)) - 2 \quad \forall\, x \in \mathbb{R}.$$

Thus stated limits for χ in Theorem 3 thus follows from Theorem 4 and L'Hopital's rule.

3. Finally, the limits of $f(U(x))/U'(x)$ as $x \to \pm\infty$ are obtained by using the limits of χ and the identity

$$\begin{aligned} \frac{f(U(x))}{U'(x)} &= c - \frac{[U(x+1) - U(x)] - [U(x) - U(x-1)]}{U'(x)} \\ &= c - \int_0^1 \Big\{ e^{\int_x^{x+z} \chi(s)ds} - e^{-\int_{x-z}^{x} \chi(s)ds} \Big\} dz. \end{aligned}$$

In the next two subsections, we show the in addition part of Theorem 3.

2.3. **The characteristic values of non-minimum speed waves.**

Lemma 2.1. *If (c, U) is a traveling wave of speed $c > c_{\min}$, then the characteristic equation $c\lambda = e^{\lambda} + e^{-\lambda} - 2 + f'(0)$ has two different real roots and $\lambda := \lim_{x\to-\infty} U'(x)/U(x)$ is the smaller root. In particular, when $f'(0) = 0$, $\lim_{x\to-\infty} U'(x)/U(x) = 0$.*

Proof. Recall from Theorem 2 of [4] that $c_{\min} \geqslant c_*$, where

$$c_* := \min_{\lambda>0} \frac{e^{\lambda} + e^{-\lambda} - 2 + f'(0)}{\lambda}.$$

Hence $c_{\min} z = e^z + e^{-z} - 2 + f'(0)$ always has a root. This implies that $cz = e^z + e^{-z} - 2 + f'(0)$ has exactly two roots, which we denote by $\lambda(c)$ and $\Lambda(c)$ with $\lambda(c) < \Lambda(c)$, for $c > c_{\min}$.

Suppose on the contrary that $\lim_{x\to-\infty} U'(x)/U(x) = \Lambda(c)$. Let $\hat{c} \in (c_{\min}, c)$ and $(\hat{c}, \hat{U})$ be a traveling wave of speed $\hat{c}$. By (1.5), $\lim_{x\to-\infty} \hat{U}'(x)/\hat{U}(x) \leqslant \Lambda(\hat{c})$. Then

$$\lim_{x\to-\infty} \frac{d}{dx}\Big(\ln \frac{\hat{U}(x)}{U(x)} \Big) = \lim_{x\to-\infty} \Big\{ \frac{\hat{U}'(x)}{\hat{U}(x)} - \frac{U'(x)}{U(x)} \Big\} \leqslant \Lambda(\hat{c}) - \Lambda(c) < 0,$$

by the strictly monotonicity of $\Lambda(c)$ in c. Thus, $\lim_{x\to-\infty}\ln[\hat{U}(x)/U(x)]=\infty$ and there exists $M>0$ such that $\hat{U}(x)>U(x)$ for all $x\leqslant -M$. Similarly,

$$\lim_{x\to\infty}\frac{d}{dx}\Big\{\int_{U(x)}^{\hat{U}(x)}\frac{ds}{f(s)}\Big\}=\lim_{x\to\infty}\Big\{\frac{\hat{U}'(x)}{f(\hat{U}(x))}-\frac{U'(x)}{f(U(x))}\Big\}$$
$$=\begin{cases} 1/\hat{c}-1/c & \text{if } f'(1)=0,\\ [\mu(\hat{c})-\mu(c)]/f'(1) & \text{if } f'(1)<0.\end{cases}$$

This quantity is positive when $f'(1)=0$; so is the case when $f'(1)<0$ since the negative root $\mu=\mu(c)$ of $c\mu=e^{\mu}+e^{-\mu}-2+f'(1)$ satisfies $\mu(\hat{c})<\mu(c)$. Thus there exists $M_1>0$ such that $\hat{U}(x)>U(x)$ for all $x\geqslant M_1$. In conclusion, $\hat{U}(\cdot+M_1)>U(\cdot-M)$.

Now both $u_1(x,t):=\hat{U}(x+M_1+\hat{c}t)$ and $u_2(x,t):=U(x-M+ct)$ are solutions of (1.2). Since $u_1(\cdot,0)\geqslant u_2(\cdot,0)$, the comparison principle for (1.2) implies $u_1(\cdot,t)\geqslant u_2(\cdot,t)$ for all $t>0$, which is impossible since $c>\hat{c}$. Thus, $\lim_{x\to-\infty}U'(x)/U(x)=\lambda(c)$. □

The asymptotic behavior of U stated in Theorem 3 immediately gives the following

Corollary 2.2. *Suppose (c_1,U_1) and (c_2,U_2) are two traveling waves where $c_1<c_2$. Then there exist $a,b\in\mathbb{R}$ such that*

$$U_1<U_2 \quad in \quad (-\infty,a), \qquad U_1>U_2 \quad in \quad (b,\infty).$$

We remark that in the case of the differential equation $cU'=U''+f(U)$ one can take $a=b$ to conclude that a smaller speed wave profile is steeper than a larger speed wave profile; namely, on the phase plane (U,U'), if one writes $U'=P(c,U)$, then $P(c_1,s)>P(c_2,s)$ for all $s\in(0,1)$ and $c_2>c_1\geqslant c_{\min}$. For (1.3), we believe that this should also be the case.

2.4. **The characteristic value of minimum speed waves.**

Lemma 2.3. *If $(c_{\min},U)$ is a wave of minimum speed, then $\lambda:=\lim_{x\to-\infty}U'(x)/U(x)$ is the larger root (if there are two) of the characteristic equation $c_{\min}z=e^{z}+e^{-z}-2+f'(0)$.*

Proof. We use a contradiction argument. Suppose λ is the smaller root. Let Λ be the larger root. This allows us to construct a super-solution Φ of wave speed c for some $c<c_{\min}$ by joining an exponential function ψ defined on $(-\infty,0]$ and another function ϕ defined on $[0,\infty)$ obtained from the wave profile U of speed $c_{\min}$. We divide this construction into the following steps.

1. Set $\omega=(\lambda+\Lambda)/2$ and $\delta:=c_{\min}\,\omega-e^{\omega}-e^{-\omega}+2-f'(0)$. Then $\delta>0$ since the function $P(z):=c_{\min}\,z-e^{z}-e^{-z}+2-f'(0)$ is concave and vanishes at λ and Λ. Also by translation, we can assume that $U(0)$ is so small that

$$\sup_{0<s\leqslant U(0)e^{\omega}}\Big|\frac{f(s)}{s}-f'(0)\Big|<\frac{\delta}{2}, \qquad \sup_{x\leqslant 1}\frac{U'(x)}{U(x)}<\omega\ .$$

Set $\psi(x) = U(0)e^{\omega x}$. For every $c \in [c_{\min} - \delta/(2\omega), c_{\min}]$,

$$\begin{aligned}\mathcal{L}\psi(x) &:= c\,\psi'(x) - \psi(x+1) - \psi(x-1) + 2\psi(x) - f(\psi(x)) \\ &= \psi(x)\Big\{c\,\omega - e^{\omega} - e^{-\omega} + 2 - \frac{f(\psi(x))}{\psi(x)}\Big\} > 0 \quad \forall x \le 1.\end{aligned}$$

2. Next, we construct $\phi(c,\cdot)$, to be used as the super-solution defined on $[0,\infty)$. For each $c \in (0, c_{\min}]$, consider the equation $\phi = \mathbf{T}^c\phi$ on $\mathbb{R}$ where

$$\mathbf{T}^c\phi := \begin{cases} e^{-mx/c}\{U(0) + c\int_0^x e^{mz/c}W[m,\phi](z)dz\} & \text{if } x \geqslant 0, \\ U(x) & \text{if } x < 0, \end{cases}$$

$$W[m,\phi](z) := \phi(z+1) + \phi(z-1) + [m-2]\phi(z) + f(\phi(z)).$$

Following [4], a solution can be obtained as follows. Define $\{\phi_n\}_{n=0}^{\infty}$ by

$$\phi_0(c,\cdot) \equiv \mathbf{1}, \qquad \phi_{n+1}(c,\cdot) := \mathbf{T}^c\phi_n(c,\cdot)\ \forall\, n \in \mathbb{N}.$$

Note that $\mathbf{T}^c$ is a monotonic operator: $\psi_1 \leqslant \psi_2 \Rightarrow \mathbf{T}^c\psi_1 \leqslant \mathbf{T}^c\psi_2$. It follows that $\phi_{n+1} \leqslant \phi_n \leqslant \mathbf{1}$. In addition, since

$$c\,(e^{mx/c}U)' - e^{mx/c}W[m,U] = (c - c_{\min})U'e^{\mu x/c} \leqslant 0,$$

integrating this inequality over $[0,x]$ gives $U \leqslant \mathbf{T}^cU$. This implies that $\phi_n \geqslant U$ for all n. Consequently, $\phi(c,\cdot) := \lim_{n\to\infty}\phi_n$ exists and is a solution to $\phi = \mathbf{T}^c\phi$. It is easy to see that $U \leqslant \phi < \mathbf{1}$ on $[0,\infty)$, $\phi(c,0) = U(0)$, and

$$c\,\phi'(c,x) = \phi(c,x+1) + \phi(c,x-1) - 2\phi(c,x) + f(\phi(c,x)) \quad \forall\, x > 0.$$

This equation implies, for $0 < c_1 < c_2 \leqslant c_{\min}$, that $\phi(c_2,\cdot) \leqslant \mathbf{T}^{c_1}\phi(c_2,\cdot)$, so that $\phi_n(c_1,\cdot) \geqslant \phi(c_2,\cdot)$ for all n and $\phi(c_1,\cdot) > \phi(c_2,\cdot)$ on $(0,\infty)$. Following an idea in [4] or the technique for the uniqueness of U presented in this paper (§4), one can further show that $\phi(c,\cdot)$ is unique. The uniqueness implies that $\phi(c,\cdot)$ is continuous in c and $\phi(c_{\min},\cdot) \equiv U$. Therefore, $\lim_{c\to c_{\min}}\phi(c,\cdot) = U$ in $C^1([0,\infty))$. This further implies that

$$\lim_{c\to c_{\min}} \frac{\phi'(c,x)}{\phi(c,x)} = \frac{U'(x)}{U(x)} \quad \text{uniformly for } x \in [0,1].$$

3. Now let $c \in [c_{\min} - \delta/(2\omega), c_{\min})$ be such that

$$\max_{x\in[0,1]} \frac{\phi'(c,x)}{\phi(c,x)} < \omega.$$

We define

$$\Phi(x) = \begin{cases} \psi(x) & \text{if } x \leqslant 0, \\ \phi(c,x) & \text{if } x > 0. \end{cases}$$

Since $\psi(0) = U(0) = \phi(c,0)$ and

$$\frac{\psi'(x)}{\psi(x)} = \omega > \frac{\phi'(c,x)}{\phi(c,x)} \quad \forall x \in (-\infty,0)\cup(0,1],$$

$\phi < \psi$ in $(0,1]$ and $\psi < \phi \equiv U$ in $(-\infty,0)$. That is,

$$\Phi = \min\{\phi\ ,\ \psi\} \qquad \text{on } (-\infty,1].$$

Consequently, considering separately $x \in (-\infty, 0)$, $(0,1]$ and $(1,\infty)$, we see that

$$c\,\Phi'(x) \geqslant \Phi(x+1) + \Phi(x-1) - 2\Phi(x) + f(\Phi(x)) \quad \forall x \in (-\infty,0)\cup(0,\infty);$$

that is, Φ is a super-solution of wave speed c.

Thus, by Proposition 1 (iii), there is a traveling wave of speed c for some $c < c_{\min}$, contradicting the minimality of $c_{\min}$. This proves the lemma. $\square$

Remark 2.2. *If $f'(\cdot) \leqslant 0$ on $[1-\delta, 1]$ for some $\delta > 0$, then a constructive proof of Lemma 2.3 can be obtained by taking*

$$\Phi(x) = [U(0)+\epsilon]e^{\omega x} \ \ \forall\, x \leqslant 0, \qquad \Phi(x) = U(x+\varepsilon - \varepsilon e^{-kx}) + \epsilon \ \ \forall x > 0$$

where $0 < \epsilon \ll \varepsilon \ll U(0) \ll 1 \ll k$. We leave the verification to the interested readers.

3. Monotonicity of Wave Profiles

This section is dedicated to the proof of the monotonicity of any wave profile U. We point out here that the limits in (1.5) are established without the knowledge of the monotonicity of U so that we can use them here.

3.1. **The method of sliding.** This traditional method is *to compare $U(\cdot+\tau)$ and $U(\cdot)$ by decreasing τ continuously from a large value down to zero, namely, to show that*

$$\inf\{\tau > 0 \mid U(\cdot+\tau) > U(\cdot) \ \text{ on } \ \mathbb{R}\} = 0. \tag{3.1}$$

This implies $U' \geqslant 0$, and from an integral equation, $U' > 0$ on $\mathbb{R}$. If we know $U' > 0$ near $x = \pm\infty$ (e.g. by (1.5) for the case $\mu < 0 < \lambda$), then (3.1) follows easily from a comparison principle (cf. [4]). When $f'(0) = 0$, it is very difficult to show directly that $U' > 0$ in a vicinity of $x = -\infty$. Similar difficulty occurs near $x = \infty$ when $f'(1) = 0$. To overcome this difficulty, we use a modification of the method, stated in the third part of the following

Lemma 3.1. (i) *If $[a,b]$ is an interval on which $U' \leqslant 0$, then $b - a < 1$.*
(ii) *If $U' > 0$ on $[\xi, \xi+1]$, then $U(\xi) < U(x)$ for all $x > \xi$.*
(iii) *If $U' > 0$ on $[\xi-1,\xi+1]\cup[\eta-1,\eta+1]$ where $\xi < \eta$, then $U' > 0$ on $[\xi,\eta]$.*

Proof. (i) Suppose otherwise $b-a \geqslant 1$. Let $\hat{x} \in [b,\infty)$ be a point such that $U(\hat{x}) \leqslant U(x)$ for all $x \geqslant b$. Then $\hat{x}$ is a global minimum of U restricted on $[a,\infty)$, since $U' \leqslant 0$ on $[a,b]$. This leads to the following contradiction

$$0 = c\,U'(\hat{x}) = U(\hat{x}+1) + U(\hat{x}-1) - 2U(\hat{x}) + f(U(\hat{x})) \geqslant f(U(\hat{x})) > 0.$$

(ii) Let $\hat{x} \geqslant \xi+1$ be a point such that $U(\hat{x}) \leqslant U(x)$ for all $x \geqslant \xi+1$. Then $U(\xi) < U(\hat{x})$ since otherwise the same contradiction as above arises. Thus $U(\xi) < U(x)$ for all $x > \xi$.

(iii) By the second assertion, $U(\eta) > U(\xi)$, so that we can define

$$\tau^* := \inf\{\,\tau \in (0,\eta-\xi] \ \mid U(\cdot) < U(\cdot+\tau) \ \text{ on } \ [\xi,\eta-\tau]\,\}.$$

Clearly, $\tau^* \in [0, \eta - \xi)$. We claim that $\tau^* = 0$. Suppose on the contrary that $\tau^* > 0$. Then there exists $\hat{x} \in [\xi, \eta - \tau^*]$ such that,

$$U(\hat{x} + \tau^*) - U(\hat{x}) = 0 \leqslant U(x + \tau^*) - U(x) \quad \forall x \in [\xi, \eta - \tau^*].$$

For $x \in [\xi-1, \xi]$: (i) if $x+\tau^* \le \xi$, then $U(x+\tau^*)-U(x) > 0$ since $U' > 0$ on $[\xi-1, \xi]$; (ii) if $x+\tau^* > \xi$, by the second assertion, $U(x+\tau^*) > U(\xi) \geq U(x)$. Thus $U(x+\tau^*) > U(x)$ for all $x \in [\xi - 1, \xi]$. Similarly, $U(x+\tau^*) > U(x)$ for all $x \in [\eta - \tau^*, \eta - \tau^* + 1]$. Hence,

$$U(\hat{x} + \tau^*) - U(\hat{x}) = 0 \leqslant U(x + \tau^*) - U(x) \qquad \forall\, x \in [\xi - 1, \eta - \tau^* + 1].$$

Consequently, $U'(\hat{x} + \tau^*) = U'(\hat{x})$. Using the equation for U, we conclude that

$$U(\hat{x} + \tau^* + 1) + U(\hat{x} + \tau^* - 1) = U(\hat{x} + 1) + U(\hat{x} - 1).$$

Since $U(\cdot + \tau^*) \geqslant U(\cdot)$ on $[\xi - 1, \eta - \tau^* + 1]$, we see that $U(\hat{x} + \tau^* \pm 1) = U(\hat{x} \pm 1)$. By induction, $U(\hat{x} + \tau^* + k) = U(\hat{x} + k)$ for all integer k satisfying $\hat{x} + k \in [\xi - 1, \eta - \tau^* + 1]$. But this is impossible since $U(x + \tau^*) > U(x)$ for all $x \in [\xi - 1, \xi]$. Thus, $\tau^* = 0$.

That $\tau^* = 0$ implies $U(\cdot + \tau) > U(\cdot)$ on $[\xi, \eta - \tau]$ along a sequence $\tau \searrow 0$. In particular, $U'(x) \geqslant 0$ on $[\xi, \eta]$. Finally, for $m = \max_{0 \leqslant s \leqslant 1} |2 - f'(s)|$ and every $x \in [\xi, \eta]$,

$$c\,U''(x) = U'(x+1) + U'(x-1) + [f'(U) - 2]U'(x) \geqslant -mU'(x).$$

It follows that $(U'(x)e^{mx/c})' \geqslant 0$ or $U'(x)e^{mx/c} \geqslant U'(\xi)e^{m\xi/c} > 0$ for all $x \in [\xi, \eta]$. □

3.2. **A linear equation from blow-up.** It remains to find a sequence $\{[\xi_j - 1, \xi_j + 1]\}$ of intervals on which $U' > 0$. For this, we shall use a blow-up technique for the functions $\rho(x)$ and $\sigma(x)$, leading to the following two linear problems:

$$\begin{cases} c\,R'(x) = R(x+1) + R(x-1) - 2R(x) & \forall\, x \le 1, \\ |R| \le 1 \quad \text{on} \quad (-\infty, 2], \qquad |R(0)| = 1; \end{cases} \tag{3.2}$$

$$\begin{cases} c\,R'(x) = R(x+1) + R(x-1) - 2R(x) & \forall\, x \ge -1, \\ |R| \le 1 \quad \text{on} \quad [-2, \infty), \qquad |R(0)| = 1. \end{cases} \tag{3.3}$$

Lemma 3.2. *(i) If R solves* (3.2)*, then $|R| > 1/2$ on $[A-1, A+1]$ for some $A < 0$.*
(ii) Any solution of (3.3) *satisfies $|R| > 1/2$ on $[A-1, A+1]$ for some $A > 0$.*

Proof. (i) Note that $|R'| \leqslant 4/c$ on $(-\infty, 1]$. Set $z(x) := R'(x)/[R(x) + 2]$. Then

$$c\,z(x) = e^{\int_x^{x+1} z(t)dt} + e^{-\int_{x-1}^{x} z(t)dt} - 2, \quad |z(x)| \le 4/c \qquad \forall\, x \le 1.$$

Following the argument used in the previous section, we conclude that $\lim_{x\to-\infty} z(x)$ exists. Since R is bounded, $\liminf_{x\to-\infty} |R'(x)| = 0$. Thus, $\lim_{x\to-\infty} z(x) = 0$, which implies that $\lim_{x\to-\infty} R'(x) = 0$.

As $R(0)$ is a global extremum of R restricted on $(-\infty, 1]$, $R(j) = R(0)$ for all integer $j \le 1$. Upon using $\lim_{x\to-\infty} R'(x) = 0$, we derive that $\lim_{x\to-\infty} R(x) = R(0)$, from which, the assertion of the lemma follows.

(ii) The proof is analogous to the case (i) and therefore is omitted. □

3.3. **The monotonicity of wave profile.** That $U' > 0$ follows from Lemma 3.1 (iii) and the following

Lemma 3.3. *There exists a sequence* $\{\xi_i\}_{i\in\mathbb{Z}}$ *such that* $U' > 0$ *on* $[\xi_i - 1, \xi_i + 1]$ *for each* $i \in \mathbb{Z}$ *and* $\lim_{i\to\pm\infty} \xi_i = \pm\infty$.

Proof. **The sequence** $\{\xi_i\}_{i\leqslant 0}$. We need consider only the case $f'(0) = 0$ and $\lim_{x\to-\infty} \rho(x) = 0$ where $\rho(x) = U'(x)/U(x)$. Define

$$\varepsilon_j = \max_{x\le j} |\rho(x)| \ \forall\, j < 0, \qquad \theta = \limsup_{j\to-\infty} \frac{\varepsilon_{j-3}}{\varepsilon_j} \in [0, 1].$$

We claim that $\theta = 1$. Suppose not. Then, for $\hat\theta = (1 + \theta)/2$, there exists $J < 0$ such that $\varepsilon_{j-3} \le \hat\theta\varepsilon_j$ for all $j \le J$. Hence, $\varepsilon_{J-3k} \le \varepsilon_J \hat\theta^k$ for every integer $k \ge 0$. Consequently, $|\rho(x)| \le \varepsilon_J \hat\theta^{(J-x)/3-1}$ for all $x \le J$. For $y < J$,

$$\ln \frac{U(J)}{U(y)} = \int_y^J \rho(x)dx \leqslant \int_y^J \varepsilon_J \hat\theta^{(J-x)/3-1} dx \leqslant \frac{3\varepsilon_J}{|\hat\theta \ln \hat\theta|}.$$

Sending $y \to -\infty$ we obtain a contradiction. Hence $\theta = 1$.

Let $\{j_k\}_{k=1}^\infty$ be a sequence such that $\lim_{k\to\infty} j_k = -\infty$ and $\lim_{k\to\infty} \varepsilon_{j_k-3}/\varepsilon_{j_k} = 1$. Let $x_k \leqslant j_k - 3$ be a point such that $|\rho(x_k)| = \varepsilon_{j_k-3}$. Define $\rho_k(x) := \rho(x_k + x)/|\rho(x_k)|$. Then $\max_{x\le 3} |\rho_k(x)| \le \varepsilon_{j_k}/\varepsilon_{j_k-3}$, $|\rho_k(0)| = 1$, and

$$\begin{aligned} c\,\rho_k'(x) &= [\rho_k(x + 1) - \rho_k(x)]e^{\rho(x_k)\int_x^{x+1} \rho_k(z)dz} + \\ &\qquad [\rho_k(x - 1) - \rho_k(x)]e^{-\rho(x_k)\int_{x-1}^{x} \rho_k(z)dz} + \rho_k(x) f_1(U(x_k + x)) \end{aligned}$$

where $f_1(s) = f'(s) - f(s)/s \to 0$ as $s \searrow 0$. This equation implies that $\{\rho_k\}_{k=1}^\infty$ is a family of bounded and equi-continuous functions on $(-\infty, 2]$. Hence, a subsequence which we still denote by $\{\rho_k\}$ converges to a limit R, uniformly in any compact subset of $(-\infty, 2]$. Clearly, R satisfies (3.2).

By Lemma 3.2 (i), there exists a constant $A < 0$ such that either $R \geqslant 1/2$ on $[A - 1, A + 1]$ or $R \leqslant -1/2$ on $[A - 1, A + 1]$. As $\lim_{k\to\infty} \rho_k \to R$ on $[A - 1, A + 1]$, there exists an integer $K > 0$ such that for every integer $k \geqslant K$, either $\rho_k > 0$ on $[A - 1, A + 1]$ or $\rho_k < 0$ on $[A - 1, A + 1]$. By Lemma 3.1 (i), the latter case is impossible. Thus $\rho_k > 0$ on $[A - 1, A + 1]$, i.e. $U' > 0$ on $[x_k + A - 1, x_k + A + 1]$. Define $\xi_i = A + x_{K+|i|}$ for all integer $i \le 0$. Then $\lim_{i\to-\infty} \xi_i = -\infty$ and $U' > 0$ on $[\xi_i - 1, \xi_i + 1]$ for every integer $i \le 0$.

The sequence $\{\xi_i\}_{i\geqslant 1}$**.** We need consider only the case $f'(1) = 0$. Define

$$\sigma(x) = \frac{U'(x)}{U(x) - 1}, \quad \delta_j = \max_{x\in[j,\infty)} |\sigma(x)|, \quad \theta = \limsup_{j\to\infty} \frac{\delta_{j+3}}{\delta_j} \in [0, 1].$$

With an analogous argument as before, we can show that $\theta = 1$. Take a sequence $\{j_k\}_{k=1}^\infty$ satisfying $\lim_{k\to\infty} j_k = \infty$ and $\lim_{k\to\infty} \delta_{j_k+3}/\delta_{j_k} = 1$. Let $x_k \ge j_k + 3$ be a point such that $\delta_{j_k+3} = |\sigma(x_k)|$. Set $\sigma_k(x) = \sigma(x + x_k)/|\sigma(x_k)|$. Then $|\sigma_k| \le \delta_{j_k}/\delta_{j_k+3}$ in $[-3, \infty)$. Same as before, a subsequence of $\{\sigma_k\}_{k=0}^\infty$ converges to a limit R satisfying (3.3). The rest of the proof follows from an analogous argument as before. This completes the proof of Lemma 3.3 and also the proof of Theorems 2 and 3. □

4. Uniqueness of Traveling Waves

In this section we prove Theorem 1. In the sequel, U and V are two traveling waves with the same speed c. We want to show that $U(\cdot) \equiv V(\cdot - \xi)$ for some $\xi \in \mathbb{R}$.

4.1. **A Comparison Principle.**
The sliding method applies on compact intervals.

Lemma 4.1. *If $V \leqslant U$ on $[a-1,a) \cup (b,b+1]$ where $a \leqslant b$, then $V \leqslant U$ on $[a,b]$.*

Proof. Let ξ be the number such that $\min_{[a-1,b+1]}\{U(\cdot) - V(\cdot - \xi)\} = 0$ and let $y \in [a-1,b+1]$ be the maximum value satisfying $U(y) - V(y-\xi) = 0$. Then $y \notin [a,b]$ since otherwise $U'(y) = V'(y-\xi)$ and the equations for $U(\cdot)$ and $V(\cdot - \xi)$ evaluated at y would imply $U(y \pm 1) = V(y - \xi \pm 1)$, contradicting the maximality of y. Thus, $y \in [a-1,a) \cup (b,b+1]$, and by the assumption, $V(y) \leqslant U(y) = V(y-\xi)$. Thus $\xi \leqslant 0$. We conclude that $U(\cdot) \geqslant V(\cdot - \xi) \geqslant V(\cdot)$ on $[a-1,b+1]$. □

The success of such a simple translation technique relies on (i) the existence of a minimal translation ξ and (ii) the existence of a maximum y, both of which attribute to the fact that a continuous function on a compact set attains its global extremes. When the domain of interest is unbounded, neither ξ nor y may exist, and therefore different techniques are needed.

4.2. **Comparison near $x = \infty$.**
We shall compare traveling waves on the unbounded domain $[0,\infty)$. Since simple translation technique does not work, we shall instead construct a family of super-solutions for which translation technique works. If one is willing to make the assumption $f' \leqslant 0$ on $[1-\delta, 1]$ for some $\delta > 0$, then for every $\varepsilon > 0$,

$$\min\{U + \varepsilon, 1\} \qquad \text{on } [-1,\infty)$$

is a super-solution on $[0,\infty)$ provided that $U(-1) \geqslant 1 - \delta$. In this manner, no asymptotic behavior of U near $x = \infty$ is needed.

When only the assumption **(A)** is made, we construct a different family of super-solutions obtained from the detailed asymptotic behavior of wave profiles and compression:

$$Z(\ell, x) := U([1+\ell]x) \qquad \forall\, x \in [-1,\infty), \ell \in (0,1].$$

The idea here is that the rate of Z approaching 1 as $x \to \infty$ is faster than that of any wave profile, and therefore is strictly bigger than any wave profile for sufficiently large x.

Since $\lim_{x\to\infty} U''(x)/U'(x) = \mu \leqslant 0 < c$ and $U'(x+h)/U'(x) = e^{\int_x^{x+h} U''(s)/U'(s)ds}$, by translation, we may assume that

$$\sup_{x \geqslant 0,\ |h| \leqslant 2} \frac{U''(x+h)}{U'(x)} < c. \tag{4.1}$$

For $\ell \in (0,1]$ and $x \geqslant 0$, writing $y = (1+\ell)x$ and $Z(\ell, x) = Z(x)$, we calculate

$$\begin{aligned}\mathcal{L}Z(x) &:= c\,Z'(x) - Z(x+1) - Z(x-1) + 2Z(x) - f(Z(x)) \\ &= c[1+\ell]\,U'(y) - U(y+1+\ell) - U(y-1-\ell) + 2U(y) - f(U(y)) \\ &= c\,\ell\,U'(y) + U(y+1) + U(y-1) - U(y+1+\ell) - U(y-1-\ell) \\ &= \ell\,U'(y)\Big\{c - \int_0^1 \int_{-1-\ell z}^{1+\ell z} \frac{U''(y+h)}{U'(y)} dh dz\Big\} > 0.\end{aligned}$$

This shows that for each $\ell \in (0,1]$, $Z(\ell, \cdot)$ is a (strict) super-solution on $[0,\infty)$.

Lemma 4.2. *Assume* (4.1). *Suppose* $V \leqslant U$ *on* $[0,1]$. *Then* $V \leqslant U$ *on* $[0,\infty)$.

Proof. Consider the function, for $x \geqslant 0, \xi \in \mathbb{R}$, and $\ell > 0$,

$$\Psi(\xi, \ell, x) := \int_{V(x-\xi)}^{U([1+\ell]x)} \frac{ds}{f(s)}.$$

Note that

$$\lim_{x\to\infty} \frac{\partial \Psi(\xi,\ell,x)}{\partial x} = \lim_{x\to\infty} \Big(\frac{(1+\ell)U'}{f(U)} - \frac{V'}{f(V)}\Big) > 0 \quad \forall\, \ell > 0, \xi \in \mathbb{R};$$

$$\inf_{x\geq 0, \xi\in\mathbb{R}, \ell\in[0,1]} \frac{\partial \Psi}{\partial \xi} = \inf_{y\in\mathbb{R}} \frac{V'(y)}{f(V(y))} > 0.$$

Thus $\lim_{x\to\infty} \Psi(\xi, \ell, x) = \infty$. For each fixed $\ell \in (0,1]$, there exists at least one ξ such that $\Psi(\xi, \ell, \cdot) \geqslant 0$ on $[0,\infty)$. Let $\xi(\ell)$ be the infimum of such numbers.

We claim that $\xi(\ell) \leqslant 0$. Suppose otherwise. Since $\lim_{x\to\infty} \Psi(\xi(\ell), \ell, x) = \infty$, there exists $y \in [0,\infty)$ such that $\Psi(\xi(\ell), \ell, y) = 0$. We must have $y > 1$, since $V(\cdot - \xi(\ell)) < V(\cdot) \leqslant U(\cdot) \leqslant U([1+\ell]\cdot)$ on $[0,1]$. Thus, for $Z(x) = U([1+\ell]x)$,

$$Z(y) = V(y - \xi(\ell)), \qquad V(\cdot - \xi(\ell)) \leqslant Z(\cdot) \;\text{ on }\; [0,\infty).$$

This implies $V'(y - \xi(\ell)) = Z'(y)$ and a contradiction

$$0 = \mathcal{L}V\Big|_{y-\xi(\ell)} \geq \mathcal{L}Z\Big|_{y} > 0.$$

This contradiction shows that $\xi(\ell) \leqslant 0$, so that $V(\cdot) \leqslant V(\cdot - \xi(\ell)) \leqslant U([1+\ell]\cdot)$ on $[0,\infty)$. Sending $\ell \searrow 0$, we obtain that $V(\cdot) \leqslant U(\cdot)$ on $[0,\infty)$. □

4.3. **Comparison near** $x = -\infty$**.** In general, on the unbounded interval $(-\infty, 0]$, it is vary hard to construct a family of super-solutions that can be used for the translation argument, such as that in the previous two subsections; this is due to the fact that the constant state $\mathbf{0}$ is unstable. Hence we compare directly two traveling waves. We shall show that wave profiles are ordered (i.e. one is bigger than the other) near $x = -\infty$, by magnifying differences between any two wave profiles.

For every $\xi \in \mathbb{R}$ and $x \in \mathbb{R}$, we define

$$W(\xi, x) = \begin{cases} \displaystyle\int_{V(x-\xi)}^{U(x)} \frac{ds}{f(s)} & \text{if } \; c > c_{\min}, \\ \ln U(x) - \ln V(x-\xi) & \text{if } \; c = c_{\min}. \end{cases}$$

Note that $W(\xi, x)$ magnifies the differences between U and V. When $c > c_{\min}$,

$$W_x(\xi, x) \ := \ \frac{\partial W(\xi, x)}{\partial x} = \frac{U'}{f(U)} - \frac{V'}{f(V)} \longrightarrow 0 \text{ as } \ x \to \pm\infty.$$

This limit shows that the magnified difference between wave profiles changes slowly. The conclusion for $c = c_{\min}$ is analogous.

Lemma 4.3. *There exist $\nu > 0$ and $A \in [-\infty, \infty]$ such that*

$$\lim_{x \to -\infty} W(\xi, x) = A + \nu\xi \quad \forall\, \xi \in \mathbb{R}. \tag{4.2}$$

Consequently, near $x = -\infty$, $U < V(\cdot - \xi)$ if $A + \nu\xi < 0$ and $U > V(\cdot - \xi)$ if $A + \nu\xi > 0$.

Proof. First, we consider the case $c > c_{\min}$. Note that

$$\lim_{x \to -\infty} \Big\{ W(\xi, x) - W(0, x) \Big\} = \lim_{x \to -\infty} \int_{x-\xi}^{x} \frac{V'(y)dy}{f(V(y)))} = \nu\xi$$

where $\nu = 1/c$ when $f'(0) = 0$ and $\nu = \lambda/f'(0)$ otherwise. Suppose $\lim_{x\to-\infty} W(\xi, x)$ does not exist. Then $A := \limsup_{x\to-\infty} W(\xi, x) > B := \liminf_{x\to-\infty} W(\xi, x)$. Taking an appropriate ξ, we can assume without loss of generality that $A > 0 > B$. Let α, β be finite numbers satisfying $B < \beta < 0 < \alpha < A$. Then there exist sequences $\{x_i\}$ and $\{y_i\}$ satisfying

$$W(\xi, x_i) = \alpha, \quad W(\xi, y_i) = \beta, \quad x_{i+1} < y_i < x_i, \quad \lim_{i\to\infty} x_i = -\infty.$$

Since $\lim_{x\to-\infty} W_x(\xi, x) = 0$, there exists a large integer i such that $W(\xi, \cdot) > 0$ in $[x_{i+1} - 1, x_{i+1}] \cup [x_i, x_i + 1]$ and $W(\xi, y_i) < 0$. This implies that $V(\cdot - \xi) < U(\cdot)$ on $[x_{i+1} - 1, x_{i+1}] \cup [x_i, x_i + 1]$ and $V(y_i - \xi) > U(y_i)$ which is impossible by Lemma 4.1. Thus $A = B$.

The case $c = c_{\min}$ is analogous. □

4.4. **Proof of Theorem 1.** Let U and V be two traveling wave profiles with the same speed c. By translation, we can assume that $V(0) = U(0)$ and that U and V satisfy (4.1). By exchanging the roles of U and V if necessary we can use Lemma 4.3 to conclude that (4.2) holds with $A \in [0, \infty]$.

Let $\eta \geqslant 0$ be the unique value such that

$$\min_{x\in[0,1]} \{U(x) - V(x - \eta)\} = 0.$$

By Lemma 4.2, $V(\cdot - \eta) \leqslant U(\cdot)$ on $[0, \infty)$. We claim that $V(\cdot - \eta) \leqslant U(\cdot)$ on $(-\infty, 0]$. Suppose not. Then $\inf_{x\in\mathbb{R}} W(\eta, x) < 0$. Since $W_\xi > 0$ and $W(\eta, \pm\infty) \geqslant 0$, there is a unique value $\xi > \eta$ such that $\min_{\mathbb{R}} W(\xi, \cdot) = 0$. This implies that there exists $y \in \mathbb{R}$ such that $W(\xi, y) = 0 = \min_{\mathbb{R}} W(\xi, \cdot)$. It further implies that $V(\cdot - \xi) \leqslant U(\cdot)$ and $V(y - \xi) = U(y)$. A comparison principle shows that this is impossible. Hence, $V(\cdot - \eta) \leqslant U(\cdot)$ on $\mathbb{R}$. Since $\min_{[0,1]}\{U(\cdot - \eta) - V(\cdot)\} = 0$, we must have $\eta = 0$ and $U \equiv V$. □

5. Asymptotic Expansions

Finally, we derive and verify asymptotic expansions for traveling wave profiles near $x = -\infty$, accurate enough to distinguish the translation differences. The idea is to construct, on $(-\infty, 1]$, sub/super solutions having special tails near $x = -\infty$ and slopes on the interval $[0, 1]$. The comparison between a wave profile and a sub/super solution near $x = -\infty$ will be made by a result similar to (4.2) in Lemma 4.3. The comparison on $[0, 1]$ will be made in a manner similar to that in the proof of Lemma 2.3, Step 3.

5.1. **Super/sub solutions.** In the sequel, a Lipschitz continuous function defined on $[a-1, b+1]$ is called a **super/sub solution (of speed** c**)** on $[a, b]$ if

$$\pm\,\mathcal{L}\,[\phi](x) \geqslant 0 \qquad \text{a.e. } x \in (a, b).$$

where $\mathcal{L}\,[\phi](x) := c\,\phi'(x) - \phi(x+1) - \phi(x-1) + 2\phi(x) - f(\phi(x))$.

Lemma 5.1. *Suppose ϕ is a subsolution (or supersolution) on $[a, b]$ and $\phi < U$ (or $\phi > U$) on $[a-1, a) \cup (b, b+1]$. Then $\phi < U$ (or $\phi > U$) on $[a, b]$.*

The proof is similar to that for Lemma 4.1 and is omitted.

Our asymptotic expansion for a wave profile is expressed in terms of a constructed function ϕ such that, for some $x_0 \in \mathbb{R}$,

$$U(x + x_0) = \phi(x + o(1)) \ \ \forall\, x \leqslant 0 \qquad \text{where} \ \lim_{x\to-\infty} o(1) = 0. \tag{5.1}$$

For this, we shall use the same idea as that of Lemma 4.3. Consider the case $\lambda \neq 0$. Suppose ϕ is either a sub-solution or a super-solution on $(-\infty, 0]$ and

$$\lim_{x\to-\infty} \frac{\phi'(x)}{\phi(x)} = \lim_{x\to-\infty} \frac{U'(x)}{U(x)} = \lambda > 0. \tag{5.2}$$

Consider the function, for $\xi \in \mathbb{R}$ and $x \leqslant 0$,

$$W(\xi, x) = \int_{\phi(x)}^{U(x+\xi)} \frac{ds}{s} = \ln \frac{U(x+\xi)}{\phi(x)}. \tag{5.3}$$

Lemma 5.2. *Suppose ϕ satisfies* (5.2) *and is either a super-solution or a sub-solution on $(-\infty, 0]$. Let W be defined as in* (5.3)*. Then* (4.2) *holds for some $A \in [-\infty, \infty]$.*

The proof is similar to that for Lemma 4.3 and therefore is omitted.

Suppose A is shown to be finite. Then for $x_0 := -A/\nu$, every $\varepsilon > 0$, and all $x \ll -1$, $W(x_0 - \varepsilon, x) < 0 < W(x_0 + \varepsilon, x)$; that is, $\phi(x - \varepsilon) < U(x + x_0) < \phi(x + \varepsilon)$ for every $\varepsilon > 0$ and all $x \ll -1$. Hence (5.1) holds. To construct sup/super solutions and to show that A is finite, we shall assume that

(B) $|f(u) - f'(0)u| \leq M u^{1+\alpha}$ for all $u \in [0, 1]$ and some positive constants M and α.

In most cases, we shall construct sub/super solutions via linear combinations of exponential functions. Note that for $\phi = ae^{\omega x}$, $\mathcal{L}\phi = P(\omega)\phi + [f'(0)\phi - f(\phi)]$ where

$$P(\omega) := c\,\omega - e^{\omega} - e^{-\omega} + 2 - f'(0).$$

Observe that $P(\cdot)$ is concave, positive between its two roots, and negative outside of these two roots. Denote by λ and Λ, where $0 \leqslant \lambda \leqslant \Lambda$, the two roots of $P(\cdot) = 0$. Among all possibilities, we divide them into four cases:

(i) $c = c_{\min}$ and λ is not a double root;
(ii) $c = c_{\min}$ and λ is a double root;
(iii) $c > c_{\min}$ and $f'(0) > 0$;
(iv) $c > c_{\min}$ and $f'(0) = 0$.

Note that $\lim_{x\to-\infty}\{U'(x)/U(x)\} > 0$ in the cases (i)–(iii). For the last case (iv), $\lambda = 0$ so that sub/super solutions have to be constructed by non-exponential functions. For this, we need extra assumptions on f.

5.2. **The case $c = c_{\min}$ and λ not a double root.** Note that $\lambda < \Lambda$. Also,

$$\lim_{x\to-\infty}\frac{U'(x)}{U(x)} = \Lambda > 0 \implies \frac{U(x)}{U(0)} = e^{\int_0^x U'/U} = e^{\Lambda x + o(x)}.$$

Choose ω_1 and ω_2 satisfying

$$\lambda < \omega_1 < \Lambda < \omega_2, \quad \omega_2 < (1+\alpha)\Lambda.$$

Then $P(\omega_1) > 0 = P(\Lambda) > P(\omega_2)$. Consider, for $\varepsilon \in [0,1]$ and small $\delta > 0$,

$$\phi^{\pm}(\varepsilon,\delta,x) := \delta\Big\{e^{\Lambda x} \pm \varepsilon(e^{\omega_1 x} - e^{\Lambda x}) \pm \delta^{\alpha/2}(e^{\Lambda x} - e^{\omega_2 x})\Big\}.$$

Note that when $\varepsilon > 0$ and $x \ll -1$, $\phi^+ \gg U$ and $\phi^- < 0$. Also, for all $x \leqslant 0$,

$$\mathcal{L}\,[\phi^+] = \delta\Big\{\varepsilon P(\omega_1)e^{\omega_1 x} - P(\omega_2)\delta^{\alpha/2}e^{\omega_2 x} + O(1)\delta^{\alpha}[\varepsilon^{1+\alpha}e^{(1+\alpha)\omega_1 x} + e^{(1+\alpha)\Lambda x}]\Big\} > 0$$

if $\varepsilon \in [0,1]$ and $\delta \in (0,\delta_0]$ for some $\delta_0 > 0$. Similarly, for every $\varepsilon \in [0,1]$ and $\delta \in (0,\delta_0]$, $\max\{0,\phi^-(\varepsilon,\delta,\cdot)\}$ is a sub-solution on $(-\infty,0]$. Taking δ_0 small enough we can assume that $\phi^{\pm}_x > 0$ for all $x \in [0,1]$, $\varepsilon \in [0,1]$ and $\delta \in [0,\delta_0]$.

Take ξ negatively large such that $\delta := U(\xi) < \delta_0$. Comparing $U(\cdot+\xi-1)$ with $\phi^+(\varepsilon,\delta,\cdot)$ on $(-\infty,0]$ for every $\varepsilon \in (0,1]$, we see that $U(x+\xi-1) \leq \phi^+(\varepsilon,\delta,x)$ for all $x \leq 0$. Here the positivity of ε guarantees that $\phi^+ > U$ near $x = -\infty$. Now sending $\varepsilon \searrow 0$ we conclude that $U(x+\xi-1) \leq \delta[1+\delta^{\alpha/2}]e^{\Lambda x}$ for all $x \leqslant 0$. Similarly, $U(x+\xi+1) > \delta[1-\delta^{\alpha/2}]e^{\Lambda x}$ for all $x \leqslant 0$.

Now applying Lemma 5.2 to $\phi = \phi^+(0,\delta_0,x)$, we see that there is the limit

$$A = \lim_{x\to-\infty}\Big\{\ln U(x) - \ln\phi^+(0,\delta_0,x)\Big\} = \lim_{x\to-\infty}\Big\{\ln U(x) - \Lambda x\Big\} - \ln[\delta_0(1+\delta_0^{\alpha/2})].$$

From the estimate in the previous paragraph, A must be finite. Hence we proved the following:

Theorem 5.1. *Assume* **(A)** *and* **(B)**. *Let* $(c_{\min}, U)$ *be a traveling wave of the minimum speed where the characteristic equation has two roots* λ, Λ, $\lambda < \Lambda$. *Then, for some* $x_0 \in \mathbb{R}$,

$$U(x) = e^{\Lambda[x+x_0+o(1)]} \quad \forall x \leqslant -1 \qquad \textit{where} \quad \lim_{x\to-\infty} o(1) = 0.$$

5.3. **The case** $c = c_{\min}$ **and** λ **a double root.** Note that $P(\lambda) = P'(\lambda) = 0$; that is,

$$c_{\min} = e^{\lambda} - e^{-\lambda}, \quad f'(0) = \lambda(e^{\lambda} - e^{-\lambda}) + (2 - e^{\lambda} - e^{-\lambda}). \tag{5.4}$$

Take $\omega \in (\lambda, [1+\alpha]\lambda)$ and consider the function, for small $\delta > 0$,

$$\phi^*(\delta, x) = \delta[-xe^{\lambda x} - \delta^{\alpha/2}(e^{\lambda x} - e^{\omega x})]. \tag{5.5}$$

Note that $\phi^* > 0$ in $(-\infty, 0)$ and $\phi^* < 0$ in $(0, \infty)$. Since $P(\omega) < 0$, for $x \le 0$,

$$\mathcal{L}\phi^* = \delta\Big\{\delta^{\alpha/2} P(\omega) e^{\omega x} + O(1)\delta^{\alpha}[|x| + 1]^{1+\alpha} e^{(1+\alpha)\lambda x]}\Big\} < 0.$$

It follows that $\phi^- := \max\{\phi^*, 0\}$ is a sub-solution for every $\delta \in (0, \delta_0]$, where $\delta_0 > 0$.

From Lemma 5.2, there exists the limit

$$A = \lim_{x\to-\infty} \Big\{\ln U(x) - \lambda\, x - \ln|x|\Big\}. \tag{5.6}$$

We claim that $A < \infty$. Suppose $A = \infty$. Then for each fixed $\xi \in \mathbb{R}$, $U(x+\xi) > \phi^-(\delta, x)$ for all $x \ll -1$. Since $\phi^- = 0$ on $[0, \infty)$ and ϕ^- is a sub-solution, a comparison gives $U(x+\xi) > \phi^-(\delta, x)$ for all $x \in \mathbb{R}$. This is impossible for every $\xi \in \mathbb{R}$. Thus $A < \infty$.

Since $P(\omega) < 0$ for every $\omega \neq \lambda$, it is very hard to construct super-solutions. As the existence of a super-solution implies the existence of a traveling wave, the construction of a super-solution is equivalent to find $c_{\min}$ which is not totally determined by the local behavior of $f(s)$ near $s = 0$. That $c_{\min}$ is the solution of (5.4) which is uniquely determined by $f'(0)$ requires special properties on the non-linearity on f. The whole non-linear structure of f on $[0,1]$ determines whether A is bounded from below. As will be seen in a moment, the answer to whether A is bounded is all we need to determine uniquely the asymptotic behavior of U as $x \to -\infty$, i.e., the alternatives in (1.9).

Case (1) $A > -\infty$. Then A is finite, so from (5.6), the first alternative in (1.9) holds.

Case (2) $A = -\infty$. Fix $\omega \in (\lambda, (1+\alpha)\lambda)$. Consider, for $\varepsilon \in [0,1]$ and small $\delta > 0$,

$$\phi^+(\varepsilon, \delta, x) = \delta\{[1 - \varepsilon x]e^{\lambda x} - \delta^{\alpha/2} e^{\omega x}\}.$$

Direct calculation shows that ϕ^+ is a super-solution on $(-\infty, 0]$ for every $\varepsilon \in [0,1]$ and $\delta \in (0, \delta_0]$. Fix a translation such that $U(1) \le \delta_0/2$. For every $\varepsilon \in (0,1]$ we compare $U(\cdot)$ and $\phi^+(\varepsilon, \delta_0, \cdot)$ on $(-\infty, 0]$. When $x \in [0,1]$, $U(x) \le U(1) < \delta_0/2 < \phi(\varepsilon, \delta_0, x)$. Since $A = -\infty$, we see that $U < \phi$ for all $x \ll -1$. It then follows that $U(\cdot) < \phi(\varepsilon, \delta_0, \cdot)$ on $(-\infty, 1]$. Sending $\varepsilon \searrow 0$ we obtain $U(x) \le \delta_0 e^{\lambda x}$ for all $x \in (-\infty, 0]$.

Also, by Lemma 5.2, there exists the limit

$$\tilde{A} := \lim_{x\to-\infty} \Big\{\ln U(x) - \ln \phi^+(0, \delta_0, x)\Big\} = \lim_{x\to-\infty} \Big\{\ln U(x) - \lambda x\Big\} - \ln \delta_0.$$

In addition, since $U(x) \le \delta_0 e^{\lambda x}$ for all $x \in (-\infty, 0]$, $\tilde{A} \le 0$.

Next we show that $\tilde{A} > -\infty$. To do this, for every $\omega_1 \in [\lambda, \omega]$, consider the function $\phi^-(\omega_1, \delta, x) := \delta[e^{\omega_1 x} + e^{\omega x}]$. It is easy to show that ϕ^- is a sub-solution on $(-\infty, 0]$ for every $\omega_1 \in [\lambda, \omega]$ and every $\delta \in (0, \delta_0]$.

Fix a translation such that $U(-1) > 2\delta_0$. For every $\omega_1 \in (\lambda, \omega]$, by comparing U and $\phi^-(\omega_1, \delta_0, x)$, we see that $U > \phi^-(\omega_1, \delta_0, x)$, since $\omega_1 > \lambda$ implies $U > \phi^-$ for all $x \ll -1$. Now sending $\omega_1 \searrow \lambda$ we see that $U(x) \geq \delta_0 e^{\lambda x}$ for all $x \leq 0$. Thus $\tilde{A}$ is finite; namely, the second alternative in (1.9) holds.

Finally, we provide two examples showing that both alternatives in (1.9) can happen.

Example 1. This example provides the second alternative in (1.9). We define

$$U(x) = \frac{e^x}{1+e^x}, \quad \lambda = 1, \quad c = e - \frac{1}{e},$$
$$f(u) = \frac{u(1-u)(e-1)[2(1-u)^2 + 2eu^2 + (e^2+1)(e+1)u(1-u)/e]}{e(1-u)^2 + eu^2 + u(1-u)(e^2+1)}.$$

Using $e^x = U(x)/[1-U(x)]$, one can verify that (c, U) is a traveling wave. Since $f'(0) = 2 - 2/e$, $\lambda = 1$ is a double root of the characteristic equation $c\omega = e^\omega + e^{-\omega} - 2 + f'(0)$. Consequently, $c_{\min} = e - 1/e$.

Example 2. We show that the first alternative in (1.9) holds if

$$f \in C^{1+\alpha}([0,1]), \quad f(0) = f(1) = 0 < f(u) \leqslant f'(0)u \quad \forall\, u \in (0,1). \tag{5.7}$$

First of all, define $(c_{\min}, \lambda)$ as in (5.4), one can show that $\min\{1, e^{\lambda x}\}$ is a super-solution with $c = c_{\min}$, so that there is a traveling wave of speed $c_{\min}$. Consequently, the minimum wave speed is given by the solution of (5.4); see, for example, [3, 4, 18].

Also, there is a super-solution given by

$$\phi^+(x) = [1 - \tfrac{\lambda}{1+\lambda}\, x]e^{\lambda x} \quad \forall\, x < 0, \qquad \phi^+(x) = 1 \ \text{ for } \ x \geqslant 0.$$

Note that, for a large constant M, $\phi^+(x+M) > \phi^*(\delta_0, x)$ on $\mathbb{R}$, where ϕ^* is as in (5.5). Following the existence proof of [3], $(\max\{\phi^*, 0\}, \phi^+)$ sandwiches a solution which satisfies the first alternative in (1.9).

We conclude the following:

Theorem 5.2. *Assume* **(A)** *and* **(B)**. *Suppose $c = c_{\min}$ and the characteristic equation has a root λ of multiplicity 2, i.e.* (5.4) *holds. Then there is the alternative* (1.9). *In addition, under* (5.7), *only the first alternative in* (1.9) *holds.*

5.4. **The case $c > c_{\min}$ and $f'(0) > 0$.** Let λ and Λ, $\lambda < \Lambda$, be two roots of the characteristic equation. Pick ω such that $\lambda < \omega < \min\{\Lambda, (1+\alpha)\lambda\}$. Then $P(\omega) > 0$. For each $\varepsilon \in (0, e^{-\omega}]$ and small δ, consider functions

$$\phi^\pm(\varepsilon, \delta, x) := \delta([1 \mp \varepsilon]e^{\lambda x} \pm \varepsilon e^{\omega x}), \quad x \leqslant 1.$$

Note that

$$\min_{0 \leqslant x \leqslant 1} \frac{\phi^+_x(\varepsilon, \delta, x)}{\phi^+(\varepsilon, \delta, x)} = \lambda + \varepsilon(\omega - \lambda), \qquad \max_{0 \leqslant x \leqslant 1} \frac{\phi^-_x(\varepsilon, \delta, x)}{\phi^-(\varepsilon, \delta, x)} = \lambda - \varepsilon(\omega - \lambda).$$

In addition, for all $x \leqslant 0$, $\varepsilon \in (0,1]$, and $\delta \in (0,1]$, using $|f(u) - f'(0)u| \le M u^{1+\alpha}$ and $0 < \phi^{\pm} \le 2\delta e^{\lambda x}$ we obtain

$$\begin{aligned} \pm\,\mathcal{L}\,[\phi^{\pm}\delta] &= \delta\varepsilon P(\omega) e^{\omega x} \pm [f(\phi^{\pm}\delta) - f'(0)\phi^{\pm}\delta] \\ &\geqslant \delta e^{\omega x}\{\varepsilon\, P(\omega) - 2^{1+\alpha} M \delta^{\alpha} e^{[(1+\alpha)\lambda - \omega]x}\}. \end{aligned}$$

Hence, we have the following:

(i) For every $\varepsilon \in (0, e^{-\omega}]$, there exists $x_\varepsilon \leqslant 0$ such that $\phi^{\pm}(\varepsilon, 1, \cdot)$ is a super/sub solution on $(-\infty, x_\varepsilon]$.

(ii) For every $\varepsilon \in (0, e^{-\omega}]$, there exists $\delta_\varepsilon > 0$ such that for every $\delta \in (0, \delta_\varepsilon]$, $\phi^{\pm}(\varepsilon, \delta, \cdot)$ is a super/sub solution on $(-\infty, 0]$.

Indeed, we need only take

$$x_\varepsilon := \min\Big\{0, \frac{\ln[\varepsilon P(\omega)] - \ln[2^{1+\alpha} M]}{(1+\alpha)\lambda - \omega}\Big\}, \qquad \delta_\varepsilon = \min\Big\{1, \Big(\frac{\varepsilon P(\omega)}{2^{1+\alpha} M}\Big)^{1/\alpha}\Big\}.$$

Theorem 5.3. *Assume* **(A)**, **(B)**, *and $f'(0) > 0$. Let (c, U) be a traveling wave with speed $c > c_{\min}$. Then $U(x) = e^{\lambda(x + x_0 + o(1))}$ for some $x_0 \in \mathbb{R}$ where $\lim_{x\to-\infty} o(1) = 0$.*

Proof. First of all, note that (4.2) holds for W defined as in (5.3) with $\phi = \phi^{+}(\varepsilon, 1, x)$.

We show that $A > -\infty$. Suppose $A = -\infty$. Fix $\varepsilon = e^{-\omega}$. Since

$$\lim_{x\to\infty} U'(x)/U(x) = \lambda,$$

there exists $\xi < 0$ such that $U'(x)/U(x) < \lambda + \varepsilon(\omega - \lambda)$ for all $x < \xi + 2$. Now we compare $U(\cdot + \xi)$ with $\phi := \phi^{+}(\varepsilon, U(\xi), \cdot)$ on $(-\infty, 0]$. By taking negatively large ξ, we may assume that $U(\xi) < \delta_\varepsilon$ so that ϕ is a super-solution on $(-\infty, 0]$.

Note that $\phi(0) = U(0 + \xi)$ and

$$\frac{\phi'(x)}{\phi(x)} > \lambda + \varepsilon(\omega - \lambda) > \frac{U'(x+\xi)}{U(x+\xi)} \quad \forall\, x \in [0,1]$$

so that $U(\cdot + \xi) < \phi(\cdot)$ on $(0, 1]$. Also, $\lim_{x\to-\infty}[\ln\phi(x) - \ln U(x+\xi)] = \infty$. It follows by comparison that $\phi(\cdot) > U(\cdot + \xi)$ on $(-\infty, 0]$, contradicting $\phi(0) = U(0 + \xi)$. Thus $A > -\infty$.

Similarly, by using the sub-solution ϕ^{-}, one can show that $A < \infty$. Thus $A = \lim_{x\to-\infty}\{\ln U(x) - \lambda x\}$ exists and is finite. This completes the proof. □

5.5. **The case $c > c_{\min}$ and $f'(0) = 0$.** We assume the following:

(B1) $0 \leqslant f f'' \leqslant M f'^2$ on $(0, \epsilon]$ for some $\epsilon > 0$ and $M > 0$; $\int_0^\epsilon f'^2(s)/f(s)ds < \infty$.

Simple examples of such functions are

$$f(u) = \kappa u^{1+q}(1-u)^p, \qquad f(u) = \kappa e^{-1/u}(1-u)^p \qquad (\kappa > 0, q > 0, p \geqslant 1).$$

Theorem 5.4. *Assume* **(A)**, **(B1)**, *and $f'(0) = 0$. Let (c, U) be a traveling wave with non-minimum speed c. Then (1.10) holds for some $x_0 \in \mathbb{R}$.*

Proof. **1. The idea.** The proof is based on the following formal calculation. When $f'(0) = 0$ and $c > c_{\min}$, it follows from Theorem 3 that $cU' \approx f(U)$. Then at least formally we should have $c^2U'' \approx cf'(U)U' \approx f(U)f'(U)$. Since by the mean value theorem $U(x+1) + U(x-1) - 2U(x) = U''(y) \approx U''(x)$, we obtain that

$$cU' \approx U'' + f(U) \approx f(U) + f(U)f'(U)/c^2.$$

This suggests that sub/super solutions can be obtained from solutions of ODEs of the form $c\phi' = f(\phi) + f(\phi)f'(\phi)/c^2 \pm o(1)$, where $o(1)$ is a small positive term large enough to offset the error of the approximation $U(x+1) + U(x-1) - 2U(x) = U''(y) \approx U''(x)$.

2. Construction of super/sub solutions. Let δ_0 be a small enough constant and be fixed. For every $\delta \in (0, \delta_0]$ and $K \in [1, 1/(4f'^2(\delta))]$, let ϕ be the solution of

$$c\phi' = f(\phi)\{1 + f'(\phi)/c^2 \pm Kf'^2(\phi)\} \quad \text{on} \quad (-\infty, 1], \qquad \phi(0) = \delta. \tag{5.8}$$

The solution is given implicitly by

$$\int_\delta^{\phi(x)} \frac{ds}{f(s)[1 + f'(s)/c^2 \pm Kf'^2(s)]} = \frac{x}{c} \quad \forall x \leqslant 1.$$

When δ_0 is small, we have $\phi \leq \delta[1 + o(1)]$ and $c\phi' = f(\phi)[1 + o(1)]$ on $(-\infty, 1]$. In the sequel, $O(1)$ is a quantity bounded by a constant independent of K and δ.

Write (5.8) as $c\phi' = (1 + g(\phi))f(\phi)$ where $g := f'/c^2 \pm Kf'^2$. In the sequel, the arguments of f, f', f'', and g are evaluated at $\phi(x)$, if not specified. Since $f'' \geqslant 0$ and $ff'' = O(1)f'^2$ on the interval of interest, we see that

$$|g| + |g'f/f'| = O(f') + O(f'^2)K.$$

Consequently,

$$c^2\phi''(x) = \{(1+g)f' + fg'\}(1+g)f = ff'\{1 + O(f') + O(f'^2)K\}.$$

Also by the mean value theorem,

$$\phi(x+1) + \phi(x-1) - 2\phi(x) = \phi''(y) \quad \text{for some} \quad y \in [x-1, x+1],$$

$$\frac{f'(\phi(y))}{f'(\phi(x))} = \exp\Big(\int_x^y \frac{(1+g)ff''}{cf'}\Big) = \exp\Big(\int_x^y O(f'(\phi(z))dz\Big).$$

This implies that

$$f'(\phi(y)) = [1 + O(f'(\phi(x)))]f'(\phi(x)).$$

Similarly,

$$f(\phi(y)) = [1 + O(f'(\phi(x)))]f(\phi(x)).$$

This follows that

$$c^2\phi''(y) = f'f\{1 + O(f') + O(f'^2)K\}\Big|_{\phi(x)}.$$

Hence, for all $x \leqslant 1$,

$$\begin{aligned} \mathcal{L}[\phi](x) &= c\phi' - f - f'f\Big\{c^{-2} + O(f') + O(f'^2)K\Big\} \\ &= ff'^2\Big\{\pm K + O(1) + O(f')K\Big\}. \end{aligned}$$

Thus we have the following

Lemma 5.3. *There exist a small positive constant* δ_0 *and a large constant* K_0 *such that for every* $\delta \in (0, \delta_0]$ *and every* $K \in [K_0, 1/(4f'^2(\delta))]$*, the solution* $\phi^\pm(\delta, x) := \phi(x)$ *of* (5.8) *is a super/sub solution on* $(-\infty, 0]$.

3. The comparison. Consider the function

$$W^\pm(\xi, x) = \int_{\phi^\pm(\delta,x)}^{U(x+\xi)} \frac{ds}{f(s)[1 + f'(s)/c^2]} \qquad x \le 1, \xi \in \mathbb{R}.$$

Following a proof similar to that for Lemma 4.3, we can show that (4.2) holds with $W = W^\pm$, $A = A^\pm \in [-\infty, \infty]$ and $\nu = 1/c$. Note that

$$\begin{aligned} W^+ - W^- \;&=\; \int_{\phi^+}^{\delta} \Big\{ \frac{1}{f[1 + f'/c^2]} - \frac{1}{f[1 + f'/c^2 + Kf'^2]} \Big\} ds \\ &\quad - \int_{\phi^-}^{\delta} \Big\{ \frac{1}{f[1 + f'/c^2]} - \frac{1}{f[1 + f'/c^2 - Kf'^2]} \Big\} ds, \end{aligned}$$

since the two integrals involving K cancel each other. Sending $x \to -\infty$ and using $\phi^\pm(-\infty) = 0$ and $\int_0^\epsilon f'^2(s)/f(s)ds < \infty$, we then obtain

$$\lim_{x \to -\infty} \{W^+(\xi, x) - W^-(\xi, x)\} = \int_0^\delta \frac{2Kf'^2}{f\{[1 + f'/c^2]^2 - [Kf'^2]^2\}}\, ds < \infty.$$

We now show that $A^+ > -\infty$. Suppose on the contrary that $A^+ = -\infty$. For each $\delta \in (0, \delta_0]$, taking $K = 1/(4f'(\delta)^2)$ we see that

$$\frac{\phi^{+\prime}(x)}{f(\phi^+(x))} = \frac{1}{c} - \frac{f'(\phi^+)}{c^3} + \frac{f'^2(\phi^+)}{4cf'^2(\delta)} \geqslant \frac{1}{c} + \frac{1}{8c} \quad \forall\, x \in [0, 1]$$

if δ_0 is small enough. As we know that $\lim_{x \to -\infty} U'/f(U) = 1/c$, there exits $\xi < 0$ such that $U'/f(U) < 1/c + 1/(8c)$ for all $x \le \xi + 1$. Now set $\delta = U(\xi)$ and compare $U(\xi + \cdot)$ and $\phi^+(\delta, \cdot)$ on $(-\infty, 0]$.

As $\phi^{+\prime}/f(\phi^+) > U'/f(U)$ on $[0, 1]$ and $\phi(0) = U(\xi + 0)$, we have $\phi^+(\cdot) > U(\xi + \cdot)$ on $(0, 1]$. Also, $A^+ = -\infty$ implies that $\phi^+(x) > U(\xi + x)$ for all $x \ll -1$. By comparison, $\phi^+ > U$ on $(-\infty, 0]$, contradicting $\phi^+(0) = U(\xi + 0)$. Thus $A^+ > -\infty$. Similarly, using ϕ^-, we can show that $A^- < \infty$. Hence $A^\pm$ are finite.

Finally, we observe that

$$\begin{aligned} \lim_{x \to -\infty} W^+(0, x) \;&=\; \lim_{x \to -\infty} \Big\{ \int_\delta^{U(x)} \frac{ds}{f(s)[1 + f'(s)/c^2]} - \frac{x}{c} \Big\} \\ &\quad - \int_0^\delta \Big\{ \frac{1}{1 + f'(s)/c^2} - \frac{1}{1 + f'(s)/c^2 + Kf'^2(s)} \Big\} \frac{ds}{f(s)}, \end{aligned}$$

the assertion of the Theorem thus follows. □

As an illustration, we consider the case when

$$f(u) = \kappa u^2 (1 - u)^p \quad (\kappa > 0,\ p \geqslant 1).$$

Then for some integral constant a

$$\int_{1/2}^{u} \frac{ds}{f(s)[1+f'(s)/c^2]} = -\frac{1}{\kappa u} + \Big(\frac{p}{\kappa} - \frac{2}{c^2}\Big) \ln u + a + O(u) \quad \text{as } u \to 0.$$

After translation, we see that, as $x \to -\infty$,

$$-\frac{1}{\kappa U(x)} + \Big(\frac{p}{\kappa} - \frac{2}{c^2}\Big) \ln U(x) = \frac{x}{c} + o(1)$$

This implies that

$$\frac{1}{U(x)} = \frac{\kappa|x|}{c} + O(\ln|x|) = \frac{\kappa|x|}{c}\Big(1+o(1)\Big), \quad \ln U(x) = \ln \frac{c}{\kappa|x|} + o(1).$$

Thus, after another translation,

$$\begin{aligned} U(x) &= \frac{c}{\kappa[|x| - x_0 + o(1)] + (p\,c - 2\kappa/c)\ln|x|} \\ &= \frac{c}{\kappa|x|} - \frac{(pc^2 - 2\kappa)\ln|x|}{\kappa^2 x^2} - \frac{cx_0 + o(1)}{\kappa x^2} \qquad \text{as } x \to -\infty. \end{aligned}$$

Note that the translation is distinguished by the third term in the Taylor's expansion.

Finally, observe that

$$\int_{1/2}^{u} \frac{ds}{f(s)[1+f'(s)/c^2]} = \int_{1/2}^{u} \frac{ds}{f(s)} - \frac{\ln f(u)}{c^2} + a + o(1) \quad \text{as } u \to 0.$$

In particular, if $f(u) = \kappa u^{1+q}[1+o(1)]$ for some $q > 0$, then $U \propto |x|^{-1/q}$ so that $\ln f(U) \approx -b\ln|x| + B + o(1)$ for some $b > 0$ and $B \in \mathbb{R}$. Therefore, it is generic that for some constants $b > 0$ and $x_0 \in \mathbb{R}$,

$$\int_{1/2}^{U(x)} \frac{ds}{f(s)} = \frac{c[x + x_0 + o(1)] - b\,\ln|x|}{c^2}.$$

In a similar manner, we can establish an asymptotic expansion near ∞. We omit the details.

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